1 Table of Basic Antiderivatives

1.1 Integration by Parts Formula
\[ \int u \, dv = uv - \int v \, du \] (1)

1.2 Power Functions
\[ \int x^\alpha \, dx = \frac{1}{1+\alpha} x^{1+\alpha}, \quad \text{if } \alpha \neq -1 \] (2)
\[ \int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln |x| \] (3)

1.3 Exponential Functions
\[ \int e^x \, dx = e^x \] (4)
\[ \int a^x \, dx = \frac{1}{\ln(a)} a^x, \quad \text{if } a > 0 \] (5)

1.4 Logarithms
\[ \int \ln(x) \, dx = x \ln(x) - x \] (6)

1.5 Trigonometric Functions
\[ \int \cos(x) \, dx = \sin(x) \] (7)
\[ \int \sin(x) \, dx = -\cos(x) \] (8)
\[ \int \tan(x) \, dx = \ln |\sec(x)| \] (9)
\[ \int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| \] (10)
\[ \int \sec(x)^2 \, dx = \tan(x) \] (11)
\[ \int \sec(x) \tan(x) \, dx = \sec(x) \] (12)
\[ \int \cos(x)^m \sin(x)^n \, dx = (*) \] (13)

1.6 Trigonometric Substitution
\begin{align*}
\text{Integrand Contains} & \quad \text{Try Substituting} \\
\sqrt{a^2 - x^2} & \quad x = a \sin(\theta) \\
\sqrt{a^2 + x^2} & \quad x = a \tan(\theta) \\
\sqrt{x^2 - a^2} & \quad x = a \sec(\theta)
\end{align*}

2 Some of the Derivations and a Few Examples

2.1 Exponential Functions
Note that (5) can be derived from (4) via a little bit of manipulation using properties of logarithms followed by a substitution. First, we write everything in terms of \(e\) and natural logarithms:
\[ \int a^x \, dx = \int e^{\ln(a^x)} \, dx = \int e^{x \ln(a)} \, dx. \]
Now make the substitution \( u = x \ln(a) \) so that \( du = \ln(a) \, dx \). It then follows that
\[ \int e^{x \ln(a)} \, dx = \int e^u \frac{du}{\ln(a)} = \frac{1}{\log(a)} \int e^u \, du = \frac{1}{\log(a)} e^u \quad \text{(use formula (4))} \]
\[ = \frac{1}{\log(a)} e^{x \ln(a)}. \]
Tracing back the equalities, we obtain the formula at (5).
A word of caution, however: \( \ln(a) \) only makes sense if \( a \) is a positive number, hence we must assume that \( a > 0 \).

2.2 Logarithms
The integral of \( \ln(x) \) is obtained via a somewhat devious integration by parts. To integrate
\[ \int \ln(x) \, dx, \]
take \( u = \ln(x) \) and \( dv = dx \). Then \( du = x^{-1} \, dx \) and \( v = x \). Applying the integration by parts formula at (1), we get
\[ \int u \, dv = uv - \int v \, du \]
\[ = \ln(x) x - \int x x^{-1} \, dx \]
\[ = x \ln(x) - x, \]
which is the formula given at (6).
2.3 An Antiderivative for Tangent

If we recall that
\[ \tan(x) = \frac{\sin(x)}{\cos(x)}, \]
then the integral at (9) can be done via a substitution. Let \( u = \cos(x) \) so that \( du = -\sin(x) \, dx \). Then we have
\[
\int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx \\
= \int -\frac{1}{u} \, du \\
= -\ln|u| \quad \text{(use formula (3))} \\
= -\ln|\cos(x)|.
\]
Using properties of logarithms, we obtain the formula at (9)—can you explain why?

2.4 An Antiderivative for Secant

The antiderivative of the secant function is somewhat mysterious, and relies on using devious trick that comes out of nowhere. Honestly, even those of us that know what we are doing generally look at a table of integrals to remind us what the formula at (10) should be. That being said, if you know the right trick, the derivation doesn’t use any tools more advanced than substitution. The first thing to observe is that we can multiply \( \sec(x) \) by 1 without changing anything. If we write 1 in a funny way, we can take advantage of several cancelations to get the job done. In particular, we have
\[ 1 = \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}. \]
Hence we have
\[
\int \sec(x) \, dx = \int \sec(x) \cdot 1 \, dx \\
= \int \sec(x) \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} \, dx \\
= \int \frac{\sec(x)^2 + \sec(x) \tan(x)}{\sec(x) + \tan(x)} \, dx.
\]
Now we make a substitution: let \( u = \sec(x) + \tan(x) \) (i.e. take \( u \) to be the denominator of the integrand) so that
\[ du = \sec(x) \tan(x) + \sec(x)^2 \, dx. \]
Notice that \( du \) is exactly the numerator of the integrand, and so we have
\[
\int \frac{\sec(x)^2 + \sec(x) \tan(x)}{\sec(x) + \tan(x)} \, dx \\
= \int \frac{du}{u} \\
= \ln|u| \quad \text{(use formula (3))} \\
= \ln|\sec(x) + \tan(x)|,
\]
which is exactly the desired result.

2.5 Products of Sines and Cosines

Observe that we have given no formula at (13). This is because integrals of this form require a little bit of thought. We are trying to evaluate the indefinite integral
\[
\int \cos(x)^m \sin(x)^n \, dx,
\]
where \( m \) and \( n \) are both integers (positive or negative whole numbers). There are three cases to consider:

(a) \( m \) is odd: If \( m \) is odd, then we can write \( m = 2k+1 \) for some integer \( k \). Then, using the Pythagorean identity \( \sin(x)^2 + \cos(x)^2 = 1 \), we obtain
\[
\int \cos(x)^m \sin(x)^n \, dx \\
= \int \left[ \cos(x)^2 \right]^k \sin(x)^n \cos(x) \, dx \\
= \int [1 - \sin(x)^2]^k \sin(x)^n \cos(x) \, dx.
\]
Now make the substitution \( u = \sin(x) \) so that \( du = \cos(x) \, dx \). Then we have
\[
\int [1 - \sin(x)^2]^k \sin(x)^n \cos(x) \, dx \\
= \int [1 - u^2]^k u^n \, du.
\]
This integral can now be evaluated by either multiplying out \( [1 - u^2]^k \) and applying formula for power functions at (2), or via another substitution. The “right” thing to do depends on how large \( k \) is—if \( k \) is relatively small, it might be easier to multiply everything out, otherwise substitution might be more straight-forward. Once this integral is evaluated, remember to replace \( u \) with \( \sin(x) \) at the end!

(b) \( n \) is odd: If \( n \) is odd, then the process is identical to the above, except that we “peel off” a sine term instead of a cosine term. That is, we replace \( \sin(x)^n \) by
\[ [1 - \cos(x)^2]^k \sin(x) \]
and make the substitution \( u = \cos(x) \).

(c) both \( m \) and \( n \) are even: This case can be quite tedious, and I hope that you will never have to integrate anything more complicated than \( \cos(x)^2 \sin(x)^2 \) by hand. The general idea is to reduce the problem to something more tractable using a half-angle or double-angle formula. Rather
than try to work out a general formula, we’ll consider the example
\[
\int \cos(x)^2 \sin(x)^2 \, dx.
\]
Recall the double-angle formula
\[
\sin(2\theta) = 2 \cos(\theta) \sin(\theta)
\]
\[
\implies \cos(x)^2 \sin(x)^2 = \frac{1}{4} \sin(2x)^2
\]
(before continuing, make sure you know why this is true). Hence we can write
\[
\int \cos(x)^2 \sin(x)^2 \, dx = \frac{1}{4} \int \sin(2x)^2 \, dx.
\]
This is still not quite something that we can integrate, so we apply another trigonometric identity. Specifically, we apply the half-angle formula
\[
\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}
\]
\[
\implies \sin(2x)^2 = \frac{1}{2} - \frac{\cos(4x)}{2}.
\]
Rewriting the original integral, we have
\[
\int \cos(x)^2 \sin(x)^2 \, dx = \frac{1}{4} \int \sin(2x)^2 \, dx
\]
\[
= \frac{1}{4} \int \left[ \frac{1}{2} - \frac{\cos(4x)}{2} \right] \, dx
\]
\[
= \frac{1}{8} \int [1 - \cos(4x)] \, dx.
\]
To finish evaluating this integral, make the substitution \( u = 4x \)—this is left as an exercise to the avid student.

It is worth mentioning that you may also be asked to integrate things of the form \( \tan(x)^m \sec(x)^n \). The techniques are similar, so I won’t get into them here. If you are curious, any first-year calculus text should have some examples for you to work through.

### 2.6 An Example of Trigonometric Substitution

Trigonometric substitution is a technique for integration, not a simple formula. Making a reasonable substitution and evaluating an integral with one of the radical expressions given above requires some practice and experience. I give one example here just to jog your memory. Consider the integral
\[
\int \sqrt{16x^2 - 9} \, dx.
\]
Suppose that you have a right triangle with hypotenuse of length 4x and one leg of length 3 (see the figure, below).

It follows from the Pythagorean theorem that the length of the remaining leg is given by
\[
b = \sqrt{(4x)^2 - 3^2} = \sqrt{16x^2 - 9},
\]
which is the numerator of the integrand that we are interested in. Using right-triangle trigonometry (i.e. the mnemonic SOH CAH TOA), we know that
\[
\sec(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{4x}{3} = \frac{4}{3} \cdot x.
\]
Hence we can make the substitution \( x = \frac{3}{4} \sec(\theta) \). Because we need to get a \( dx \) term from somewhere, we can differentiate on both sides of this identity to get
\[
dx = 3 \sec(\theta) \tan(\theta) \, d\theta.
\]
Now that we have the \( dx \) term, we need to also write the integrand in terms of \( \theta \). Using the figure and the mnemonic again, we have
\[
\frac{\sqrt{16x^2 - 9}}{x} = 4 \cdot \frac{\sqrt{16x^2 - 9}}{4x} = 4 \cdot \frac{\text{opp}}{\text{hyp}} = 4 \sin(\theta).
\]
Combining these results, we can now write the original integral in terms of \( \theta \):
\[
\int \frac{\sqrt{16x^2 - 9}}{x} \, dx = \int 4 \sin(\theta) \cdot \frac{3}{4} \sec(\theta) \tan(\theta) \, d\theta
\]
\[
= 3 \left( \sec(\theta)^2 - 1 \right) \, d\theta \quad (\text{do you know why?})
\]
\[
= 3 (\tan(\theta) - \theta) + C.
\]
Finally, we need to get everything back in terms of \( x \) (rather than \( \theta \)). The tangent term is handled by once again referring to the figure:
\[
\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{16x^2 - 9}}{3}.
\]
To deal with the other term, recall that we chose \( \theta \) so that \( \sec(\theta) = \frac{4}{3} x \). This implies that \( \theta = \arccsc\left(\frac{4}{3} x\right) \). Thus we have the closed form
\[
\int \frac{\sqrt{16x^2 - 9}}{x} \, dx = \sqrt{16x^2 - 9} + 3 \arccsc\left(\frac{4}{3} x\right) + C.
\]