Comments on the Lifting Criterion

A basic step in the proof of the Lifting Criterion in Section 79 of Munkres is the following observation:

Suppose that \( f : (Y, y_0) \to (B, b_0) \) is a continuous map of arcwise connected, locally arcwise connected spaces, and let \( p : (E, e_0) \to (B, b_0) \) be a base point preserving covering space projection such that \( E \) is also connected and the image of the associated map of fundamental groups \( f_* \) is contained in the image of \( p_* \). Let \( \alpha \) and \( \beta \) be continuous curves joining \( y_0 \) to \( y \in Y \), and let \( \tilde{f}\alpha \) and \( \tilde{f}\beta \) denote the unique liftings of \( f \circ \alpha \) and \( f \circ \beta \) to continuous curves in \( E \) whose values at 0 are \( e_0 \). Then \( \tilde{f}\alpha(1) = \tilde{f}\beta(1) \).

**Proof of assertion.** Let \( \varphi = \alpha + (-\beta) \), which is a closed curve based at \( y_0 \). Then \( f \circ \varphi \) is a closed curve in \( B \), and the fundamental group condition plus the Covering Homotopy Property show that the unique lifting \( \gamma = \tilde{\varphi} \) of \( \varphi \) to a curve in \( E \) with initial condition \( e_0 \) is also a closed curve. It follows immediately that the curve \( \gamma_1(t) = \gamma(t) \) (for \( t \in [0,1] \)) is a lifting of \( f \circ \alpha \) to \( E \) with initial condition \( e_0 \) and \( \gamma_2(t) = \gamma(1 - \frac{1}{2}t) \) (for \( t \in [0,1] \)) is a lifting of \( f \circ \beta \) to \( E \) with initial condition \( e_0 \). Therefore \( \gamma_1 = \tilde{f}\alpha \) and \( \gamma_2 = \tilde{f}\beta \). By these formulas, the values of these curves at \( t = 1 \) are given by \( \gamma_1(1) = \gamma(\frac{1}{2}) \) and \( \gamma_2(1) = \gamma(\frac{1}{2}) \) respectively, and the conclusion of the assertion follows directly from these equations.