CLARIFICATIONS TO COMMENTARIES – II

Reduced paths and simple circuits. The definitions of these on page 50 of the commentaries were somewhat informal, and they should be amplified as follows.

Definition. Let $E_1, \cdots, E_n$ be an edge-path sequence such that the vertices of $E_i$ are $v_{i-1}$ and $v_i$. This sequence is said to be reduced if $v_1, \cdots, v_n$ are distinct and either $n \neq 2$ or else $v_0 \neq v_2$ (the latter case is just a sequence with $E_2 = E_1$, physically corresponding to going first along $E_1$ in one direction and then back in the opposite direction).

The following result is more or less predictable but still important.

PROPOSITION. If two distinct vertices $x$ and $y$ can be connected by an edge-path sequence, then they can be connected by a reduced sequence.

Proof. Take a sequence with a minimum number of edges. We claim it is automatically reduced. If not, then there is a first vertex which is repeated, and a first time at which it is repeated. In other words, there is a minimal pair $(i, j)$ such that $i < j$ and $v_i = v_j$, which means that if $(p, q)$ is any other pair with this property we have $p \geq i$ and $q > j$. If we remove $E_{i+1}$ through $E_j$ from the edge-path sequence, we obtain a shorter sequence which joins the given two vertices.

Note that the converse is false. For example, take $X$ to be the triangle graph in the plane whose vertices are the three noncollinear points $a, b$ and $c$, and whose edges are the three line segments joining these pairs of points. Then $\{ab, bc\}$ and $\{ac\}$ are two reduced edge-path sequences joining $a$ to $c$ such that one consists of two edges and the other consists of one edge.

Another consequence of the definitions worth noting is that every simple circuit in a graph contains at least three edges.