CLARIFICATIONS TO COMMENTARIES

PROOF OF THE SEIFERT-VAN KAMPEN THEOREM. (pp. 48–49) Here are some additional details and modifications. We begin with material in the final two paragraphs of page 48:

We claim that $E$ is arcwise connected ... by construction, if $e_1$ and $e_2$ are two points in $F$ such that $g \cdot e_1 = e_2$ for some $g$ in the image of the fundamental group of $U$, it follows that $e_1$ can be joined to $e_2$ by a continuous curve whose image lies in the inverse image of $U$ in $E$; a similar conclusion holds if we replace $U$ be $V$ in the preceding statement.

It is easier to prove the connectedness of $E$ if we modify the preceding assertion as follows: Suppose that $e_0$ is the base point of $E$ and $g \in \Gamma$, and let $h \in \Gamma$ be an element which lies in the image of either $\pi_1(U)$ or $\pi_1(V)$. Then $ge_0$ and $gh_0e_0$ lie in the same component of $E$. — Given this, one can use the fact that the images of $\pi_1(U)$ and $\pi_1(V)$ generate $\Gamma$ to conclude that every point in $F$ lies in the same component as $e_0$ and hence $E$ is connected. Specifically, if we write $g = h_1 \cdots h_k$ for $h_i$ satisfying the given conditions and lets $g_0$ denote the product of the first $i$ factors for $0 \leq i \leq k$ (with $g_0 = 1$), then by induction we have that each $g_i \cdot e_0$ lies in the same component of $E$ as $e_0$.

We shall only consider the case where $h$ comes from the fundamental group of $U$; the other case follows by systematically replacing $U$ with $V$ throughout the discussion. It will help to have some notation. Let $k_U : \tilde{U} \to E$ be the inclusion map given by the construction of $U$, and let $u_0$ denote the base point of $\tilde{U}$, so that $k_U$ maps $u_0$ to $e_0$. Suppose that $h \in \Gamma$ lies in the image of $\pi_1(U)$, and let $h'$ map to $h$. By construction we know that $k_u$ sends $h' u_0$ to $h e_0$. Let $\eta$ be the curve in $\tilde{U}$ joining $u_0$ to $h' u_0$. Then it follows that $k_U \circ \eta$ joins $e_0$ to $h e_0$, proving the assertion when $h$ comes from $\pi_1(U)$; as noted before, a similar argument applies if $h$ comes from $\pi_1(V)$, and by the remarks in the preceding paragraph it follows that $E$ is connected as required.

Next, we shall examine the following statements from page 49 more closely:

[We have] the diagram of morphisms displayed below, in which the square is commutative (all compositions of morphisms between two objects in this part of the diagram are equal).

\[
\begin{array}{cccccccc}
\pi_1(U \cap V) & \to & \pi_1(U \cap V) \\
\downarrow & & \downarrow \\
\pi_1(U \cap V) & \to & J(U) & \to & \Gamma & \to & \pi_1(X) & \to & \partial & \to & \Gamma \\
\end{array}
\]

The map $\Phi$ is the homomorphism given by the universal mapping property of the pushout group $\Gamma$ (see the commentary to Section 70). If we can show that $\partial \circ \Phi$ is the identity, then it will follow that $\Phi$ is injective. Since we already know that $\Phi$ is surjective (see Section 70), it will follow that $\Phi$ is an isomorphism, and the proof will be complete.

In the subsequent discussion on page 49, the key point is to prove that the composites

\[
\pi_1(U) \to P \to \pi_1(X) \to \Gamma \quad \pi_1(V) \to P \to \pi_1(X) \to \Gamma
\]

are just the standard maps $J(U)$ and $J(V)$ from $\pi_1(U)$ and $\pi_1(V)$ into the pushout $\Gamma$. It will be helpful to let $i_{U*}$ and $i_{V*}$ denote the maps of fundamental groups induced by the inclusions of $U$ and $V$ in $X$; by construction we have $i_{U*} = \Phi \circ J(U)$ and $i_{V*} = \Phi \circ J(V)$.
As before, it suffices to show that $\partial \circ \Phi \circ J(U) = J(U)$, for the argument in the other case will follow by systematic substitution of $V$ for $U$ throughout. — Let $h'$ be an element in $\pi_1(U)$, and let $h$ be its image in $\Gamma$. By construction, the covering space transformation determined by $\partial \circ \Phi(h) \in \Gamma$ sends the base point $e_0$ to $\Phi(h) \cdot e_0 = i_{U*}(h') \cdot e_0$. On the other hand, we also know that the covering space transformation of $\tilde{U}$ associated to $h'$ sends $u_0$ to $h' \cdot u_0$, and if we apply the mapping $k_U$ from the previous discussion, it follows that the covering space transformation of $E$ associated to $J(U)(h')$ sends $e_0 = k_U(u_0)$ to $i_{U*}(h') \cdot e_0$.

The preceding argument shows that $\partial \circ i_{U*} = J(U)$, and the identity in the first sentence of the preceding paragraph then follows because $i_{U*} = \Phi \circ J(U)$. As noted above, we have a similar identity involving $V$. Taken together, these imply that the restrictions of $\partial \circ \Phi$ to the images of $J(U)$ and $J(V)$ are the identity, and since these images generate $\Gamma$ it follows that $\partial \circ \Phi$ must be the identity, as claimed. ■