Computing the Second Fundamental Form

Plane \( z = 0 \)
\[
\overrightarrow{X}(u, v) = (u, v, 0).
\]
\[
\overrightarrow{X}_1(u, v) = (1, 0, 0) \quad \overrightarrow{X}_2(u, v) = (0, 1, 0)
\]
\[
\overrightarrow{X}_1 \times \overrightarrow{X}_2 = (0, 0, 1)
\]
Since this is a unit vector, we may take this to be \( \overrightarrow{N} \).

SFF coefficients.
\[
e = \overrightarrow{N} \cdot \overrightarrow{X}_{11} = \overrightarrow{N} \cdot \overrightarrow{0} = 0
\]
Likewise for \( f \) and \( g \). So we have
\[
SFF = 0.
\]

Cylinder \( x^2 + y^2 = 1 \).
\[
\overrightarrow{X}(\cos u, \sin u, v).
\]
\[
\overrightarrow{X}_1 = (-\sin u, \cos u, 0) \quad \overrightarrow{X}_2 = (0, 0, 1)
\]
So \[
\overrightarrow{X}_1 \times \overrightarrow{X}_2 = \text{(calculations)} = (-\cos u, -\sin u, 0).
\]
This is again a unit vector, so we can take this to be \( \overrightarrow{N} \) (inward normal in this case).
In both cases \( \vec{X}_1 \) and \( \vec{X}_2 \) are orthonormal so that they have the same FFF:

\[
E = \vec{X}_1 \cdot \vec{X}_1 = 1, \quad F = \vec{X}_1 \cdot \vec{X}_2 = 0, \quad G = \vec{X}_2 \cdot \vec{X}_2 = 1
\]

However, if we compute the SFF for the cylinder, here is what we get:

\[
\vec{X}_{11} = (-\cos u, -\sin u, 0) \\
\vec{X}_{12} = \vec{X}_{22} = 0.
\]

Hence \( f = g = 0 \) immediately and \( e = (-\cos u, -\sin u, 0) \cdot (-\cos u, -\sin u, 0) = 1 \) so that \( \text{SFF} = \text{du}du, \text{ nonzero}. \)

**Another Example**

Consider the cone \( x^2 + y^2 = z^2, \ z > 0 \), so that one has a parametrization

\[
\vec{X}(u,v) = (v \cos u, v \sin u, v)
\]
In this case:
\[ \overrightarrow{X_1} = (-v \sin u, v \cos u, 0) \]
\[ \overrightarrow{X_2} = (\cos u, \sin u, 1) \]

Here are the coefficients of the FFF:
\[ E = v^2 \quad F = 0 \quad G = 2 \]

To compute the SFF, first consider:
\[ \overrightarrow{D_0} = \overrightarrow{X_1} \times \overrightarrow{X_2} = \begin{vmatrix} i & j & k \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = \]
\[ (v \cos u, v \sin u, -v) \]

Therefore:
\[ N = \frac{1}{18} \overrightarrow{D_0} = \frac{1}{\sqrt{2}} \overrightarrow{D_0} = \frac{1}{\sqrt{2}} (\cos u, \sin u, -1) \]

New:
\[ \overrightarrow{X_{11}} = (-v \cos u, -v \sin u, 0) \]
\[ \overrightarrow{X_{12}} = (- \sin u, \cos u, 0) \]
\[ \overrightarrow{X_{22}} = (0, 0, 0) \]
These lead to
\[ e = -\frac{v}{\sqrt{2}}, \quad f = 0, \quad g = 0. \]

Recommended exercise

What happens if we take a cylinder of radius \( r > 0 \) with parametrization \((r \cos \frac{u}{r}, r \sin \frac{u}{r}, v)\) or a cone of the form \( x^2 + y^2 = a^2 z^2 \), where \( a > 0 \), with parametrization 
\((a \varepsilon \cos u, a \varepsilon \sin u, v)\)?