Math 209A Homework 3

Edward Burkard

2. **Integration**

2.5. **Product Measures.**

**Problem 46.** Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = B_{[0,1]}$, $\mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x, x) | x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\int \int \chi_D \, d\mu \, d\nu$, $\int \int \chi_D \, d\nu \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.

**Proof.** First note that $D_x = \{(x, x)\}$ and $D^y = \{(y, y)\}$. Then:

\[
\int \int \chi_D \, d\mu \, d\nu = \mu(D_x) \, d\nu = \int 0 \, d\nu = 0.
\]

\[
\int \int \chi_D \, d\nu \, d\mu = \nu(D^y) \, d\mu = \int 1 \, d\mu = 1.
\]

Let $\{A_i \times B_i\}$ be a countable covering of $D$ by rectangles in $\mathcal{M} \times \mathcal{N}$. Then given any $x \in [0,1]$, $(x, x) \in A_i \times B_i$ for some $i$. Thus $x \in A_i \cap B_i$. So we have that $[0,1] \subset \bigcup_{i=1}^{\infty} (A_i \cap B_i)$. Since $\mu([0,1]) = 1 > 0$ we must have that $\mu(A_j \cap B_j) > 0$ for some $j$. Thus $\mu(A_j) > 0$ and $\mu(B_j) > 0$ which gives that $\nu(B_j) = \infty$ since $B_j$ is an uncountable set. So given any covering of $D$ by rectangles in $\mathcal{M} \times \mathcal{N}$, we necessarily have that $\sum_{i=1}^{\infty} \mu \times \nu(A_i \times B_i) = \infty$. Therefore $\mu \times \nu(D) = \infty$, and in particular:

\[
\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D) = \infty
\]

\[\square\]

**Problem 48.** Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \wp(\mathbb{N})$, and $\mu = \nu =$ counting measure. Define

\[
f(m, n) = \begin{cases} 
1 & m = n, \\
-1 & m = n + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Then $\int |f| \, d(\mu \times \nu) = \infty$, and $\int \int f \, d\mu \, d\nu$ and $\int \int f \, d\nu \, d\mu$ exist and are unequal.

**Proof.** First break down the domain of $f$ into a disjoint union. Let $A = \{(m, m) | m \in \mathbb{N}\}$, $B = \{(m + 1, m) | m \in \mathbb{N}\}$, and $C = \mathbb{N} \times \mathbb{N} \setminus (A \cup B)$ then:

\[D(f) = A \cup B \cup C\]

where $f|_A \equiv 1$, $f|_B \equiv -1$, and $f|_C \equiv 0$. So:

\[
\int |f| \, d(\mu \times \nu) = \int_A |f| \, d(\mu \times \nu) + \int_B |f| \, d(\mu \times \nu) + \int_C |f| \, d(\mu \times \nu)
\]

\[= \mu \times \nu(A) + \mu \times \nu(B) + \mu \times \nu(C)
\]

\[= \infty + \infty + 0 = \infty.
\]
\[ \int_{x} \int_{y} f(x,y) \, dx \, dy = \int_{E} \mu(dy) \int_{E} \nu(dx) = \int_{E} \nu(dx) \int_{E} \mu(dy). \]

Clearly these last two integrals are not equal.

\square

**Problem 49.** Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

a. If \( E \in \mathcal{M} \times \mathcal{N} \) and \( \mu \times \nu(E) = 0 \), then \( \nu(E_x) = \mu(E_y) = 0 \) for a.e. \( x \) and \( y \).

b. If \( f \) is \( \mathcal{L} \)-measurable and \( f = 0 \) \( \lambda \)-a.e., then \( f_x \) and \( f_y \) are integrable for a.e. \( x \) and \( y \), and \( \int f_x \, \nu = \int f_y \, \mu = 0 \) for a.e. \( x \) and \( y \). (Here the completeness of \( \mu \) and \( \nu \) is needed.)

**Proof.** Theorem 2.39 - Let \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) be complete, \( \sigma \)-finite measure spaces, and let \((X \times Y, \mathcal{L}, \lambda)\) be the completion of \((X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)\). If \( f \) is \( \mathcal{L} \)-measurable and either

(i) \( f \geq 0 \), or

(ii) \( f \in L^1(\lambda) \),

then \( f_x \) is \( \mathcal{N} \)-measurable for a.e. \( x \) and \( f_y \) is \( \mathcal{M} \)-measurable for a.e. \( y \), and in case (ii) \( f_x \) and \( f_y \) are also integrable for a.e. \( x \) and \( y \). Moreover \( x \mapsto \int f_x \, \nu \) and \( y \mapsto \int f_y \, \mu \) are measurable, and in case (ii) also integrable, and

\[ \int f \, d\lambda = \int f(x,y) \, d\mu(x) \, d\nu(y) = \int f(x,y) \, d\nu(y) \, d\mu(x). \]

a. Let \( E \in \mathcal{M} \times \mathcal{N} \) with \( \mathcal{M} \times \mathcal{N}(E) = 0 \). Then

\[ 0 = \mu \times \nu(E) = \int \chi_{E} \, d(\mu \times \nu) = \int \chi_{E_x} \, d\nu = \int \chi_{E_y} \, d\mu, \]

which gives that \( \nu(E_x) = \mu(E_y) = 0 \).

b. Let \( f \) be \( \mathcal{L} \)-measurable and \( f = 0 \) \( \lambda \)-a.e. Let \( A = \{(x,y) | f(x,y) \neq 0 \} \). By the assumption on \( f \) we have that \( \lambda(A) = 0 \). Thus we can find a set \( E \in \mathcal{M} \otimes \mathcal{N} \) with \( A \subseteq E \) and \( \mu \times \nu(E) = 0 \). This means that \( \nu(E_x) = \mu(E_y) = 0 \). Since we have that \( A_x \subseteq E_x \) and \( A^y \subseteq E_y \), and the fact that \( \mu \) and \( \nu \) are complete
we get that, \( A_x \) and \( A^y \) are measurable with measure zero. Thus \( f_x \) and \( f^y \) are integrable. So using Fubini’s theorem:

\[
0 = \int f \, d(\mu \times \nu) = \int f_x \, d\nu = \int f^y \, d\mu.
\]

Now to prove Theorem 2.39: Let \( f \) be a \( \mathcal{L} \)-measurable function. Then by Proposition 2.12 there is a \( \mu \times \nu \)-measurable function \( g \) such that \( f = g \cdot \lambda \) a.e. If (i) \( f \geq 0 \), or (ii) \( f \in L^1(\lambda) \), then, by applying lemma (b) to \( f - g \) we have that \( f_x \) is \( \mathcal{N} \)-measurable and \( f^y \) is \( \mathcal{M} \)-measurable for a.e. \( x \) and \( y \), respectively. Then applying Fubini’s theorem we get that the maps \( x \mapsto \int g_x \, d\nu \) and \( y \mapsto \int g^y \, d\mu \) are measurable. Then by lemma (b) we have that \( \int (f_x - g_x) \, d\nu = 0 \), which gives that \( \int (g_x) \, d\nu = \int (f_x) \, d\nu \). Similarly we get that \( \int (g^y) \, d\mu = \int (f^y) \, d\mu \). In particular, if we are in case (ii), applying lemma (b) and Fubini’s theorem to \( f - g \) we have:

\[
\int f \, d\lambda = \iint f(x,y) \, d\mu(x) \, d\nu(y) = \iint f(x,y) \, d\nu(y) \, d\mu(x).
\]

\( \square \)

3. Signed Measures and Differentiation

3.1. Signed Measures.

**Problem 1.** Prove Proposition 3.1 - Let \( \nu \) be a signed measure on \( (X, \mathcal{M}) \). If \( \{E_j\} \) is an increasing sequence in \( \mathcal{M} \), then \( \nu(\bigcup_{i=1}^{\infty} E_j) = \lim_{n \to \infty} \nu(E_j) \). If \( \{E_j\} \) is a decreasing sequence in \( \mathcal{M} \) and \( \nu(E_1) < \infty \), then \( \nu(\bigcap_{j=1}^{\infty} E_j) = \lim_{n \to \infty} \nu(E_j) \).

**Proof.** Let \( \{E_j\} \) be an increasing sequence in \( \mathcal{M} \). Let \( E_0 = \emptyset \), and define \( F_n = E_n \setminus E_{n-1} \). Then \( \{F_n\} \) is a sequence of disjoint sets with \( \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n \). Applying the additivity of \( \nu \) we have:

\[
\nu\left( \bigcup_{j=1}^{\infty} E_j \right) = \nu\left( \bigcup_{j=1}^{\infty} F_j \right) = \sum_{j=1}^{\infty} \nu(F_j) = \lim_{n \to \infty} \sum_{j=1}^{n} \nu(F_j) = \lim_{n \to \infty} \nu(E_n)
\]

as desired.

Let \( \{G_j\} \) be a decreasing sequence in \( \mathcal{M} \) with \( \nu(G_1) < \infty \). Define \( H_j = G_1 \setminus G_j \), then \( H_1 \subset H_2 \subset H_3 \subset \cdots \), and \( \nu(G_1) = \nu(H_j) + \nu(G_j) \), and \( \bigcup_{j=1}^{\infty} H_j = G_1 \setminus \bigcap_{j=1}^{\infty} G_j \). By the previous proof we get:

\[
\nu(G_1) = \nu\left( \bigcap_{j=1}^{\infty} G_j \right) + \lim_{j \to \infty} \nu(H_j)
\]

\[
= \nu\left( \bigcap_{j=1}^{\infty} G_j \right) + \lim_{j \to \infty} \nu(H_j)
\]

\[
= \nu\left( \bigcap_{j=1}^{\infty} G_j \right) + \lim_{j \to \infty} [\nu(G_1) - \nu(G_j)]
\]

\[
= \nu\left( \bigcap_{j=1}^{\infty} G_j \right) + \nu(G_1) - \lim_{j \to \infty} \nu(G_j).
\]

So subtracting \( \nu(G_1) \) (which is ok since \( \nu(G_1) < \infty \) from both sides and moving the limit to the otherside we get \( \nu(\bigcap_{j=1}^{\infty} G_j) = \lim_{j \to \infty} \nu(G_j) \), as desired. \( \square \)

**Problem 2.** If \( \nu \) is a signed measure, (a) \( E \) is \( \nu \)-null iff \( |\nu|(E) = 0 \). Also, (b) if \( \nu \) and \( \mu \) are signed measures, (i) \( \nu \perp \mu \) iff (ii) \( |\nu| \perp \mu \) iff (iii) \( \nu^+ \perp \mu \) and \( \nu^- \perp \mu \).

**Proof.**

a. \((\Rightarrow)\) Suppose that \( E \in \mathcal{M} \) is \( \nu \)-null. Then \( \nu(E) = 0 \). Let \( P, N \in \mathcal{M} \) be the sets given by the Jordan decomposition of \( \nu \). Then \( |\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) = 0 - 0 = 0 \).

\((\Leftarrow)\) Suppose that \( |\nu|(E) = 0 \). Then \( |\nu|(E) = \nu^+(E) - \nu^-(E) = 0 \), which forces \( \nu^+(E) = \nu^-(E) = 0 \) since \( \nu^+ \) and \( \nu^- \) are positive measures. Thus \( \nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0 \). Therefore \( E \) is \( \nu \)-null.
Problem 11. Let $\nu \perp \mu$. Then there are sets $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$, $\nu(E) = 0$, and $\mu(F) = 0$. Since $\nu$ is a signed measure, by the Jordan Decomposition Theorem we can write $\nu = \nu^+ - \nu^-$ where $\nu^+ \perp \nu^-$ where $P, N \in \mathcal{M}$ are the sets satisfying the conditions that $P \cap N = \emptyset$, $P \cup N = X$, $\nu^+(N) = 0$, and $\nu^-(P) = 0$. Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

(iii) $\Rightarrow$ (ii) Suppose that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Let $E, F \in \mathcal{M}$ be the sets corresponding to $\nu^+ \perp \mu$ and let $G, H \in \mathcal{M}$ be the sets corresponding to $\nu^- \perp \mu$. Consider the sets $A := E \cap G$ and $B := F \cup H$. It can be checked by set theory that $A \cup B = X$ and $A \cap B = \emptyset$. Since $A \subseteq E$ and $A \subseteq G$ and $A \cup B = X$, we get that $\nu^+(A) = \nu^- (A) = 0$ and thus $|\nu|(A) = \nu^+(A) + \nu^- (A) = 0 + 0 = 0$. Therefore $|\nu| \perp \mu$.

(ii) $\Rightarrow$ (i) Suppose that $|\nu| \perp \mu$. Then we have the sets $E, F \in \mathcal{M}$ with $E \cap F = \emptyset$, $E \cup F = X$, $|\nu|(E) = 0$, and $\mu(F) = 0$. From part (a) we have that $|\nu|(E) = 0 \implies \nu(E) = 0$. Thus $\nu \perp \mu$.

\[ \square \]

3.2. The Lebesgue-Radon-Nikodym Theorem.

Problem 8. (i) $\nu \ll \mu$ iff (ii) $|\nu| \ll \mu$ iff (iii) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

**Proof.** Let our measure space be $(X, \mathcal{M})$.

(i) $\Rightarrow$ (ii) Suppose that $\nu \ll \mu$. From Problem 2a, we have that $|\nu|(E) = 0$, $\forall E \in \mathcal{M}$ such that $\nu(E) = 0$ which are all the sets for which $\mu(E) = 0$. Thus $|\nu| \ll \mu$.

(ii) $\Rightarrow$ (iii) Suppose $|\nu| \ll \mu$. Since $|\nu| = \nu^+ + \nu^-$, the sum of two positive measures, for any $E \in \mathcal{M}$ with $\mu(E) = 0$ we must have that $\nu^+(E) = \nu^-(E) = 0$ since $|\nu|(E) = 0$. Thus we have $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

(iii) $\Rightarrow$ (i) Suppose $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Then we have that for any $E \in \mathcal{M}$ with $\mu(E) = 0$, $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0$. Therefore $\nu \ll \mu$.

\[ \square \]

Problem 10. Theorem 3.5 may fail when $\nu$ is not finite. (Consider $d\nu(x) = \frac{dx}{x}$ and $d\mu(x) = dx$ on $(0,1)$, or $\nu$ = counting measure and $\mu$ = a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

**Proof.** Let $\nu(E) = \int_E \frac{1}{x} \, dx$ and $\mu(E) = \int_E \, dx$.

\[ \square \]

Problem 11. Let $\mu$ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha \in A} \subset L^1(\mu)$ is called uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\int_E f_\alpha| < \varepsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.

a. Any finite subset of $L^1(\mu)$ is uniformly integrable.

b. If $\{f_\alpha\}$ is a sequence in $L^1(\mu)$ that converges in the $L^1$ metric to $f \in L^1(\mu)$, then $\{f_\alpha\}$ is uniformly integrable.

**Proof.**