LOOP GROUPS AND CATEGORIZED
GEOMETRY

Notes for talk at Streetfest
(joint work with John Baez, Alissa Crans and Urs Schreiber)

Lie 2-groups

A (strict) Lie 2-group is a small category $\mathcal{G}$ such that

- the set of objects $\mathcal{G}_0$ and
- the set of morphisms $\mathcal{G}_1$

are Lie groups;

- source and target $s, t: \mathcal{G}_1 \to \mathcal{G}_0$,
- the identity assigning function $i: \mathcal{G}_0 \to \mathcal{G}_1$,
- composition $\circ: \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$

are all homomorphisms of Lie groups.

By Lie group we mean a group $G$ which is also a manifold such that the map

\[ G \times G \to G \]
\[ (g, h) \mapsto gh^{-1} \]

is smooth, but by manifold we could mean an infinite dimensional manifold modelled on a locally convex topological vector space, for example a Fréchet space. In particular, we will consider Fréchet Lie groups and hence Fréchet Lie 2-groups.

Remarks

1. There is a notion of weak Lie 2-group.

2. Lie 2-groups are the same as Lie crossed modules

\[ \partial: H \to G \]
\[ \alpha: G \to \text{Aut}(H) \]
3. Every Lie 2-group $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ has an associated Lie 2-algebra $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1)$

$$\mathfrak{g}_0 = \text{Lie}(\mathcal{G}_0)$$
$$\mathfrak{g}_1 = \text{Lie}(\mathcal{G}_1)$$

The differentials $ds, dt, di$ and $d \circ \mathfrak{g}_1 \times_{\mathfrak{g}_0} \mathfrak{g}_1 \to \mathfrak{g}_1$ are all Lie algebra homomorphisms.

Let $G$ be a compact, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$. For any $k \in \mathbb{R}$ Baez and Crans construct a (weak) Lie 2-algebra $\mathfrak{g}_k$ with

$$\text{Ob}(\mathfrak{g}_k) = \mathfrak{g}$$
$$\text{Mor}(\mathfrak{g}_k) = \mathfrak{g} \oplus i\mathbb{R}$$

but where the ‘Jacobiator’ is given by the basic 3-form $k\langle x, [y, z] \rangle$ on $G$.

**Question:** Is $\mathfrak{g}_k$ the Lie 2-algebra of some Lie 2-group??

For any $k \in \mathbb{Z}$ Baez and Lauda construct a (weak) 2-group $G_k$ but this is not a Lie 2-group.

We will explain how to construct a Fréchet Lie 2-group $\mathcal{P}_kG$ whose Lie 2-algebra is equivalent to $\mathfrak{g}_k$. $\mathcal{P}_kG$ is closely related to central extensions of loop groups and to the basic gerbe on $G$.

**Loop Groups**

Let $G$ be a compact, simple, simply connected Lie group. We define

$$LG = \{f : [0, 2\pi] \to G \mid f \text{ is } C^\infty, f(0) = f(2\pi)\}$$

the loop group

and

$$\Omega G = \{f \in LG \mid f(0) = 1\}$$

the based loopgroup

$LG$ and $\Omega G$ are Fréchet Lie groups. $\Omega G$ behaves like a compact Lie group in many ways except that there exist topologically non-trivial central extensions

$$1 \to \mathbb{T} \to \hat{\Omega G} \to \Omega G \to 1$$

the Kac-Moody group

where $\hat{\Omega G}$ is the Kac-Moody group. There is a corresponding central extension of Lie algebras

$$0 \to i\mathbb{R} \to \Omega_k\mathfrak{g} \to \Omega\mathfrak{g} \to 0$$
This is easier to understand: it is determined up to isomorphism by the Kac-Moody $2$-cocycle

$$\omega(f, g) = 2 \int_0^{2\pi} \langle f(\theta), g'(\theta) \rangle d\theta$$

where $f, g \in \Omega G = \{f: [0, 2\pi] \to g| f \text{ is } C^\infty, f(0) = f(2\pi) = 0\}$ and $\langle , \rangle$ denotes the Killing form normalised so that $|h_\theta| = \frac{1}{\sqrt{2\pi}}$ where $h_\theta$ is the co-root associated to the longest root $\theta$.

**Kac-Moody Central Extension**

Let $G$ be a simply connected Lie group forming part of a central extension

$$1 \to \mathbb{T} \to \hat{G} \to G \to 1$$

so that $G$ is the total space of a principal $\mathbb{T}$-bundle over $G$ which is also a group containing $\mathbb{T}$ as a central subgroup. Suppose also that $G$ is equipped with a connection $\nabla$ whose curvature $2$-form is $F_\nabla$. Denote by

$$P_0 G = \{f: [0, 2\pi] \to G| f \text{ is } C^\infty, f(0) = 1\}$$

the group of **based paths** in $G$. $P_0 G$ is a group under pointwise multiplication of paths in $G$. Note that there is a homomorphism $\pi: P_0 G \to G$ which evaluates a path at its endpoint: $\pi(f) = f(2\pi)$. The kernel of $\pi$ is just the based loop group $\Omega G$.

We can use the homomorphism $\pi$ to pullback the central extension $\hat{G}$ to obtain a new group $\pi^* \hat{G}$ which is a central extension of $P_0 G$

$$\pi^* \hat{G} \longrightarrow \hat{G}$$

$$\downarrow$$

$$P_0 G \longrightarrow \pi^* \hat{G}$$

$P_0 G$ is **contractible** and hence the central extension $1 \to \mathbb{T} \to \pi^* \hat{G} \to P_0 G \to 1$ is **split**. In fact a splitting can be constructed explicitly as follows using the connection $\nabla$: if $f$ is a based path in $P_0 G$ denote by $\hat{f}$ the unique horizontal lift of $f$ to a path in $\hat{G}$ starting at $1$. Then $\sigma: P_0 G \to \pi^* \hat{G}$ defined by $\sigma(f) = (f, \hat{f})$ provides such a splitting. In particular we obtain an isomorphism $\pi^* \hat{G} \cong P_0 G \times \mathbb{T}$. The product on the group $P_0 G \times \mathbb{T}$ is determined however by a $\mathbb{T}$-valued $2$-cocycle

$$c: P_0 G \times P_0 G \to \mathbb{T}$$
If $G = \Omega G$ then
\[
c(f, g) = \exp \left( 2ik \int_0^{2\pi} \int_0^{2\pi} \langle f(t)^{-1} f'(t), g'(\theta) g(\theta)^{-1} \rangle d\theta dt \right)
\]
where $f = f(t, \theta), g = g(t, \theta) \in P_0 \Omega G$.

So we have the commutative diagram of groups and group homomorphisms
\[
\begin{array}{ccc}
P_0G \times \mathbb{T} & \xrightarrow{\hat{\pi}} & \hat{G} \\
\downarrow & & \downarrow \\
P_0G & \xrightarrow{\pi} & G
\end{array}
\]
where $\hat{\pi}(f, z) = \hat{f}(2\pi) z$. The kernel $\ker \hat{\pi}$ is the normal subgroup
\[
\ker \hat{\pi} = \{(f, z) | \hat{f}(2\pi) = z^{-1}, f(2\pi) = 1\}
\]
where $\ker \hat{\pi} = \{(f, z) | z^{-1} = \text{Hol}_f(\nabla)\}$

where $\text{Hol}_f(\nabla)$ denotes the holonomy of $\nabla$ around the loop $f$. Since $G$ is simply connected,
\[
\text{Hol}_f(\nabla) = \exp \left( 2\pi i \int_{D_f} F_{\nabla} \right)
\]
where $D_f$ is any disc with boundary the loop $f$. If $G = \Omega G$ then $F_{\nabla}$ is the left invariant 2-form whose value at the identity is just the Kac-Moody 2-cocycle:
\[
F_{\nabla}(\xi, \eta) = 2 \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle d\theta
\]

The point of this construction is that we can recover $\hat{G}$ by

1. equipping $P_0G \times \mathbb{T}$ with the product coming from the 2-cocycle $c$, and
2. letting $\hat{G}$ be the quotient
\[
\hat{G} = \frac{P_0G \times \mathbb{T}}{N}
\]

where $N$ is the normal subgroup
\[
N = \{(\gamma, z) | \gamma \in \Omega G, z^{-1} = \exp \left( 2\pi i \int_{D_\gamma} F_{\nabla} \right) \}
\]
Construction of $\mathcal{P}_kG$

As above, let $P_0G = \{ f : [0, 2\pi] \to G \mid f \text{ is } C^\infty, f(0) = 1 \}$. $P_0G$ acts on $\Omega G$ by conjugation and induces an action on $P_0\Omega G$. Define an action of $P_0G$ on $P_0\Omega G \times \mathbb{T}$ by

\[ p \cdot (f, z) = (p^{-1}fp, z \exp \left( ik \int_0^{2\pi} \int_0^{2\pi} \langle f(t)^{-1}f'(t), p'(\theta)p(\theta)^{-1} \rangle d\theta dt \right)) \]

where $p = p(\theta) \in P_0G$ and $f = f(t, \theta) \in P_0\Omega G$. This action of $P_0G$ preserves the normal subgroup $N$ and induces an action of $P_0G$ on $\hat{\Omega}_kG$ by automorphisms, so that we have a Fréchet Lie crossed module

\[ \alpha : P_0G \to \text{Aut}(\hat{\Omega}_kG) \]
\[ \partial : \hat{\Omega}_kG \to P_0G \]

where $\partial$ is defined as the composite $\hat{\Omega}_kG \xrightarrow{P} \Omega G \xrightarrow{i} P_0G$.

Let $\mathcal{P}_kG$ have Fréchet Lie group $\text{Ob}(\mathcal{P}_kG)$ of objects given by

\[ \text{Ob}(\mathcal{P}_kG) = P_0G \]

and Fréchet Lie group $\text{Mor}(\mathcal{P}_kG)$ of morphisms given by the semi-direct product

\[ \text{Mor}(\mathcal{P}_kG) = P_0G \rtimes \hat{\Omega}_kG \]

Then $\mathcal{P}_kG = (P_0G, P_0G \rtimes \hat{\Omega}_kG)$ is a Fréchet Lie 2-group when source, target, composition etc are defined as follows:

**source** $s(p, \hat{\gamma}) = p$

**target** $t(p, \hat{\gamma}) = p\partial(\hat{\gamma})$

**composition** $(p_1, \hat{\gamma}_1) \circ (p_2, \hat{\gamma}_2) = (p_1, \hat{\gamma}_1 \hat{\gamma}_2)$ when $t(p_1, \hat{\gamma}_1) = s(p_2, \hat{\gamma}_2)$, i.e. $p_2 = p_1\partial(\hat{\gamma}_1)$.

**identities** $i(p) = (p, 1)$.

where $p \in P_0G$ and $\hat{\gamma} \in \hat{\Omega}_kG$.

**Theorem 1.** The Lie 2-algebra of $\mathcal{P}_kG$ is equivalent to $\mathfrak{g}_k$. 

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Topology of $\mathcal{P}_k G$

The nerve of any topological 2-group $G = (\mathcal{G}_0, \mathcal{G}_1)$ is a simplicial topological group and we therefore obtain on passing to geometric realisations a topological group

\[ |\mathcal{P}_k G| \]

In fact more is true: there is an exact sequence of topological 2-groups

\[ 1 \rightarrow \mathcal{L}_k G \rightarrow \mathcal{P}_k G \rightarrow G \rightarrow 1 \]

where $\mathcal{L}_k G$ is the topological 2-group with

\[
\text{Ob}(\mathcal{L}_k G) = \Omega G \\
\text{Mor}(\mathcal{L}_k G) = \Omega G \ltimes \widehat{\Omega_k G}
\]

and where $G$ is considered as a discrete 2-group (i.e. there only exist identity morphisms). Here by exact sequence we mean only that the sequences of groups $1 \rightarrow \text{Ob}(\mathcal{L}_k G) \rightarrow \text{Ob}(\mathcal{P}_k G) \rightarrow \text{Ob}(G) \rightarrow 1$ and $1 \rightarrow \text{Mor}(\mathcal{L}_k G) \rightarrow \text{Mor}(\mathcal{P}_k G) \rightarrow \text{Mor}(G) \rightarrow 1$ are both exact.

Applying the geometric realisation functor $| \cdot |$ we get a short exact sequence of topological groups

\[ 1 \rightarrow |\mathcal{L}_k G| \rightarrow |\mathcal{P}_k G| \rightarrow G \rightarrow 1 \]

We note the following two facts:

- $|\mathcal{L}_k G|$ has the homotopy type of a $K(\mathbb{Z}, 2)$
- $|\mathcal{P}_k G| \rightarrow G$ is a locally trivial fibre bundle with fibre $K(\mathbb{Z}, 2)$.

Recall that $K(\mathbb{Z}, 2)$ bundles on a space $X$ are classified up to isomorphism by their Dixmier-Douady invariants in $H^3(X; \mathbb{Z})$. We have

**Theorem 2.** The Dixmier-Douady class of the $K(\mathbb{Z}, 2)$-bundle $|\mathcal{P}_k G| \rightarrow G$ is $k$ times the generator of $H^3(G; \mathbb{Z}) = \mathbb{Z}$. When $k = \pm 1$, $|\mathcal{P}_k G|$ is $\hat{G}$ — the topological group obtained by killing the third homotopy group of $G$.

Fundamental particles (for example electrons) have extra degrees of freedom (spin) — so we need to enlarge the group of symmetries to Spin($n$)

\[ 1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1 \]

For string theory we need to enlarge the group of symmetries to an even bigger group String($n$)

\[ 1 \rightarrow K(\mathbb{Z}, 2) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n) \rightarrow 1 \]

When $G = \text{Spin}(n)$ and $k = \pm 1$, we obtain the important result that $|\mathcal{P}_k G| = \text{String}(n)$.  

\( \mathcal{P}_kG \) and gerbes

Note that the Lie 2-group \( \mathcal{P}_kG = (P_0G, P_0G \ltimes \hat{\Omega}_kG) \) fits into a short exact sequence of groupoids

\[
\begin{array}{cccccc}
\mathbb{T} & \rightarrow & P_0G \ltimes \hat{\Omega}_kG & \rightarrow & P_0G \ltimes \Omega G & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
* & \rightarrow & P_0G & \rightarrow & P_0G & \rightarrow & 0
\end{array}
\]

This exhibits \( \mathcal{P}_kG \) as a \( \mathbb{T} \)-central extension of groupoids. So \( \mathcal{P}_kG \) is a \( \mathbb{T} \)-bundle gerbe in the sense of Murray. In this way \( \mathcal{P}_kG \) provides a realisation of the basic gerbe on \( G \).

In fact \( \mathcal{P}_kG \) is a multiplicative \( \mathbb{T} \)-bundle gerbe. Recall that a multiplicative gerbe on \( G \) consists of the following data:

- a \( \mathbb{T} \)-gerbe \( \mathcal{G} \) on \( G \)
- a morphism \( \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \)
- a coherent natural isomorphism

\[
\begin{array}{ccc}
\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G} & \xrightarrow{m \otimes 1} & \mathcal{G} \otimes \mathcal{G} \\
1 \otimes m & \downarrow & m \\
\mathcal{G} \otimes \mathcal{G} & \xrightarrow{m} & \mathcal{G}
\end{array}
\]

Multiplicative gerbes play a role in Chern-Simons theory and twisted \( K \)-theory. It is interesting to note that \( \mathcal{P}_kG \) is a strictly multiplicative gerbe.

**String structures**

Suppose that \( P \xrightarrow{\mathcal{G}} M \) is a principal \( G \)-bundle where \( G \) is as above. Let \( [\nu] \) denote the generator of \( H^3(G; \mathbb{Z}) = \mathbb{Z} \). \([\nu]\) is universally transgressive: let \( [c] \in H^4(M; \mathbb{Z}) \) denote the transgression of \([\nu]\) in the fibre bundle \( P \to M \). If \( G = \text{Spin}(n) \) then \( 2[c] = p_1 \).

A string structure for \( P \) is a lift of the structure group of the bundle \( \Omega P \) to \( \hat{\Omega}_kG \):

\[
\begin{array}{ccc}
\Omega P & \xrightarrow{\hat{\Omega}G} & \Omega P \\
\Omega M & \xleftarrow{\Omega G} & \Omega M
\end{array}
\]
where \( \Omega P \to \Omega M \) is the principal \( \Omega G \)-bundle one gets by applying the based loops functor \( \Omega \) to the pointed spaces \( P \) and \( M \).

Assume \( M \) is 2-connected. Then a string structure for \( P \) exists iff \( [c] = 0 \). In this case, one can construct explicitly a non-abelian gerbe for the crossed module \( \Omega k G \to P_0 G \) in the sense of Breen. The existence of a string structure can also be interpreted, when \( G = \text{Spin}(n) \), as a solution to the obstruction problem

\[
\begin{array}{ccc}
B\text{String}(n) & \to & B\text{Spin}(n) \\
\downarrow & & \downarrow \\
M & \longrightarrow & \end{array}
\]

In my opinion, the gerbe referred to above is analogous to an extension of the structure group of \( P \) to \( \text{String}(n) \).