1. Determine if the following improper integral converges or diverges:

\[
\int_1^e \frac{1}{x \sqrt{\ln x}} \, dx
\]

Note \( \int_1^e \frac{1}{x \sqrt{\ln x}} \, dx = \lim_{b \to 1} \int_b^e \frac{1}{x \sqrt{\ln x}} \, dx. \)

Using substitution, with \( u = \ln x, \ du = \frac{1}{x} \, dx, \ u(b) = \ln b, \ u(e) = 1. \)

We see that \( \int_1^e \frac{1}{x \sqrt{\ln x}} \, dx = \lim_{b \to 1} \int_{\ln b}^1 \frac{1}{\sqrt{u}} \, du = \lim_{b \to 1} 2 - \sqrt{\ln b} = 2. \) Hence, \( \int_1^e \frac{1}{x \sqrt{\ln x}} \, dx \) converges.

2. Consider the sequence \( a_n = \frac{1 + (-1)^n}{n}. \)

(a) Is \( a_n \) bounded below? If so, find a lower bound.

As \( 0 \leq \frac{1 + (-1)^n}{n} \leq \frac{2}{n} \), \( a_n \) is bounded below by 0.

(b) Is \( a_n \) non increasing? Explain your answer.

\( a_n \) is not non increasing as \( a_{2n-1} = 0 \) for \( n \geq 1 \), but \( a_{2n} = \frac{2}{n} \) so \( a_n \) increases from every odd to the next even.

(c) Find \( \lim_{n \to \infty} a_n. \)

Using the sandwich theorem, as \( 0 \leq \frac{1 + (-1)^n}{n} \leq \frac{2}{n} \) and \( \lim_{n \to \infty} 0 = \lim_{n \to \infty} \frac{2}{n} = 0 \), \( \lim_{n \to \infty} a_n = 0. \)

3. List the first three partial sums of \( \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}. \)

\( S_1 = \frac{1}{6}, \ S_2 = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}, \ S_3 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{10}. \)
4. Suppose \( a_n \) and \( b_n \) are sequences and \( \sum_{n=1}^{\infty} a_n + b_n \) converges. Is it true that \( \sum_{n=1}^{\infty} a_n \) converges? Why or why not?

Let \( a_n = \frac{1}{n} \) and \( b_n = -\frac{1}{n+1} \). \( a_n + b_n = \frac{1}{n} - \frac{1}{n+1} \) and \( \sum_{n=1}^{\infty} a_n + b_n \) is a converging telescoping series, but \( \sum_{n=1}^{\infty} a_n \) is a diverging \( p \)-series. So the answer is false.

5. Find the sum of the series \( \sum_{n=0}^{\infty} \frac{2(-1)^n}{5^n} \)

\[
\sum_{n=0}^{\infty} \frac{2(-1)^n}{5^n} = \frac{2}{1 + \frac{1}{5}} = \frac{5}{3}
\]

6. Suppose \( a_n \) is a sequence which is negative, increasing and \( a_n = f(n) \) for \( n \geq 1 \) for some continuous function \( f(x) \) on \([1, \infty)\). How might one determine if \( \sum_{n=1}^{\infty} a_n \) is convergent or not?

Taking the absolute value of \( a_n \), \( |a_n| = -a_n > 0 \). Since \( a_n \leq a_{n+1}, |a_n| = -a_n \geq -a_{n+1} = |a_{n+1}| \), so \( |a_n| \) is a decreasing positive function that agrees with the continuous function \( |f(x)| \) on \([1, \infty)\). Hence, \( |a_n| \) satisfies the hypotheses of the integral test. Now if \( \sum_{n=1}^{\infty} |a_n| \) is convergent then so is \( \sum_{n=1}^{\infty} a_n \) is convergent and if \( \sum_{n=1}^{\infty} |a_n| \) is divergent, then \( \sum_{n=1}^{\infty} a_n \) is divergent since \( \sum_{n=1}^{\infty} a_n = -\sum_{n=1}^{\infty} |a_n| \).

7. Determine if the following series converge or diverge.

(a) \( \sum_{n=1}^{\infty} \frac{2n + 1}{n^2 - n + 1} \)

Compare with the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) which is a diverging \( p \)-series.

\[
\lim_{n \to \infty} \frac{2n + 1}{n^2 - n + 1} = \lim_{n \to \infty} \frac{n(2n + 1)}{n^2 - n + 1} = 2 \neq 0.
\]

So by the limit comparison test, \( \sum_{n=1}^{\infty} \frac{2n + 1}{n^2 - n + 1} \) diverges.

(b) \( \sum_{n=2}^{\infty} \frac{\ln n}{n^3} \)

Compare with \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) which is a converging \( p \)-series.

\[
\lim_{n \to \infty} \frac{n^2}{\ln n} = \lim_{n \to \infty} \frac{n \ln n}{n^2} = 0.
\]

So by the limit comparison test, \( \sum_{n=2}^{\infty} \frac{\ln n}{n^3} \) converges.

(c) \( \sum_{n=1}^{\infty} \frac{5}{n^n} \)

Using the root test, \( \lim_{n \to \infty} \sqrt[n]{\frac{5}{n^n}} = \lim_{n \to \infty} \frac{\sqrt[n]{5}}{n} = 0 < 1 \). We conclude that \( \sum_{n=1}^{\infty} \frac{5}{n^n} \) converges by the root test.
(d) \[ \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \]

Here we will use the ratio test. We need to compute \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{(2n+2)!} \frac{(2n+2)!}{(n+1)!} = \frac{(n+1)!}{(2n)!} \frac{(2n)!}{n!(2n+2)!} = 0 < 1. \) Hence, \( \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \) converges by the ratio test.

(e) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \]

We will use the alternating series test. \( a_n = \frac{1}{\sqrt{n}} > 0 \) is decreasing and \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0. \) Therefore, by the alternating series test, \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges.

(f) \[ \sum_{n=1}^{\infty} n \sin \left( \frac{1}{n} \right) \]

\[ \lim_{n \to \infty} n \sin \left( \frac{1}{n} \right) = \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1 \text{ as } \frac{1}{n} \to 0 \text{ as } n \to \infty. \] So by the \( n \)-th term test, \( \sum_{n=1}^{\infty} n \sin \left( \frac{1}{n} \right) \) diverges.

8. Determine if the following series are conditionally convergent, absolutely convergent or divergent:

(a) \[ \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \]

Note \( \left| \frac{(-3)^n}{n!} \right| = \frac{3^n}{n!} \). Using the ratio test, \( \lim_{n \to \infty} \frac{3^{n+1}}{3^n} = \lim_{n \to \infty} \frac{3}{n+1} = 0 < 1. \) Hence, \( \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \) is absolutely convergent by the ratio test.

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \]

\[ \frac{|(-1)^n n|}{n^2 + 1} = \frac{n}{n^2 + 1} \geq 1 \cdot \frac{n}{2n} = \frac{1}{2} \times \text{the divergent p series } \sum_{n=1}^{\infty} \frac{1}{n}. \text{ Now by the direct comparison test, we obtain } \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \text{ is a diverging series. But } \frac{n}{n^2 + 1} > 0 \text{ is a decreasing sequence with } \lim_{n \to \infty} \frac{n}{n^2 + 1} = 0. \] Hence, \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \) converges by the alternating series test. Thus, \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \) is conditionally convergent.

(c) \[ \sum_{n=1}^{\infty} \frac{(2n)^2}{(2n^2 + 1)^n} \]

Note \( \left| \frac{(2n)^2}{(2n^2 + 1)^n} \right| = \left| \frac{(2n)^2}{(2n^2 + 1)^n} \right| \). We will use the root test. We see that
\[
\lim_{n \to \infty} \sqrt[4]{\frac{(2n)^{2n}}{(2n^2 + 1)^n}} = \lim_{n \to \infty} \frac{4n^2}{(2n^2 + 1)^{1/n}} = 2.
\]
Thus, \[ \sum_{n=1}^{\infty} \frac{(2n)^{2n}}{(2n^2 + 1)^n} \] diverges.