M6. Further hints. (i) The binary operation on the set of closed curves is defined by pointwise multiplication. Therefore, checking the identities $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and $1 \otimes f = f = f \otimes 1$ reduce to checking that the functions obtained by evaluating both sides at an arbitrary element of $G$ are the same. Similarly, if we define $f^{-1}$ by $f^{-1}(x) = (f(x))^{-1}$, then checking that $f \otimes f^{-1} = 1 = f^{-1} \otimes f$ reduces to evaluating all the relevant expressions at an arbitrary point $x$.

(ii) We need to show that if $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $f_0 \otimes g_0 \simeq f_1 \otimes g_1$, where $\simeq$ denotes endpoint preserving homotopy. Let $H$ and $K$ be the homotopies for the maps $f_i$ and $g_i$ respectively. Consider the homotopy $L = H \cdot K$ defined by the algebraic product of $H$ and $K$ viewed as maps into the topological group $G$.

(iii) Let $\simeq$ be as above, and let $C$ denote the constant closed curve sending all points to 1. Given $a, b \in \pi_1(G, 1)$, choose representatives $\alpha$ and $\beta$ respectively. Verify each of the relationships in the chain $\alpha + \beta \simeq (\alpha + C) \otimes (C + \beta) \simeq \alpha \otimes \beta$.

(iv) In the notation of the previous paragraph, verify each of the relationships in the chain $\alpha \otimes \beta \simeq (\alpha + C) \otimes (C + \beta) \simeq (C + \beta) \otimes (\alpha + C) \simeq \beta \otimes \alpha$. Recall that $1 \cdot g = g = g \cdot 1$ for all $g \in G$.

A1. Correction. In the second sentence, “[P,Y]” should be replaced by “[P,X].”

A4. Further hints. (i) For each point $x \in U$ there is a $\delta(x) > 0$ such that the open disk $N_{\delta(x)}(x; \mathbb{R}^n)$ is contained in $U$; it follows that the closures of the disks $V_x N_{\delta(x)/2}(x; \mathbb{R}^n)$ are compact and contained in $U$. The disks $V_x$ form an open covering of $U$, and since $U$ is second countable it has the Lindelöf property: Every open covering has a countable subcovering. Let $\{V_m\}$ denote a countable subcovering extracted from the $V_x$’s, and let $F_m$ be the closure of $V_m$ (hence $F_m$ is a closed disk contained in $U$). Let

$$C_m = \bigcup_{j=1}^{m} F_m$$

and verify this family of compact subsets has all the required properties.

(ii) Choose $f : X \to Y$ representing $u$. Why is the image of $f$ contained in some subset $C_m$?
(iii) Suppose that $f$ and $g$ are homotopic maps from $X$ into $C_m$ and $H$ is the homotopy between $j_m \circ f$ and $j_m \circ g$. Why is the image of $H$ contained in some subset $C_n$, and why can we choose $n$ to be greater than or equal to $m$? Recall that if $X$ is compact then so is $X \times [0,1]$.

Munkres, Section 53

M2. Further hints. How are the slices of the covering map $p$ over the open set $U$ related to the connected components of $p^{-1}[U]$?

M4. Further hints. Let $z \in Z$, write $p^{-1}\{z\} = \{z_1, \ldots, z_m\}$, and let $U$ be an open neighborhood of $z$ in $Z$ which is evenly covered with respect to $p$. Then $p^{-1}[U]$ is a union of pairwise disjoint open subsets $U_i$ ($1 \leq i \leq m$) such that $z_i \in U_i$. Why are there open subsets $V_i \subset U_i$ such that $z_i \in V_i$ and $V_i$ is evenly covered with respect to $q$? Let

$$W = \bigcap_{i=1}^{m} p[V_i]$$

and explain why $W$ is an open neighborhood of $z$ that is evenly covered with respect to the composite.

M6. Further hints. (i) There are several separate conclusions depending upon the topology of the codomain $B$, and it is best to handle each one individually. Another fact along these lines worth noting is that if $B$ is $T_1$, then so is $E$ (try proving this — the argument is easier than any of the following).

Suppose that $B$ is HAUSDORFF. It is convenient to split things into two cases, depending upon whether or not the distinct points $x, y \in E$ map to the same point in $B$. If they do, try to construct disjoint neighborhoods using slices of an evenly covered neighborhood of $p(x) = p(y)$. On the other hand, if $p(x) \neq p(y)$, let $U$ and $V$ be disjoint open neighborhoods of these image points in $B$ and consider $p^{-1}[U]$ and $p^{-1}[V]$.

Suppose that $B$ is REGULAR. Let $x \in E$, and let $U$ be an open neighborhood of $x$ in $E$. Let $W$ be an open neighborhood of $p(x)$ in $B$ that is evenly covered, let $W_0$ be the unique slice in $p^{-1}[W]$ containing $x$, and let $W_1 = U \cap W_0$, so that $p$ maps $W_1$ homeomorphically to an open neighborhood of $p(x)$ which is contained in $W$. Since $B$ is regular, there is an open neighborhood $V_0$ of $p(x)$ in $B$ such that

$$p(x) \in V_0 \subset \overline{V_0} \subset p[W_1].$$

Let $V = W_1 \cap p^{-1}[V_0]$. By construction we have $x \in V$, so it is only necessary to show that the closure of $V$ in $E$ is contained in $W_1$. — If we let $A = W_1 \cap p^{-1}[\overline{V_0}]$, then it will suffice to prove that $A$ is closed in $E$ or equivalently that $E - A$ is open in $E$. Let $W'$ be the union of all the slices for $p$ over $p[W_1]$, except for $W_1$ itself. Check that $E - A$ is the union of $W'$ and $E - \overline{V_0}$, and explain how this is relevant.

Suppose that $B$ is COMPLETELY REGULAR. Let $x \in E$, and let $U$ be an open neighborhood of $x$ in $E$. Let $W$ be an open neighborhood of $p(x)$ in $B$ that is evenly covered, let $W_0$ be the unique slice in $p^{-1}[W]$ containing $x$, and let $W_1 = U \cap W_0$, so that $p$ maps $W_1$ homeomorphically to an open neighborhood of $p(x)$ which is contained in $W$. Since $B$ is completely regular, by the preceding discussion we know that $E$ is also regular, so choose an open neighborhood $V$ of $x$ such that the closure of $V$ is contained in $W_1$. Since $B$ is completely regular and $p$ maps $W_1$ homeomorphically to a subspace of $B$, we know that $W_1$ is also completely regular (recall that a subspace of a completely regular space is also completely regular). Let $f_0$ be a continuous function from $W_1$ to $[0,1]$ which
Let \( p \) be the unique slice in \( e \). Then the sets \( W_b' \) form an open covering of \( B \) and hence there is a finite subcovering by some sets \( W_{b(k)} \) for \( 1 \leq k \leq m \). Let \( V \) be the family of all open sets \( V_e \) where \( e \) runs through all points such that \( p(e) = b(k) \) for some \( k \). Why is this set finite? Explain why

\[
p^{-1}[W_{b(k)}] = \bigcup_{p(e)=b(k)} V_e
\]

and use this to show that \( V \) is a finite open covering of \( E \). For each \( V_e \) in \( V \) choose the \( U_\alpha \) in \( U \) such that \( V_e \subset N_e \subset U_\alpha \), and explain why this collection defines a finite subcovering of \( E \).

**A1. Correction.** There should be an additional hypothesis that \( E_2 \) is **connected**. There are simple counterexamples if this condition is not met. Specifically, let \( E = \bigcup_{i=1}^\infty E_i \) denote the (topological) disjoint union of two copies of \( E \), let \( q : E \to B \) be a covering space projection, let \( p_1 : E \to B \) be the map whose restriction to each copy of \( E \) is given by \( q \), and let \( p : E \to \bigcup_{i=1}^\infty E_i \) be the map which sends \( E \) to the first disjoint copy of \( E \) on the right hand side. Then \( p_1 = p \) and \( p_2 \) are both covering space projections, but \( p \) is not because it is not surjective.

**Hints.** First of all, we need the following basic fact:

**Lemma.** Let \( X \) be a connected space, and let \( B \) be a base for the topology on \( X \). Then for each \( u, v \in X \) there is a finite sequence of open sets \( U_0, \cdots, U_m \) in \( B \) such that \( u \in U_0, v \in U_m \) and \( U_i \cap U_{i-1} \neq \emptyset \) for all \( i > 0 \).

**Sketch of proof.** Given an arbitrary \( X \) and \( B \), define a binary relation \( \mathcal{R} \) by \( u \mathcal{R} v \) if and only if the condition in the Lemma holds, show that it is an equivalence relation, explain why the
equivalence classes are open, using this explain why they must also be closed, and finally conclude that there is only one equivalence class if $X$ is connected.

Returning to the original problem, let $x \in E_2$, let $U_1$ be an evenly covered open neighborhood of $p_2(x)$ in $X$ with respect to $p_2$, let $U_2$ be an evenly covered open neighborhood of $p_2(x)$ with respect to $p_1$, and let $U = U_1 \cap U_2$, so that $U$ is evenly covered with respect to both maps. Therefore we know that $p_1^{-1}[U]$ and $p_2^{-1}[U]$ are both isomorphic to disjoint unions of copies of $U$. More precisely, we know that $p_1^{-1}[U]$ is homeomorphic to $U \times A$ for some discrete space $A$, while $p_2^{-1}[U]$ is homeomorphic to $U \times B$ for some discrete space $B$, and under these homeomorphisms the map $p$ sends a slice $U \times \{a\}$ to a slice $U \times \{h(a)\}$ by the standard mapsending $(u, a)$ to $(u, h(a))$, where $h : A \to B$ is some mapping of indexing sets. If $x$ lies in the slice corresponding to $U \times \{b\}$, then $x$ will be evenly covered, with slices given by all $U \times \{a\}$ such that $h(a) = b$. It follows that $p$ will be a covering map provided it is surjective. The point of the connectedness condition is that it should imply the surjectivity of $p$.

To show this, proceed as follows: Suppose that we have $z \in E_2$ and $z$ lies in an open set $U$ such that $p_2(U)$ is evenly covered with respect to both $p_1$. Use the discussion of the preceding paragraph to show that if $z$ lies in the image of $p_1$ then so does every point in $U$. Let $B$ be the base of open sets in $E_2$ satisfying the condition in the second sentence of this paragraph, let $R$ be as in the lemma, and show that if two points lie in the same equivalence class with respect to $R$ and one of them lies in the image of $p$, then so does the other. Why does this imply that $p$ is onto? Recall that $E_2$ is connected.

**A2. Hints.** (i) Given $y \in Y$, let $V \subset X$ be an evenly covered open neighborhood of $f(y)$, and let $U$ be an open neighborhood of $y$ such that $f[U] \subset V$.

(ii) Let $h$ be the continuous mapping sending $(y, e)$ to $e$; explain why the image of $h$ lies in the inverse image of $Y$, verify the functional identities in the exercise, and show that an inverse to $h$ is given by $k : p^{-1}[Y] \to Y \times X$ $E$ is given by $k(z) = (p(z), e)$. One needs to check that the formula determines an element in the subspace $Y \times X$ $E \subset Y \times E$.

**A3. Hints.** By the assumptions, the topology for $X$ has a base of open subsets that are also closed; explain why it has a base of evenly covered open subsets that are also closed. Consider the slices in $E$ which lie above such open subsets of $X$. We claim they form a base for the topology on $E$; explain why it suffices to show that each slice is closed in $E$. To show that such a slice $V$ is closed, write its complement $E \setminus V$ as a union of the sets $E \setminus p^{-1}[p[V]]$ and all the other slices $V'$ such that $p[V'] = p[V]$? Why are all these subsets open in $E$?

**Munkres, Section 54**

**M8. Hints.** Before proceeding, it is useful to note the following general result:

**PROPOSITION.** If $p : E \to B$ is a covering space projection, then $p$ is an open mapping.

**Proof.** Let $U \subset E$ be open, for each $y \in U$ let $V_y$ be an open neighborhood of $f(y)$ which is evenly covered, and let $U_y$ be an open neighborhood of $y$ such that $U_y \subset U$ and $U_y$ is contained in a slice over $V_y$. Then by definition the map $p$ sends $U_y$ to an open subset of $B$, and hence

$$p[U] = \bigcup_{y \in U} p[U_y]$$

is open in $X$.■
Returning to the original problem, we know that \( p \) is open, continuous and onto, so it is only necessary to check that \( p \) is 1–1. The general results on covering spaces imply that for each \( b \in B \) the inverse image \( p^{-1}([b]) \) is in 1–1 correspondence with the set of cosets \( \pi_1(B, b)/\text{Image } p_* \). Why do this and simple connectivity imply about \( p^{-1}([b]) \)?

**A1. Hints.** Let \( p_n : R^n \to T^n \) be the Cartesian product of \( n \) copies of \( p : R \to S^1 \). Then \( p_n \) is equivalent to \( p_{n-1} \times p \) under the natural identifications of \( \prod^n X \) with \( \left( \prod^{n-1} X \right) \times X \) when specialized to \( X = R \) or \( S^1 \). We know that \( p = p_1 \) is a covering space projection, and by a result from Section 53 we know that \( p_n \) will be a covering space projection if \( p_{n-1} \) is.

**A2. Further hints.** We know that \( \pi_1(T^n, e) \cong Z^n \). Why are the automorphisms of this group equal to the set of matrices described in the problem? To work the second part, take the map described in the original hint. The associated map will then be given by an \( n \times n \) matrix over the integers. To determine the \( (i, j) \) entry of this matrix, let \( \theta_j : S^1 \to T^n \) denote the injection map whose projection onto the \( j^{th} \) coordinate is the identity and whose projection onto the other coordinates is the constant map with value 1, and let \( p_i \) denote projection onto the \( i^{th} \) coordinate. Find a relationship between \( p_i \circ f_A \circ \theta_j \) and the entries of \( A \).