1. Let $V$ and $W$ be finite dimensional vector spaces and suppose $T: V \to W$ is linear. Prove that $T$ is injective if and only if $\dim(V) = \text{rank}(T)$.

Proof: ($\Rightarrow$) Suppose $T$ is injective, then $\ker T = \{0\}$, so $\text{Null}(T) = \text{Dim}(\ker T) = 0$ so by the rank + nullity theorem

$$\dim(V) = \text{Rank}(T) + \text{Null}(T) = \text{Rank}(T).$$

($\Leftarrow$) Suppose $\text{Rank}(T) = \dim(V)$ then by rank + nullity theorem $\text{Null}(T) = 0$ so $\ker T = \{0\}$ and this says $T$ is injective.

2. Let $V$ and $W$ be finite dimensional vector spaces and suppose $T: V \to W$ is linear. Prove that $T$ cannot be surjective if $\dim(V) < \dim(W)$

Proof: Suppose $\dim(V) < \dim(W)$ and assume for a contradiction that $T$ is surjective. Then by definition $T(V) = W$. This says

$$\dim(W) = \dim(T(V)) = \text{Rank}(T)$$

By the rank + nullity theorem we now know

$$\dim(V) = \dim(W) + \text{Null}(T).$$

But by assumption, $\dim(V) < \dim(W)$ and so

$$\dim(W) > \dim(W) + \text{Null}(T)$$

or in other words, $\text{Null}(T) < 0$. But the $\text{Null}(T) = \dim(\ker T)$ and dimension is the number of elements in a set. So $\text{Null}(T) < 0$ doesn’t make sense, it’s a contradiction. Therefore $T$ cannot be surjective.

3. Let $V$ and $W$ be finite dimensional vector spaces and suppose $T: V \to W$ is linear. Prove that $T$ cannot be injective if $\dim(V) > \dim(W)$.

Proof: Suppose $\dim(V) > \dim(W)$ and assume for a contradiction that $T$ is injective. Then we know $\ker T = \{0\}$ so $\text{Null}(T) = 0$ so by the rank + nullity theorem

$$\dim(V) = \text{Rank}(T) = \dim(T(V)).$$

But we know that $\dim(V) > \dim(W)$ so this says

$$\dim(W) < \dim(T(V)).$$

However, $T(V)$ is a subspace of $W$ and so we know $\dim(T(V)) < \dim(W)$ and this is a contradiction. Therefore, $T$ cannot be injective.

4. Let $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = p'(x).$$
Let $B_1 = \{1, x\}$ and $B_2 = \{1 + x, 1 - x\}$ be two bases for $P_1(\mathbb{R})$. Use the fact that
\[
\begin{pmatrix}
  1 & 1 \\
  1 & -1
\end{pmatrix}^{-1} = \begin{pmatrix}
  \frac{1}{2} & \frac{1}{2} \\
  \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]
to find $[T]_{B_2}^{B_1}$

Solution: Since $T(1) = 0 = 0 \cdot 1 + 0 \cdot x$ and $T(x) = 1 = 1 \cdot 1 + 0 \cdot x$ we know
\[
[T]_{B_1}^{B_1} = \begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix}
\]
Now note $Id(1 + x) = 1 + x = 1 \cdot 1 + 1 \cdot x$ and $Id(1 - x) = 1 \cdot 1 - 1 \cdot x$. Therefore
\[
[Id]_{B_1}^{B_1} = Q = \begin{pmatrix}
  1 & 1 \\
  1 & -1
\end{pmatrix}
\]
So by proposition 13 $[T]_{B_2}^{B_1} = Q^{-1}[T]_{B_1}^{B_1}Q$, that is,
\[
[T]_{B_2}^{B_1} = \begin{pmatrix}
  \frac{1}{2} & \frac{1}{2} \\
  \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 1 \\
  1 & -1
\end{pmatrix}
\begin{pmatrix}
  \frac{1}{2} & \frac{1}{2} \\
  \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]

5. Let $V = P_1(\mathbb{R})$, and, for $p(x) \in V$, define $f_1, f_2 \in V^*$ by
\[
f_1(p(x)) = \int_0^1 p(t) dt \quad \text{and} \quad f_2(p(x)) = \int_0^2 p(t) dt.
\]
Prove that $\{f_1, f_2\}$ is a basis for $V^*$, and find a basis for $V$ for which it is the dual basis.

Solution: Let $B = \{p_1(x), p_2(x)\} = \{a_1 + b_1 x, a_2 + b_2 x\}$. Now if we want $\{f_1, f_2\}$ to behave like a "dual basis", then
\[
1 = f_1(p_1(x)) = \int_0^1 a_1 + b_1 t dt = a_1 + \frac{1}{2} b_1
\]
\[
0 = f_2(p_1(x)) = \int_0^2 a_1 + b_1 t dt = 2a_1 + 2b_1
\]
\[
0 = f_1(p_2(x)) = \int_0^1 a_2 + b_2 t dt = a_2 + \frac{1}{2} b_2
\]
\[
1 = f_2(p_2(x)) = \int_0^2 a_2 + b_2 t dt = 2a_2 + 2b_2.
\]
This tells us $a_1 = 2, b_1 = -2, a_2 = -1/2$, and $b_2 = 1$ so $B = \{2 - 2x, -1/2 + x\}$. Now let’s check that $B$ really is a basis. Since it’s got 2 elements, all we need to show is that it’s linearly independent. Suppose there are $c_1$ and $c_2$ such that
\[
c_1(2 - 2x) + c_2(-1/2 + x) = 0
\]
then
\[
(2c_1 - 1/2c_2) \cdot 1 + (-2c_1 + c_2) \cdot x = 0.
\]
We already know $\{1, x\}$ is a basis for $P_1(\mathbb{R})$ so
\[
2c_1 - 1/2c_2 = 0
\]
\[
-2c_1 + c_2 = 0
\]
Equivalently, $c_1 = c_2 = 0$ and therefore, $B$ is a basis and we’re done.