Question: Consider the category of abelian groups. We want to define a "ring-like" structure, that is, "ring-like" in the sense that a groupoid is "group-like". What would be the morphisms in this case? How could we define a partial addition and a partial multiplication on this category.

Answer: Let the morphism between any two elements $G_i$ and $G_j$ of the category be the abelian group of homomorphisms between the two groups. Define the addition on this category as addition of homomorphisms. This gives us a partially defined addition since two morphisms can only be added if they are in the same abelian group of homomorphisms. Define the multiplication on this category as the function composition of two homomorphisms. Then this gives a partially defined multiplication in the sense that two homomorphisms can only be composed if the range space of the first homomorphism is a subset of the domain of the second (notice that this is similar to the criterion for two elements in a groupoid to be composable). Recall that in a ring, not every element is invertible (in general). Now notice that every morphism here is not necessarily invertible as not all homomorphisms are bijections so inverses are not necessarily defined. The other ring properties hold as well, that is, when they make sense (i.e. distributivity and associativity).

So we can roughly think of a ringoid as a "collection" of elements with operations of multiplication and addition defined on certain ordered pairs and having the ring properties, whenever the operations are defined.

Thinking of it in a categorical sense:

Definition 1 An enriched category is a category whose hom-sets are replaced by objects from another category, in a well-behaved manner.

Definition 2 A ringoid $R$ is a category enriched over the category of abelian groups.

Example 3 Will do in the talk...

Remark 4 A map between ringoids is a functor that is an abelian group homomorphism on each hom-set.

Some examples:

Exercise 5 Given two ringoids $R$ and $S$, there is a category $\text{hom}(R,S)$ where the objects are maps from $R$ to $S$, and the morphisms are natural transformations between such maps. Then $\text{hom}(R,S)$ is again a ringoid.
Example 6 The category of (left) modules over a ring $R$ (in particular the category of vector spaces over a field $F$).

Now to start following the article:

Preliminaries

Axiom 7 A collection, $R$, of elements is said to be a ringoid if the operations of addition and multiplication are defined for certain pairs of elements and satisfy the following axioms:

1) $R = \bigsqcup_{i \in I} G_i$, where the $G_i$ are non-trivial, additive abelian groups and $I$ is an index set.

2) Addition is only defined between elements that belong to the same $G_i$ and (the rules for multiplication are similar to that of addition in a groupoid) multiplication, which can occur between any two $G_i$, is defined whenever it makes sense. (Admittedly this is a little vague, so I will give an example of this...)

3) For $a, b, c \in R$, the following hold if either side is defined (i.e. if one side is defined the other side is also, and they are equal:
   i) $a(bc) = (ab)c$
   ii) $a(b + c) = ab + ac$
   iii) $(a + b)c = ac + bc$

4) Define for $a \in R$, the left multiplicitive set $L(a) = \{x \in R \mid xa \text{ is defined}\}$, and the right multiplicitive set $R(a) = \{x \in R \mid ax \text{ is defined}\}$. $R$ satisfies the following conditions:
   $\alpha$. For every $a \in R$, $L(a) \neq \phi$ and $R(a) \neq \phi$.
   $\beta$. If $L(a) \cap L(b) \neq \phi$ and $R(a) \cap R(b) \neq \phi$, then $a + b$ is defined.

Notation 8 I will write $0_x$ to mean the zero of the element $x \in R$, i.e. $x + 0_x = x$ holds.

Lemma 9 In a ringoid $R$ if $a + b$ is defined, then $L(a) = L(b)$ and $R(a) = R(b)$.

Proof. Try it! You can ask me how to do it if you like...
Example 10 Given a ringoid $R$, the set of polynomials $R[x]$ constructed in the usual manner also form a ringoid.

Example 11 Rings are of course ringoids.

Lemma 12 Let $a, x, r \in R$. Suppose $a^2, a^3, ax, xr$ are defined. Then $(ax)r$ (and thus $a(xr)$) is defined.

Proof. Will do in talk... ■

Definition 13 A subset $I$ of a ringoid $R$ is said to be a right ideal in $R$ if:

$$I - I = \{r - s | r, s \in I, r - s \text{ is defined}\}$$

and

$$IR = \{ir | i \in I, r \in R, ir \text{ is defined}\}$$

are contained in $I$.

Left ideals and two-sided are defined similarly.

Definition 14 An ideal consisting of zeros alone will be called a null ideal. We denote this by $N$.

The Factor Ringoid

Definition 15 Will do in talk...

Semi-Simple Ringoids

Definition 16 An ideal $I \subset R$, $R$ a ringoid, is called nilpotent if there exists an $n \in \mathbb{N}$ such that:

$$I^n = \left\{ \sum_{\text{finite sum}} x_1x_2\cdots x_n | x_i \in I, \text{and } x_1x_2\cdots x_n \text{ is defined} \right\} = \emptyset \text{ or } N.$$ 

Notation 17 I will use the notation $I_N$ to denote a nilpotent ideal.

Definition 18 Will do in talk...
Definition 19 A ringoid $R$ is said to be semi-simple if:

See talk for definition...

Definition 20 Let $I$, $J$ be two right ideals of the ringoid $R$. Suppose that $\forall r \in R$

(a) either $r = i + j, i \in I, j \in J$,
   or $r = i, i \in I$,
   or $r = j, j \in J$;
and (b) $I \cap J = N$ or $\varnothing$.

Then we say that $R$ is the direct sum of $I$ and $J$ and we write $R = I \oplus J$.

Lemma 21 If $I$ is a minimal right ideal in the ringoid $R$, then $I^2 = N$ or $\varnothing$; or $I = eR$, $e^2 = e$.

Lemma 22 Peirce decomposition:

Given an idempotent $e \in R$, $R = eR \oplus R'$, where $R'$ is a right ideal.

Proof. Let $r \in R$. Set $r = \begin{cases} er + (r - er) & \text{if the right hand side is defined.} \\ r & \text{otherwise.} \end{cases}$

Let $R' = \{(r - er) | r \in R\} \cup \{r \ (r - er) \text{ is not defined}\}$

Then $R'$ is a right ideal and $eR \cap R' \subset N$. Hence $R = eR \oplus R'$. ■

Theorem 23 A semi-simple ringoid $R$ is a direct sum of a finite number of minimal right ideals $e_iR$, $e_i^2 = e_i$.

Proof. Will do in talk... ■

Lemma 24 Let $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR \oplus e_{n+1}R \oplus \cdots \oplus e_mR$, such that $e_i^2 = e_i \ \forall i$ and $e_1 = e_i$ is defined iff $j \leq n$. Then:

1. $e_1e_j$ is defined for $i, j \leq n$ and $e_i e_r$ is not defined for $i \leq n$ and $r > n$.
2. $e_1R \oplus e_2R \oplus \cdots \oplus e_nR$ has a left identity.
3. $e_1R \oplus \cdots \oplus e_nR = f_1R \oplus \cdots \oplus f_nR$, for some $f_i$ such that $f_r f_s = \delta_{rs} f_i$ (where $\delta_{rs}$ is the Kronecker delta).
4. $f_i e_k (e_s f_i)$ is defined iff $e_i e_k (e_s e_i)$ is defined.
Remark 25 A semi-simple ringoid can be written as:

\[ R = \{ e_1^{a_1} R \oplus e_2^{a_1} R \oplus \cdots \oplus e_{a_1}^{a_1} R \} \oplus \{ e_1^{a_2} R \oplus e_2^{a_2} R \oplus \cdots \oplus e_{a_2}^{a_2} R \} \oplus \cdots \oplus \{ e_1^{a_n} R \oplus e_2^{a_n} R \oplus \cdots \oplus e_{a_n}^{a_n} R \}, \]

such that \( e_i^{a_s} \) is not multipliable with \( e_j^{a_s} \) for \( s \neq t \) and \( e_i^{a_s} e_j^{a_s} = \delta_{ij} e_i^{a_s} \) for \( s = 1, 2, ..., n \); \( i, j = 1, 2, ..., a_s \).

Remark 26 Let \( R \) be as in the above remark. Then \( R \) satisfies the condition that for every \( x \in R \), there exists a unique \( r \) such that:

\[ 1_x = e_1^{a_r} + e_2^{a_r} + \cdots + e_{a_r}^{a_r} \text{ has the property that } x \cdot 1_x = x. \]

Notation 27 Instead of writing \( e_i^{a_j} R \) in the decomposition of a semi-simple ringoid, I will instead write \( e_i^{a_j} a_{i,j} R \). This is because we will be dealing with matrices in the following theorems.

Simple Ringoids

Definition 28 A ringoid \( R \) is said to be simple if:

Details in talk... =)

Before we continue, let me give a general definition of the Matrix Ringoid.

Definition 29 The Matrix Ringoid, \( M \), is defined as follows: Guess where the details are!!!

Theorem 30 In talk...

Corollary 31 Let \( M = \bigcup_{m,n \in I} M_{m,n} (D) \), and let \( D \) be a division ring, where \( a_i \)

are not necessarily all different. Suppose addition is defined only for matrices in \( M_{a_i,a_j} (D) \). Also suppose multiplication is defined only for

\[ M_{a_i,a_j} (D) \cdot M_{a_j,a_k} (D) \to M_{a_i,a_k} (D). \]

(This allows for matrices of same dimension but of different colours i.e. for \((i,j) \neq (r,s)\) , \( a_i = a_r \), \( a_j = a_s \) but \( a_i \times a_j \) matrices not addible to \( a_r \times a_s \), matrices and for \( j \neq r \), \( a_j = a_r \) but \( a_i \times a_j \) matrices not multipliable on the left to \( a_r \times a_s \) matrices.) Then \( M \) contains no proper two-sided ideals.
Corollary 32 If $I$ is finite then $M$ is simple.

Definition 33 Two right ideals $I$ and $J$ in a ringoid $R$ are said to be $R$-isomorphic if there is a one-to-one map $\zeta : I \rightarrow J$ such that for $x, y \in I$ and $r \in R$:

\[
\begin{align*}
\zeta (x + y) &= \zeta (x) + \zeta (y) \\
\zeta (xr) &= \zeta (x) r
\end{align*}
\]
whenever either side is defined.

Remark 34 Let $R = I_1 \oplus I_2$ be a direct sum of two right ideals then either $r = i_1 + i_2$, or $r = i_1$, or $r = i_2$.

Define $\zeta_1 (r) = i_1$, $\zeta_2 (r) = i_2$ in the first case,
\[ \zeta_1 (r) = r, \quad \zeta_2 (r) = r \]
in the second and third cases.

Let $I \subset R$ be a minimal right ideal in $R$ then:
\[ \{ x \in I \mid \zeta_i (x) \text{ is defined} \} \text{ is either } I \text{ or } N. \]
In any case, either $\zeta_i (I) = N$ or $\zeta_i (I)$ is $R$-homomorphic to $I$.

Lemma 35 Let $I$ be a right ideal in a ringoid $R$. Then $U$, the union of all right ideals $R$-homomorphic to $I$, is a two-sided ideal.

(a) Let $R$ be a simple ringoid, then:
\[ R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R, \quad e_i^2 = e_i, \]
and all $e_i R$ are $R$-isomorphic.

(b) Each $e_i R e_i$ is a division ring.

Proof. (of part b only)
In talk... ■

Lemma 36 Let $e_1 R$ be $R$-isomorphic to $e_2 R$ then:

(1) There exist elements $e_{12}$ and $e_{21}$ of $R$ such that:
\[ e_1 e_{12} = e_{12}, \quad e_{12} e_2 = e_{12}, \quad e_{21} e_1 = e_{21}, \quad e_2 e_{21} = e_{21} \]
and
\[ e_{12} e_{21} = e_1, \quad e_{21} e_{12} = e_2, \]
(2) $e_1 Re_1$ is isomorphic to $e_2 Re_2$.

Now here is the Wedderburn Theorem for Simple Rings:

**Theorem 37** Every simple ring that is finite-dimensional over a division ring is a matrix ring.

Now finally the Wedderburn Theorem for Simple Ringoids:

**Theorem 38** Now why would I give this away here?!?!?!

**Corollary 39** Suppose in a simple ringoid $R$, $x^2$ is defined for all $x \in R$, then $R$ is a simple ring.

**Theorem 40** A semi-simple ringoid $R$ is a direct sum of simple ringoids.