APPENDIX B

THE JOIN IN AFFINE GEOMETRY

In Section II.5 we defined a notion of join for geometrical incidence spaces; specifically, if $P$ and $Q$ are geometrical subspaces of an incidence space $S$, then the join $P \star Q$ is the unique smallest geometrical subspace which contains them both. From an intuitive viewpoint, the name “join” is meant to suggest that $P \star Q$ consists of all points on lines of the form $xy$, where $x \in P$ and $y \in Q$. If $S$ is a projective $n$-space over some appropriate scalars $\mathbb{F}$, this is shown in Exercise 16 for Section III.4, and the purpose of this Appendix is to prove a similar result for an affine $n$-space over some $\mathbb{F}$.

Formally, we begin with a generalization of the idea described above.

**Definition.** Let $(S, \Pi, d)$ be an abstract geometrical incidence $n$-space, and let $X \subset S$. Define $J(X)$ to be the set

$$X \cup \{ y \in S \mid y \in uv \text{ for some } u, v \in X \}.$$ 

Thus $J(X)$ is $X$ together with all points on lines joining two points of $X$. Note that the construction of $J(X)$ from $X$ can be iterated to yield a chain of subsets $X \subset J(X) \subset J(J(X)) \cdots$.

The preceding discussion and definition lead naturally to the following:

**Question.** If $S$ is a geometrical incidence $n$-space and $P$ and $Q$ are geometrical subspaces of $S$, what is the relationship between $P \star Q$ and $J(P \cup Q)$? In particular, are they equal, at least if $S$ satisfies some standard additional conditions?

The exercise from Section III.4 shows that the two sets are equal if $S$ is a standard projective $n$-space. In general, the next result implies that the two subsets need not be equal, but one is always contained in the other.

**Theorem B.1.** In the setting above, we have $J(P \cup Q) \subset P \star Q$. However, for each $n \geq 2$ there is an example of a regular geometrical incidence spaces such that, for some choices of $P$ and $Q$, the set $J(P \cup Q)$ is strictly contained in $P \star Q$.

**Proof.** The inclusion relationship follows from $\textbf{G}(-2)$ and the fact that $P \star Q$ is a geometrical subspace of $S$. On the other hand, if we take the affine incidence space structure associated to $\mathbb{Z}_2^n$ for $n \geq 2$, then for every subset $X \subset \mathbb{Z}_2^n$ we automatically have $J(X) = X$ because every line consists of exactly two points. Thus if $W$ and $U$ are vector subspaces of $\mathbb{Z}_2^n$ such that neither contains the other, then $J(W \cup U)$ is not a vector subspace. Since $0 \in W \cap U$, we know that $W \star U$ is the vector subspace $W + U$ by Theorem II.36, and it follows in this case that $J(W \cup U)$ is strictly contained in $W \star U$.

Note that the examples constructed in the proof are in fact affine incidence spaces. The main objective of this appendix is to prove that $J(P \cup Q) = P \star Q$ if $V$ is a vector space of dimension $\geq 2$ over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$. 

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**Theorem B.2.** Let $V$ be a vector space of dimension $\geq 2$ over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$, and suppose that $P = a + U$ and $Q = b + W$ are geometrical subspaces of $V$. Then the following hold:

(i) The join $P \ast Q$ is the affine span of $P \cup Q$.
(ii) $P \ast Q = J(P \cup Q)$.

**Proof.** **FIRST STATEMENT.** If $R$ is the affine span of $P$ and $Q$, then $R$ is an affine subspace containing $P$ and $Q$ by Theorem II.19, Theorem II.16 and Exercise 1 for Section II.2 (this is where we use the assumption that $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$). Therefore it follows that $R$ also contains $P \ast Q$. On the other hand, if $R'$ is a geometrical subspace containing $P$ and $Q$, then by Theorem II.18 it contains all affine combinations of points in $P \cup Q$, and hence $R'$ must contain $R$. Combining these observations, we conclude that $R$ must be equal to $P \ast Q$.

**SECOND STATEMENT.** By the previous theorem we know that $J(P \cup Q) \subset P \ast Q$, so it suffices to show that we also have the converse inclusion $P \ast Q \subset J(P \cup Q)$.

Let $x \in P \ast Q$, and let $\{d_0, \cdots, d_p\}$ and $\{c_0, \cdots, c_q\}$ be affine bases for $P$ and $Q$ respectively. Then by the conclusion of the first part of the theorem we may write

$$x = \sum_{i=0}^{p} r_i d_i + \sum_{j=0}^{p} s_j c_j$$

where $\sum r_i = \sum s_j = 1$. Let $t = \sum r_i$, so that $\sum s_j = 1 - t$. There are now two cases, depending upon whether either or neither of the numbers $t$ and $1 - t$ is equal to zero. If $t = 0$ or $1 - t = 0$ (hence $t = 1$), then we have $x \in P \cup Q$. Suppose now that both $t$ and $1 - t$ are nonzero. If we set

$$\alpha = \sum_{i=0}^{p} \frac{r_i}{t} d_i, \quad \beta = \sum_{j=0}^{q} \frac{s_j}{1 - t} c_j,$$

then $\alpha \in P$, $\beta \in Q$, and $x = t \alpha + (1 - t) \beta$; therefore it follows that $x \in J(P \cup Q)$. $\blacksquare$