Higher Gauge Theory (II)

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More details at:
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Gauge Theory

Ordinary gauge theory describes how point particles transform as they move along paths in spacetime:

In the simplest setup, a ‘transformation’ is an element of a smooth group $G$, and ‘spacetime’ is a smooth space $M$.

(We work in a convenient category of ‘smooth spaces’, including smooth manifolds as a full subcategory, but cartesian closed, with all limits and colimits.)

A **connection** is a $g$-valued 1-form $A$ on $M$. This lets us compute a **holonomy** $\text{hol}(\gamma) \in G$ for each path $\gamma : [0, 1] \to M$, as follows. Solve this differential equation:

$$\frac{d}{dt}g(t) = A(\gamma'(t))g(t)$$

with initial value $g(0) = 1$. Then let

$$\text{hol}(\gamma) = g(1).$$
Holonomy as a Functor

The holonomy along a path doesn’t depend on its parametrization. When we compose paths, their holonomies multiply:

When we reverse a path, we get a path with the inverse holonomy:

So, let $\mathcal{P}_1(M)$ be the path groupoid of $M$:

- objects are points $x \in M$: $\bullet_x$
- morphisms are thin homotopy classes of smooth paths $\gamma: [0, 1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$:

This is a smooth groupoid: it has a smooth space of objects and a smooth space of morphisms, with all groupoid operations being smooth.

**Theorem.** Given connection on a smooth space $M$, its holonomies along paths determine a smooth ‘holonomy functor’:

$$\text{hol}: \mathcal{P}_1(M) \to G.$$
Bundles

The story so far is oversimplified. It’s evil to demand that holonomies are group elements – we should only demand that each point in $M$ have a neighborhood in which holonomies can be regarded as group elements.

So, define a bundle over $M$ to be:

- a smooth space $P$ (the total space),
- a smooth space $F$ (the standard fiber),
- a smooth map $p: P \rightarrow M$ (the projection),

such that for each point $x \in M$ there exists an open neighborhood $U$ equipped with a diffeomorphism

$$f: p^{-1}U \rightarrow U \times F,$$

(the local trivialization) such that

commutes.
Principal Bundles

If $F$ is a smooth space, $\text{Aut}(F)$ is a smooth group. Given a bundle $P \to M$ with standard fiber $F$, the local trivializations over neighborhoods $U_i$ covering $M$ give:

- smooth maps (transition functions)

$$g_{ij}: U_i \cap U_j \to \text{Aut}(F)$$

such that:

- $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$,
- $g_{ii}(x) = 1$.

For any smooth group $G$, we say a bundle $P \to M$ has $G$ as its structure group when the maps $g_{ij}$ factor through an action $G \to \text{Aut}(F)$.

If furthermore $F = G$ and $G$ acts on $F$ by left multiplication, we say $P$ is a principal $G$-bundle.
Connections

What’s a connection on a principal $G$-bundle $P \to M$? In each neighborhood $U_i$ it’s a $\mathfrak{g}$-valued 1-form $A_i$, but we demand compatibility:

$$A_i = g_{ij}A_jg^{-1}_{ij} + g_{ij}dg^{-1}_{ij}$$

on the intersections $U_i \cap U_j$.

What is the holonomy of such a connection along a path? There is a smooth groupoid $\text{Trans}(P)$, the transport groupoid, for which:

- objects are the fibers $P_x = p^{-1}(x)$ for $x \in M$, which are $G$-torsors: right $G$-spaces isomorphic to $G$.
- morphisms are $G$-torsor morphisms $f: P_x \to P_y$.

**Theorem.** Any connection on a principal $G$-bundle $P \to M$ gives a smooth ‘holonomy functor’:

$$\text{hol}: \mathcal{P}_1(M) \to \text{Trans}(P).$$
Higher Gauge Theory

Higher gauge theory should describe how strings transform as move them along surfaces in spacetime:

\[
\begin{array}{c}
\bullet \\
\beta \quad \alpha
\end{array}
\]

So, let’s categorify all the above and get a theory of 2-connections on principal 2-bundles!

The crucial trick is ‘internalization’. Given a familiar gadget \( x \) and a category \( K \), we define an ‘\( x \) in \( K \)’ by writing the definition of \( x \) using commutative diagrams and interpreting these in \( K \).

We will need these examples:

- A **smooth group** is a group in \([\text{Smooth Spaces}]\).
- A **smooth groupoid** is a groupoid in \([\text{Smooth Spaces}]\).
- A **smooth category** is a category in \([\text{Smooth Spaces}]\).
- A **smooth 2-group** is a 2-group in \([\text{Smooth Spaces}]\).
- A **smooth 2-groupoid** is a 2-groupoid in \([\text{Smooth Spaces}]\).

Here 2-groups and 2-groupoids come in two flavors: *strict* and *coherent*. In the former all laws hold as equations; in the latter, they hold up to specified isomorphisms which satisfy coherence laws of their own. For details, see my paper with Aaron Lauda and references therein.
2-Bundles

Toby Bartels has developed a theory of 2-bundles, which we roughly sketch here.

We can think of a smooth space $M$ as a smooth category with only identity morphisms. A 2-bundle over $M$ consists of:

- a smooth category $P$ (the total space),
- a smooth category $F$ (the standard fiber),
- a smooth functor $p: P \to M$ (the projection),

such that each point $x \in M$ is equipped with an open neighborhood $U$ and a smooth equivalence:

$$f: p^{-1}U \to U \times F$$

(the local trivialization) such that:

$$\begin{array}{ccc}
p^{-1}U & \xrightarrow{f} & U \times F \\
p|_{p^{-1}U} & \downarrow & \\
U & \end{array}$$

commutes.
Principal 2-Bundles

If $F$ is a smooth category, $\mathcal{G} = \text{Aut}(F)$ is a smooth 2-group. Given a 2-bundle $P \to M$ with standard fiber $F$, the local trivializations over open sets $U_i$ covering $M$ give:

- smooth maps
  \[ g_{ij} : U_i \cap U_j \to \text{Ob}(\mathcal{G}) \]

- smooth maps
  \[ h_{ijk} : U_i \cap U_j \cap U_k \to \text{Mor}(\mathcal{G}) \]
  with
  \[ h_{ijk}(x) : g_{ij}(x)g_{jk}(x) \to g_{ik}(x) \]

- smooth maps
  \[ k_i : U_i \to \text{Mor}(\mathcal{G}) \]
  with
  \[ k_i(x) : g_{ii}(x) \to 1 \in \mathcal{G}. \]

Furthermore:

- $h$ satisfies an equation on quadruple intersections $U_i \cap U_j \cap U_k \cap U_l$:

  \[
  h_{ijl} h_{jkl} h_{ikl} = g_{ij} g_{kl} = g_{ij} g_{ik} g_{kl}
  \]

  (the **associative law**)
• $k$ satisfies two equations on double intersections $U_i \cap U_j$:

\[
\begin{align*}
g_{ij} &\ circ k_i \circ 1 \circ g_{ij} \quad = \quad g_{ii} \circ \underbrace{h_{iij}}_{k_i} \circ g_{ij} \\
g_{ij} &\ circ 1 \circ g_{jj} \quad = \quad g_{ij} \circ \underbrace{h_{ijj}}_{k_j} \circ g_{ij}
\end{align*}
\]

(the **left unit law**) and

For any smooth 2-group $G$, we say a 2-bundle $P \to M$ has $G$ as its **structure 2-group** when $g_{ij}$, $h_{ijk}$, and $k_i$ factor through an action $G \to \text{Aut}(F)$.

If furthermore $F = G$ and $G$ acts on $F$ by left multiplication, we say $P$ is a **principal $G$-2-bundle**.
2-Connections

So far Urs Schreiber and I have only handled 2-connections on principal 2-bundles where the structure 2-group $\mathcal{G}$ is strict.

A smooth strict 2-group $\mathcal{G}$ is determined by:

- the smooth group $G$ consisting of all objects of $\mathcal{G}$,
- the smooth group $H$ consisting of all morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t: H \to G$ sending each morphism in $H$ to its target,
- the action $\alpha$ of $G$ on $H$ defined using conjugation in the group $\text{Mor}(\mathcal{G})$ via
  \[
  \alpha(g)h = 1_g h 1_g^{-1}
  \]

The system $(G, H, t, \alpha)$ satisfies equations making it a ‘crossed module’. Conversely, any crossed module of smooth groups gives a strict smooth 2-group.
Let $\mathcal{G}$ be a strict smooth 2-group.

Let $(G, H, t, \alpha)$ be its crossed module.

Let $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ be the corresponding ‘differential crossed module’ — the Lie algebra analogue of a crossed module.

If $P \to M$ is a principal 2-bundle with structure 2-group $\mathcal{G}$ and $U_i$ is an open cover of $M$ by neighborhoods equipped with local trivializations of $P$, we can describe a 2-connection on $P$ in terms of:

- a $\mathfrak{g}$-valued 1-form $A_i$ on each open set $U_i$,
- an $\mathfrak{h}$-valued 2-form $B_i$ on each open set $U_i$,

together with some extra data and equations for double and triple intersections. The details are in our paper; as we’ll see, these 2-connections are closely related to Breen and Messing’s connections on nonabelian gerbes.

If $P$ is trivial ($P = M \times \mathcal{G}$) all this reduces to:

- a $\mathfrak{g}$-valued 1-form $A$ on $M$,
- an $\mathfrak{h}$-valued 2-form $B$ on $M$. 
Holonomy as a 2-Functor

Let’s consider a 2-connection on a trivial 2-bundle and ponder the existence of a holonomy 2-functor

$$\text{hol}: \mathcal{P}_2(M) \to \mathcal{G}$$

where the path 2-groupoid $\mathcal{P}_2(M)$ is defined so that:

- objects are points of $M$: $\bullet_x$
- morphisms are smooth paths $\gamma: [0, 1] \to M$ such that $\gamma(t)$ is constant in a neighborhood of $t = 0$ and $t = 1$:

$$\begin{array}{c}
\bullet_x \\
\gamma \\
\bullet_y
\end{array}$$

- 2-morphisms are thin homotopy classes of smooth maps $f: [0, 1]^2 \to M$ such that $f(s, t)$ is independent of $s$ in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$:

$$\begin{array}{c}
\bullet_x \\
\gamma_1 \\
\parallel f \\
\gamma_2 \\
\bullet_y
\end{array}$$
Recall: $G$ is a strict smooth 2-group with crossed module $(G, H, t, \alpha)$. A 2-connection on a trivial principal $G$-2-bundle over $M$ consists of:

- a $g$-valued 1-form $A$ on $M$,
- an $h$-valued 2-form $B$ on $M$.

**Theorem.** A 2-connection on a trivial principal $G$-2-bundle determines a smooth ‘holonomy 2-functor’:

$$\text{hol}: \mathcal{P}_2(M) \to G$$

if and only if its **fake curvature** vanishes:

$$F_A - dt(B) = 0,$$

where $F_A$ is the usual curvature of $A$, namely the $g$-valued 2-form $F_A = dA + A \wedge A$.

Vanishing fake curvature guarantees that parallel transport along a surface $f: [0, 1]^2 \to M$ is invariant under thin homotopies — in particular, invariant under reparametrizations of $[0, 1]^2$.

All this generalizes to nontrivial principal $G$-2-bundles using the **transport 2-groupoid** $\text{Trans}(P)$, for which:

- objects are the fibers $P_x$ (which are $G$-2-torsors),
- morphisms are 2-torsor morphisms $f: P_x \to P_y$,
- 2-morphisms are 2-torsor 2-morphisms $\theta: f \Rightarrow g$.

**Theorem.** A 2-connection on a principal $G$-2-bundle $P \to M$ determines a smooth ‘holonomy 2-functor’:

$$\text{hol}: \mathcal{P}_2(M) \to \text{Trans}(P)$$

if and only if the fake curvature vanishes.
2-Bundles, Stacks and Gerbes

Just as a bundle has a sheaf of sections, a 2-bundle has a ‘stack of sections’. This must be defined carefully, using the local trivializations. In certain cases this stack is a gerbe!

Any smooth category $X$ determines a smooth 2-group $\text{Aut}(X)$, in which:

- objects are smooth equivalences $f: X \to X$.
- morphisms are smooth natural isomorphisms $\theta: f \Rightarrow g$.

A smooth group $H$ is a special sort of smooth category, so it gives a smooth 2-group $\text{Aut}(H)$.

**Theorem.** The stack of sections of a principal $\text{Aut}(H)$-2-bundle $P \to M$ is a nonabelian $H$-gerbe. A connection on this nonabelian gerbe (in the sense of Breen and Messing) is the same as a 2-connection on $P$.

Toby Bartels is working on:

**Conjecture.** The 2-category of principal $\text{Aut}(H)$-2-bundles over $M$ is biequivalent to the 2-category of nonabelian $H$-gerbes over $M$. 
