Remarks on spheres

We begin with a basic decomposition result concerning spheres. Let $\text{Int } D^m$ denote the open disk of unit 1 in $\mathbb{R}^m$.

**THEOREM.** Suppose that $p, q > 0$. Then the sphere $S^{p+q+1}$ is a union of two open subsets $U$ and $V$ such that $U$ is homeomorphic to $S^p \times \text{Int } D^{q+1}$, $V$ is homeomorphic to $\text{Int } D^{p+1} \times S^q$, and their intersection is homeomorphic to $S^p \times S^q \times (0, 1)$.

**Proof.** View $S^{p+q+1}$ as the unit sphere in $\mathbb{R}^{p+q+2} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$, and let $S_p$ and $S_q$ denote the unit spheres in $\mathbb{R}^{p+1} \times \{0\}$ and $\{0\} \times \mathbb{R}^{q+1}$ respectively. Let $U = S^{p+q+1} - S_q$ and $V = S^{p+q+1} - S_p$. Then there are homeomorphisms

$$h : S^p \times \text{Int } D^{q+1} \rightarrow U, \quad k : \text{Int } D^{p+1} \times S^q \rightarrow V$$

defined by the following formulas:

$$h(x, y) = \left( \sqrt{1-y^2} \cdot x, y \right), \quad k(x, y) = \left( x, \sqrt{1-x^2} \cdot y \right)$$

In each case it is a straightforward exercise to write down a formula for the inverse which shows that the inverse is continuous. Also, one has a similar homeomorphism $\varphi : S^p \times (0, 1) \times S^q \rightarrow S^{p+q+1} - (S_1 \cup S_2) = U \cap V$ which sends $(x, t, y)$ to $(tx, sy)$ where $s = \sqrt{1-t^2}$.

**SPECIALIZATION TO $S^3$.** In this case $p = q = 1$, and it is instructive to look at the fundamental groups of the various spaces constructed above, for they give an example of a space $X = U \cup V$ such that $(i)$ $U$ and $V$ are open arcwise connected subspaces with an arcwise connected intersection, $(ii)$ the fundamental groups of $U$ and $V$ are nontrivial, $(iii)$ the fundamental group of $X = U \cup V$ is trivial. This is true because the fundamental groups of $U$ and $V$ are infinite cyclic, while the fundamental group of $X = U \cup V$ is trivial.

Here is a more detailed explanation of the situation when $p = q = 1$: The diagram of fundamental groups

$$\pi_1(U) \quad \pi_1(U \cap V) \quad \pi_1(V)$$

corresponds to the algebraic diagram

$$\mathbb{Z} \leftarrow \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$$

where the left and right arrows represent projections onto the first and second factors. Since the generators of $\pi_1(U)$ and $\pi_1(V)$ lift to a pair of free generators for $\pi_1(U \cap V)$, and the respective generators map to the trivial elements in $\pi_1(V)$ and $\pi_1(U)$ respectively, it follows that these generators must map to zero in $\pi_1(X)$, and this in turn yields another proof that $\pi_1(S^3)$ is trivial.