Math 132 - HW 5
due January 23

January 16, 2015

1. True/False: Determine which of the following statements are true. For each true statement, give a proof and for each false statement produce a counterexample.

(a) A linear operator \( T \) on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue \( \lambda \) equals the dimension of \( E_\lambda \).

(b) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

2. (a) Let \( D \) be a diagonal matrix:

\[
D = \begin{pmatrix}
d_1 & 0 & \cdots & 0 \\
0 & d_2 \\
\vdots & \ddots & \vdots \\
0 & \cdots & d_n
\end{pmatrix}
\]

Compute \( D^n \).

(b) For

\[
A = \begin{pmatrix}
1 & 4 \\
2 & 3
\end{pmatrix}
\]

find an expression for \( A^n \), where \( n \) is an arbitrary positive integer. (Hint: If a matrix \( A \) is diagonalizable, then there is an invertible matrix \( Q \) such that \( Q^{-1}AQ \) is a diagonal matrix \( D \), or in other words \( A = QDQ^{-1} \).)

3. Two linear operators \( T \) and \( U \) on a finite-dimensional vector space \( V \) are called simultaneously diagonalizable if there exits an ordered basis \( \beta \) for \( V \) such that both \( [T]_\beta \) and \( [U]_\beta \) are diagonal matrices. Prove that if \( T \) and \( U \) are simultaneously diagonalizable operators, then \( T \) and \( U \) commute (i.e. \( TU = UT \)).