An example involving arc length

Here is an example in which the differentiation and antidifferentiation can be handled efficiently using the methods of single variable calculus classes:

**Problem.** Let

\[ x(t) = \left( (t + 1)^{3/2}, (t + 2)^{3/2} \right) \]

for \( t \geq 0 \). Find the arc length \( s(t) \) of the curve from time 0 to time \( t \), and express \( t \) as a function of the arclength \( s \).

**Solution.** First use the arc length formula to find \( s(t) \):

\[
s(t) = \int_0^t \sqrt{x_1'(u)^2 + x_2'(u)^2} \, du = \int_0^t \sqrt{\left(\frac{3}{2}\sqrt{u + 1}\right)^2 + \left(\frac{3}{2}\sqrt{u + 2}\right)^2} \, du =
\]

\[
\frac{3}{2} \int_0^t \sqrt{2u + 3} \, du = \frac{3}{2} \left[ \frac{2}{3} (2u + 3)^{3/2} \right]_0^t = \frac{1}{2} \left[ (2t + 3)^{3/2} - 3^{3/2} \right].
\]

To express \( t \) in terms of \( s \) solve this equation for \( s \).

\[
s = \frac{1}{2} \left[ (2t + 3)^{3/2} - 3^{3/2} \right] \implies 2s = (2t + 3)^{3/2} - 3^{3/2} \implies 2s + 3^{3/2} = (2t + 3)^{3/2} \implies \left(2s + 3^{3/2}\right)^{2/3} = 2t + 3 \implies t = \frac{1}{2} \left[ \left(2s + 3^{3/2}\right)^{2/3} - 3 \right].
\]