Practice Midterm 2 Solutions
MATH 9C
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1. Determine the interval of convergence for the power series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n} (x + 2)^n \)

We need to use the ratio test (or root test) to determine the radius of convergence. So we look at \( \lim_{n \to \infty} \frac{1}{n} \left| \frac{x + 2}{3} \right|^n \) = \( \lim_{n \to \infty} \frac{|x + 2|}{3} \). By the ratio test the power series converges when \( \frac{|x + 2|}{3} < 1 \). Or when \( |x + 2| < 3 \). Thus \( f(x) \) converges on the interval \((-5, 1]\).

We need to test convergence at the endpoints. Plugging in -5 into the power series we get the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n} (-5 + 2)^n = \sum_{n=1}^{\infty} \frac{1}{n} \) which is a divergent p series.

Plugging in 1 into the power series, we get the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n} (1 + 2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) which converges by the alternating series test as \( \frac{1}{n} > 0 \), \( \frac{1}{n+1} > \frac{1}{n+1} \) and \( \lim_{n \to \infty} \frac{1}{n} = 0 \). Thus the interval of convergence for the power series is \((-5, 1]\).

2. Determine the interval of convergence for the power series \( \sum_{n=0}^{\infty} \frac{2^n}{n!} (3x - 1)^n \)

We need to use the ratio test (or root test) to determine the radius of convergence. So we look at \( \lim_{n \to \infty} \frac{2^{n+1}}{n+1} \left| \frac{3x - 1}{3x - 1} \right|^n \) = \( \lim_{n \to \infty} \frac{2}{n+1} \left| \frac{3x - 1}{n+1} \right| = 0 < 1 \). By the ratio test the power series converges for all real numbers \( x \).

3. If \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) has a radius of convergence equal to 5, does \( f(5) \) necessarily converge?

If \( f(x) \) has radius of convergence 5, we know that \( f \) converges on the open interval \((-5, 5)\). We can’t determine if \( f(5) \) converges or not. We need to know the behavior of \( \sum_{n=0}^{\infty} a_n 5^n \). Not knowing \( a_n \), we cannot say anymore.

4. Find the 3rd Taylor polynomial for \( f(x) = \frac{1}{x} \) centered at \( x = 2 \).

To find the 3rd Taylor polynomial we need to compute 3 derivatives of \( f \) and evaluate them at \( x = 2 \).
\[ f(x) = \frac{1}{2}, \quad f(2) = \frac{1}{2} \]
\[ f'(x) = -\frac{1}{2}, \quad f''(2) = -\frac{1}{4} \]
\[ f''(x) = \frac{2}{x^2}, \quad f''(2) = \frac{1}{4} \]
\[ f''(x) = -\frac{6}{2x}, \quad f''(2) = -\frac{3}{8}. \]

Thus \( P_3(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{16} \). 

5. Find the 2nd Taylor polynomial for \( f(x) = \sin(2x) \) centered at \( x = \frac{\pi}{3} \).

To find the second Taylor polynomial at \( x = \frac{\pi}{3} \), we need to compute two derivatives and evaluate them at \( x = \frac{\pi}{3} \).
\[ f(x) = \sin(2x), \quad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \]
\[ f'(x) = 2 \cos(2x), \quad f'(\frac{\pi}{3}) = -1. \]
\[ f''(x) = -4 \sin(2x), \quad f''\left(\frac{\pi}{3}\right) = -2\sqrt{3}. \]

Thus \( P_2(x) = \frac{\sqrt{3}}{2} - \left(x - \frac{\pi}{3}\right) - \sqrt{3}\left(x - \frac{\pi}{3}\right)^2. \)

6. Find the Taylor series for \( f(x) = \ln x \) centered at \( x = 1 \).

The Taylor series for \( f(x) = \ln x \) centered at \( x = 1 \) is \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \). You can derive this as we derived the Taylor polynomials above, or you can use the Maclaurin series for \( \ln(x+1) \) plugging in \( x-1 \) for \( x \) as \( \ln(x-1+1) = \ln(x) \).

7. Find the Maclaurin series for \( f(x) = \frac{\sin(2x)}{x} \).

The Maclaurin series for \( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \). Thus \( \frac{\sin(2x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1} \).

8. Find the Maclaurin series for \( f(x) = e^{x^2} \).

The Maclaurin series for \( e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \). Thus \( e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \).

9. Show that \( y = xe^x \) is a solution to the differential equation \( y' = \frac{(x+1)y}{x} \).

\[ y' = xe^x + e^x = y + \frac{y}{x} = \frac{(x+1)y}{x}. \] Thus \( y = xe^x \) is a solution to the above differential equation.

10. Find the general solution to \( y' = (y^2 + 1) \sec^2(x) \).

This is a separable differential equation. Separating the variables we obtain \( \frac{dy}{y^2 + 1} = \sec^2 x dx \). Integrating both sides we obtain, \( \arctan y = \tan x + C \). Or \( y = \tan(\tan x + C) \).
11. Find the solution to the initial value problem \( xy' - y = x^2e^x, \ y(1) = 2 \).

This is a linear differential equation with \( p(x) = -\frac{1}{x} \) and \( q(x) = xe^x \). The integrating factor \( v(x) = e^{-\int \frac{1}{x} \, dx} = e^{\ln|x|} = x \). Now \( y = ux \). To solve for \( u \) we look at \( u'x = xe^x \) or \( u' = e^x \). Thus \( u = e^x + C \) and \( y = xe^x + Cx \). To solve for \( C \), \( 2 = e + C \) or \( C = 2 - e \). Thus \( y = xe^x + (2 - e)x \).

12. Find the series solution to \( y' - 4y = e^{3x} \).

This is a linear differential equation with \( p(x) = -4 \) and \( q(x) = e^{3x} \). The integrating factor \( v(x) = e^{-\int -4 \, dx} = e^{4x} \). Now \( y = ue^{4x} \). To solve for \( u \) we look at \( u' e^{4x} = e^{3x} \) or \( u' = e^{-x} \). Thus \( u = -e^{-x} + C \) and \( y = -e^{3x} + Ce^{4x} \).

13. A 20 gallon container contains 10 gallons of water and one half a pound of salt. Pure water is poured into the container at a rate of 2 gallons a minute. If the subsequent mixture leaves the container at a rate of 1 gallon a minute, set up a differential equation with initial conditions to solve for the amount of salt in the container at any time \( 0 < t < 10 \).

Let \( Q(t) \) be the amount of salt in the container at any time \( 0 < t < 10 \). The rate the salt is entering the container is 0 as there is no salt in pure water. The rate at which the salt is exiting the container is \( \frac{Q(t)}{10+t} \) pounds per minute. Thus a differential equation to solve for \( Q(t) \) is

\[
Q'(t) = -\frac{Q(t)}{10+t} \text{ with initial condition } Q(0) = \frac{1}{2}.
\]

14. A population of bacteria grows at a rate proportional to the amount present. If the population doubles in two hours, find the general solution for the amount of bacteria present.

Let \( P(t) \) be the bacteria population at any time \( t \). The word problem describes growth at a rate \( P'(t) = kP(t) \). The solution to this differential equation is \( P(t) = P(0)e^{kt} \).

15. A bicyclist is coasting down the road at a rate of 3 meters per second. If the bicyclist and his bicycle weigh 70 kg and the resistance is given by twice the speed he is traveling, find his velocity after 1 minute.

We use Newton’s second law \( F = ma = m\frac{dv}{dt} \) and the resistive force is \( F = -kv \) where \( k = 2 \). Thus \( \frac{dv}{dt} = -\frac{2}{70}v \). Solving this differential equation, we obtain \( v = v_0e^{-\frac{t}{35}} \). Thus \( v(60) = 3e^{-\frac{60}{35}} = 3e^{-\frac{12}{7}} \).

16. Find the series solution to \( y'' - y = 0 \) with initial conditions \( y'(0) = 0 \) and \( y(0) = 1 \).

Set \( y = \sum_{n=0}^{\infty} a_nx^n \). Taking derivatives we obtain \( y' = \sum_{n=0}^{\infty} (n + 1)a_n+1x^n \) and \( y'' = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}x^n \). Thus \( \sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} - a_n]x^n = 0 \). Hence, \( (n + 2)(n + 1)a_{n+2} - a_n = 0 \) for \( n \geq 0 \). From the initial conditions we have \( a_0 = 1 \) and \( a_1 = 0 \). Using the recurrence relation above we see that \( 2a_2 - a_0 = 0 \) or \( a_2 = \frac{1}{2} \).

\( 3 \cdot 2a_3 - a_1 = 0 \) or \( a_3 = 0 \). \( 4 \cdot 3a_4 - a_2 = 0 \) thus \( a_4 = \frac{1}{4} \). In general \( a_{2n} = \frac{1}{(2n)!} \) and \( a_{2n+1} = 0 \). Hence \( y = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} \). This is equivalent to \( y = \frac{e^x + e^{-x}}{2} \).
17. Use series to evaluate \( \lim_{x \to 0} \frac{\ln(1 + x)}{\sin x} \).

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{\sin x} = \lim_{x \to 0} \frac{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}}{\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)! x^{2n+1}}{x^n}} = \lim_{x \to 0} \frac{x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^n}{n}}{1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)! x^{2n}}{x^n}}.
\]

The terms in both series are all 0; thus the limit is 1.