Some Topology Related Definitions

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References. The definitions contained here were compiled (with minor editing) from the following books:

- *Topology and Geometry* by Glen E. Bredon
- *Topology* by Klaus Jänich
- *Introduction to Smooth Manifolds* by John M. Lee

Definition 1 (Topological space). A *topological space* is a pair \((X, \mathcal{T})\) consisting of a set \(X\) together with a collection \(\mathcal{T}\) of subsets of \(X\). This collection \(\mathcal{T}\) is called the *topology* of the topological space \((X, \mathcal{T})\).

Elements of \(\mathcal{T}\) are called *open sets* and are required to satisfy the following properties:

1. Any union of open sets is open.
2. The intersection of any two open sets is open.
3. The set \(X\) and the empty set \(\emptyset\) are open.

Remark. Depending on the desired emphasis, it is often customary to drop the topology from the notation and simply speak of the topological space \(X\), especially when confusion is unlikely.

Definition 2. Let \(X\) be a topological space.

1. \(C \subset X\) is called *closed* when \(X \setminus C\) is open.
2. \(N \subset X\) is called a *neighborhood* of \(x \in X\) if there is an open set \(U\) with \(x \in U \subset N\).
3. Let \(A \subset X\) be any subset. A point \(x \in X\) is called an *interior* or *exterior* or *boundary* point of \(A\), respectively, according to whether \(A, X \setminus A\), or neither is a neighborhood of \(x\).
4. The set \(A^\circ\) of interior points of \(A\) is called the *interior* of \(A\). Alternatively defined as the largest open set contained in \(A\).
5. The set \(\overline{A}\) of the points of \(X\) which are not exterior points of \(A\) is called the *closure* of \(A\). Alternatively defined as the smallest closed set containing \(A\).
6. The set \(\partial A\) of boundary points of \(A\) is called the *boundary* or *frontier* of \(A\). Alternatively defined as \(\partial A = \overline{A} \cap X \setminus \overline{A}\).
7. \(A \subset X\) is called *clopen* if it is both open and closed in \(X\).
8. \(A \subset X\) is called *dense* in \(X\) if \(\overline{A} = X\).
9. \(A \subset X\) is called *nowhere dense* in \(X\) if \((A)^\circ = \emptyset\).

Definition 3 (Equivalent definition of topological space). A *topological space* is a pair \((X, \mathcal{C})\) consisting of a set \(X\) together with a collection \(\mathcal{C}\) of subsets of \(X\). Elements of \(\mathcal{C}\) are called *closed sets* and are required to satisfy the following properties:

1. Any intersection of closed sets is closed.
2. The union of any two closed sets is closed.
3. The set \(X\) and the empty set \(\emptyset\) are closed.

Definition 4. Let \((X, \mathcal{T})\) be a topological space.

1. \(\mathcal{T}\) is called the *discrete topology* on \(X\) if \(\mathcal{T} = \mathcal{P}(X)\), that is, if \(\mathcal{T}\) consists of all subsets of \(X\). This topology is often called the *finest* or *strongest* or *largest* topology on \(X\). In particular, single points are both open and closed. When endowed with the discrete topology, \(X\) is called a *discrete space*.
2. \(\mathcal{T}\) is called the *trivial topology* on \(X\) if \(\mathcal{T} = \{X, \emptyset\}\). This topology is often called the *coarsest* or *weakest* or *smallest* topology on \(X\).
Definition 5 (Support of a function). The support of a real-valued function \( f : X \to \mathbb{R} \) on a topological space \( X \) is
\[
\text{supp}(f) = \{x : f(x) \neq 0\}.
\]
Notice the topological content here. The support of \( f \) is the closure in \((X, T)\) of \( X \setminus \ker(f)\).

Definition 6 (Continuity). Let \( f : X \to Y \) be a function between topological spaces.

1. We call \( f \) continuous if for every open set \( U \subseteq Y \), the preimage \( f^{-1}(U) \subseteq X \) is open.
2. Given \( x \in X \). We call \( f \) continuous at \( x \) if for any neighborhood \( N \) of \( f(x) \) in \( Y \), there is a neighborhood \( M \) of \( x \) in \( X \) such that \( f(M) \subseteq N \). Equivalently, \( f \) is continuous at \( x \) if \( f^{-1}(N) \) is a neighborhood of \( x \) for each neighborhood \( N \) of \( f(x) \).

Definition 7 (Discrete valued map). A discrete valued map is a continuous function from a topological space \( X \) to a discrete space \( D \).

Definition 8 (Homeomorphism). Let \( f : X \to Y \) be a bijective function between topological spaces.

1. \( f \) is called a homeomorphism if both \( f \) and \( f^{-1} \) are continuous. In this case, \( X \) and \( Y \) are called homeomorphic, often denoted \( X \approx Y \).
2. \( f \) is called an embedding if \( f \) is a homeomorphism onto its image.

Definition 9. A function \( f : X \to Y \) between topological spaces is called open if for every open set \( U \subset X \), \( f(U) \subset Y \) is again open. It is called closed if for every closed set \( C \subset X \), \( f(C) \subset Y \) is again closed.

Definition 10 (Basis). Let \( X \) be a topological space.

1. A set \( \mathcal{B} \) of open sets is called a basis for the topology if every open set is a union of sets in \( \mathcal{B} \).
2. A set \( \mathcal{S} \) of open sets is called a subbasis for the topology if every open set is a union of finite intersections of sets in \( \mathcal{S} \). Equivalently, the collection \( \mathcal{S} \) is a subbasis if finite intersections of sets in \( \mathcal{S} \) form a basis.
3. Given \( x \in X \). A collection \( \mathcal{B}_x \) of subsets of \( X \) containing \( x \) is called a neighborhood basis at \( x \) if each neighborhood of \( x \) contains some element of \( \mathcal{B}_x \) and each element of \( \mathcal{B}_x \) is a neighborhood of \( x \).

Definition 11 (Countability). Let \( X \) be a topological space.

1. \( X \) is called first countable if each point has a countable neighborhood basis.
2. \( X \) is called second countable if the topology on \( X \) has a countable basis.

Definition 12 (Subspace topology). If \((X, T)\) is a topological space and \( W \subset X \) a subset, then the topology on \( W \) defined by \( \mathcal{S} = |_{\mathcal{S}} = \{U \cap W \mid U \in T\} \) is called induced or relative or subspace topology, and the topological space \((W, \mathcal{S})\) is called a subspace of \((X, T)\).

Definition 13 (Product topology). Let \( X \) and \( Y \) be topological spaces. A subset \( W \subset X \times Y \) is called open in the product topology if for each point \((x, y) \in W\) there are neighborhoods \( U \) of \( x \) in \( X \) and \( V \) of \( y \) in \( Y \) such that \( U \times V \subset W \).

Equivalently, the product topology defined on \( X \times Y \) is the topology generated by the subbasis \( U \times V \), where \( U \subset X \) and \( V \subset Y \) are open. Since \((U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)\), this subbasis is in fact a basis. Therefore, the open sets are precisely the arbitrary union of such “rectangles”.

A product topology on any finite product of topological spaces is defined similarly.

The product topology on an infinite product is defined as the topology generated by the basis sets \( \prod U_\alpha \), where \( U_\alpha \) are open and \( U_\alpha = X_\alpha \) for all but a finite number of terms in the product. Note the collection of sets \( U_\alpha \times \prod_{\beta \neq \alpha} X_\beta \) is a subbasis for the product topology. This topology is also called the Tychonoff topology.

Definition 14 (Connectivity). Let \( X \) be a topological space.

1. \( X \) is called disconnected if it is the disjoint union of two nonempty open subsets. Otherwise, it is called connected.
(2) The **components or connected components** of $X$ are the equivalence classes determined by the equivalence relation “$p$ and $q$ belong to a connected subset of $X$”.

(3) The **quasi-components** of $X$ are the equivalence classes determined by the equivalence relation “$d(p) = d(q)$ for every discrete valued map $d$ on $X$”.

(4) $X$ is called **path connected** or **arcwise connected** if for any two points $p$ and $q$ in $X$ there exists a continuous function $\lambda: [0, 1] \to X$ with $\lambda(0) = p$ and $\lambda(1) = q$. When $\lambda$ exists, it is called a **path**.

(5) $X$ is called **locally path connected** if every neighborhood of any point in $X$ contains a path connected neighborhood.

(6) A **path component** is a maximal path connected subset.

**Definition 15** (Separation axioms). Let $X$ be a topological space.

- **(T₀)** $X$ is called **$T₀$** if for any two points $x \neq y$ there is an open set containing one of them but not the other.
- **(T₁)** $X$ is called **$T₁$** if for any two points $x \neq y$ there is an open set containing $x$ but not $y$ and another open set containing $y$ but not $x$.
- **(T₂)** $X$ is called **$T₂$** or **Hausdorff** if for any two points $x \neq y$ there are disjoint open sets $U$ and $V$ with $x \in U$ and $y \in V$.
- **(T₃)** $X$ is called **$T₃$** or **regular** if $X$ is $T₁$ and for any point $x$ and closed set $F$ not containing $x$ there are disjoint open sets $U$ and $V$ with $x \in U$ and $F \subseteq V$.
- **(T₄)** $X$ is called **$T₄$** or **normal** if $X$ is $T₃$ and for any two disjoint closed sets $F$ and $G$ there are disjoint open sets $U$ and $V$ with $F \subseteq U$ and $G \subseteq V$.

See also [Wikipedia](https://en.wikipedia.org/wiki/Separation_axioms) for more.

**Definition 16** (Completely Regular). A Hausdorff space $X$ is said to be **completely regular**, or **$T₃½$**, if for each point $x \in X$ and closed set $C \subseteq X$ with $x \notin C$, there is a continuous function $f: X \to [0, 1]$ such that $f(x) = 0$ and $f \equiv 1$ on $C$.

**Definition 17** (Convergence). Let $X$ be a topological space, and let $(xₙ)_{n \in \mathbb{N}}$ be a sequence in $X$. A point $a \in X$ is called a **limit of the sequence** if for every neighborhood $U$ of $a$ there is an $N \in \mathbb{N}$ such that $xₙ \in U$ for all $n \geq N$.

**Definition 18** (Coverings). A **covering** of a topological space $X$ is a collection of sets whose union is $X$. It is an **open covering** if the sets are open. A **subcover** is a subset of this collection which still covers $X$.

**Definition 19** (Refinements). If $U$ and $V$ are coverings of a space, then $U$ is said to be a **refinement** of $V$ if each element of $U$ is a subset of some element of $V$.

**Definition 20** (Locally finite). A collection $U$ of subsets of a topological space $X$ is said to be **locally finite** if each point $x \in X$ has a neighborhood $N$ which meets nontrivially with only a finite number of the members of $U$.

**Definition 21** (Compactness). Let $X$ be a topological space.

1. $X$ is said to be **compact** if every open covering of $X$ has a finite subcover.
2. $X$ is said to be **locally compact** if every point has a compact neighborhood.
3. $X$ is called **paracompact** if it is a Hausdorff space such that every open covering has an open, locally finite refinement.
4. $X$ is called **σ-compact** if it is the union of countably many compact subspaces.

**Definition 22** (Proper map). A continuous function $f: X \to Y$ between topological spaces is said to be **proper** if $f^{-1}(C)$ is compact for each compact subset $C$ of $Y$.

**Definition 23** (Locally closed). A subspace $A$ of a topological space is said to be **locally closed** if each point $a \in A$ has an open neighborhood $U_a$ such that $U_a \cap A$ is closed in $U_a$. 

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Definition 24 (Partition of unity). Let \( \{ U_\alpha : \alpha \in A \} \) be an open covering of the space \( X \). A partition of unity subordinate to this covering is a collection of maps (continuous functions),
\[
\{ f_\beta : X \to [0,1] : \beta \in B \},
\]
such that:

1. There is a locally finite open refinement \( \{ V_\beta : \beta \in B \} \) such that \( \text{supp}(f_\beta) \subset V_\beta \) for all \( \beta \in B \); and
2. \( \sum_\beta f_\beta(x) = 1 \) for each \( x \in X \).

Definition 25 (Quotient space). Let \( X \) be a topological space, \( Y \) a set, and \( f : X \to Y \) an onto function. Then we define a topology on \( Y \) called the quotient topology induced by \( f \), or the quotient topology, by specifying a set \( V \subset Y \) to be open if and only if \( f^{-1}(V) \) is open in \( X \). Note that this is the largest topology on \( Y \) which makes \( f \) continuous.

Definition 26 (Identification map). A continuous function from \( X \) to \( Y \) is called an identification map if it is onto and \( Y \) has the induced quotient topology.

Definition 27 (Quotient space). Let \( X \) be a topological space.

1. Let \( \sim \) be an equivalence relation on \( X \). Let \( Y = X/\sim \) be the set of equivalence classes and \( \pi : X \to Y \) the canonical map taking \( x \in X \) to its equivalence class \([x] \in X/\sim \). Then \( Y \), with the topology induced by \( \pi \), is called a quotient space of \( X \).
2. If \( A \subset X \), then \( X/A \) denotes the quotient space obtained via the equivalence relation whose equivalence classes are \([A]\) and the single point sets \([x] = \{x\}, x \in X \setminus A\).
3. Let \( Y \) be a topological space and let \( A \subset X \) be closed. Let \( f : X \to Y \) be a continuous function. We denote by \( Y \cup_f X \), the quotient space of the disjoint union \( X \cup Y \) by the equivalence relation \( \sim \) where for \( u, v \in X \cup Y \), \( u \sim v \) if one of the following is true: (a) \( u = v \), (b) \( u \in A \), and \( f(u) = f(v) \), (c) \( u \in A \) and \( v = f(u) \in Y \). The equivalence relation is often abbreviated by saying it is generated by \( a \sim f(a) \), for \( a \in A \).
   
   Note that if \( Y \) is a one-point space, then \( Y \cup_f X = X/A \).

This kind of quotient space is sometimes called an attachment since the space \( Y \) is attached to \( X \) to form the quotient. The next two definitions are common special cases of attachments.

4. If \( f : X \to Y \) is a continuous function between topological spaces, then the mapping cylinder of \( f \) is the space \( M_f = Y \cup_{f_0} X \times I \), where \( f_0 : X \times \{0\} \to Y \) is \( f_0(x,0) = f(x) \).
5. If \( f : X \to Y \) is a continuous function between topological spaces, then the mapping cone of \( f \) is the space \( C_f = M_f/(X \times \{1\}) \).

Definition 28 (Saturation). If \( A \subset X \) and if \( \sim \) is an equivalence relation on \( X \), then the saturation of \( A \) is \( \{ x \in X : x \sim a \text{ for some } a \in A \} \).

Definition 29 (Retraction). If \( A \) is a subspace of a topological space \( X \), then a map (continuous function) \( f : X \to A \) such that \( f(a) = a \) for all \( a \in A \), is called a retraction, and \( A \) is said to be a retract of \( X \).

Definition 30 (Homotopy).

1. If \( X \) and \( Y \) are topological spaces, then a homotopy of continuous functions from \( X \) to \( Y \) is a continuous function \( F : X \times I \to Y \), where \( I = [0,1] \).
2. Two continuous functions \( f_0, f_1 : X \to Y \) are said to be homotopic if there exists a homotopy \( F : X \times I \to Y \) such that \( F(x,0) = f_0(x) \) and \( F(x,1) = f_1(x) \) for all \( x \in X \).

Remark. The relation “\( f \) is homotopic to \( g \)” is an equivalence relation on the set of all continuous functions from \( X \) to \( Y \) and is denoted by \( f \sim g \).

3. A continuous function \( f : X \to Y \) is said to be a homotopy equivalence with homotopy inverse \( g \) if there is a continuous function \( g : Y \to X \) such that \( g \circ f \sim 1_X \) and \( f \circ g \sim 1_Y \). This relationship is denoted by \( X \simeq Y \). One also says, in this case, that \( X \) and \( Y \) have the same homotopy type.
(4) If $A \subset X$, then a homotopy $F: X \times I \to Y$ is said to be relative to $A$ (or rel $A$) if $F(a,t)$ is independent of $t$ for $a \in A$. A homotopy that is rel $X$ is said to be a constant homotopy.

(5) If $F: X \times I \to Y$ and $G: X \times I \to Y$ are two homotopies such that $F(x,1) = G(x,0)$ for all $x$, then define a homotopy $F \ast G: X \times I \to Y$, called the concatenation of $F$ and $G$, by

$$(F \ast G)(x,t) = \begin{cases} F(x,2t) & \text{if } t \leq \frac{1}{2}, \\ G(x,2t - 1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Remark. Homotopies can be combined at any point, not only $t = \frac{1}{2}$, and with any speed. This fact is called the Reparameterization Lemma.

(6) If $F: X \times I \to Y$ is a homotopy, then we define $F^{-1}: X \times I \to Y$ by $F^{-1}(x,t) = F(x,1 - t)$.

**Definition 31** (Contractible). A topological space is said to be contractible if it is homotopy equivalent to the one-point topological space.

**Definition 32** (Deformation). A subspace $A$ of a topological space $X$ is called a strong deformation retract of $X$ if there is a homotopy $F: X \times I \to X$ (called a deformation) such that:

- $F(x,0) = x$,
- $F(x,1) \in A$,
- $F(a,t) = a$, for $a \in A$ and all $t \in I$.

If the last equation only holds for $t = 1$, then $A$ is simply called a deformation retract of $X$.

**Definition 33** (Topological group). A topological group is a Hausdorff topological space $G$ together with a group structure on $G$ such that:

1. group multiplication $(g,h) \mapsto gh$ of $G \times G \to G$ is continuous; and
2. group inversion $g \mapsto g^{-1}$ of $G \to G$ is continuous.

**Definition 34** (Topological subgroup). A topological subgroup $H$ of a topological group $G$ is a subspace which is also a subgroup in the algebraic sense.

**Definition 35** (Homomorphism). If $G$ and $H$ are topological groups, then a homomorphism $f: G \to H$ is a group homomorphism which is also continuous.

**Definition 36** (Translations). If $G$ is a topological group and $g \in G$, then left translation by $g$ is the continuous function $L_g: G \to G$ given by $L_g(h) = gh$. Similarly, right translation by $g$ is the continuous function $R_g: G \to G$ given by $R_g(h) = hg^{-1}$.

**Definition 37** (Symmetric subset). A subset $A$ of a topological group is called symmetric if $A = A^{-1}$, where $A^{-1} = \{a^{-1} : a \in A\}$.

**Definition 38** (Group action). If $G$ is a topological group and $X$ is a topological space, then an action of $G$ on $X$ is a continuous function $G \times X \to X$, $(g,x) \mapsto g(x)$, such that:

1. $(gh)(x) = g(h(x))$; and
2. $e(x) = x$.

**Definition 39.** Let $x \in X$ be a point in a topological space $X$. Let $G$ be a topological group acting on $X$.

1. The set $G(x) = \{g(x) : g \in G\}$ is called the orbit of $x$.
2. The subgroup $G_x = \{g \in G : g(x) = x\}$ is called the isotropy or stability group at $x$.
3. The action is called transitive if there is only one orbit, i.e., $G(x) = X$ for any $x \in X$.
4. The action is called effective if

$$\forall x \in X, g(x) = x \implies g = e,$$

the identity element in $G$.  

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Definition 40 (Metric space). A metric space is a set $X$ together with a function $d: X \times X \to \mathbb{R}$, called a metric, such that for all $x, y, z \in X$, the following hold:

1. (positivity) $d(x, y) \geq 0$, with $d(x, y) = 0 \iff x = y$;
2. (symmetry) $d(x, y) = d(y, x)$; and
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 41 (Cauchy sequence). A Cauchy sequence in a metric space $(X, d)$ is a sequence $(x_1, x_2, x_3, \ldots)$ such that $\forall \varepsilon > 0, \exists N > 0$ such that $n, m > N \implies d(x_n, x_m) < \varepsilon$.

Definition 42 (Complete). A metric space $X$ is complete if every Cauchy sequence in $X$ converges in $X$.

Definition 43 (Totally bounded). A metric space $X$ is totally bounded if, for each $\varepsilon > 0$, $X$ can be covered by a finite number of $\varepsilon$-balls.

Definition 44 (Banach space). A Banach space is a complete normed vector space.

Definition 45 (Fréchet derivative). Let $X$ and $Y$ be Banach spaces. Let $A \subset X$ and let $x \in A$ be an interior point. A function $f: A \subset X \to Y$ is said to be (Fréchet) differentiable at $x$ if there exists a bounded linear operator $L_x: X \to Y$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $h \in X$ and $\|h\| \leq \delta$, then

$$\|f(x+h) - f(x) - L_x(h)\| \leq \varepsilon \|h\|.$$ 

When it exists, $L_x$ is unique, and is called the (Fréchet) derivative of $f$ at $x$, sometimes also denoted $Df(x)$.

In the case $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, $Df(x)$ is given in coordinates by the Jacobian matrix of $f$ at $x$, sometimes denoted $J_f(x)$.

Definition 46 (Lipschitz function). Let $f: (X, d_X) \to (Y, d_Y)$ be a function between metric spaces $X$ and $Y$. This function is called Lipschitz or Lipschitz continuous if there exists a real constant $K \geq 0$ such that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

The smallest such $K$ is called the Lipschitz constant of $f$. If $K = 1$, the function is called a short map, and if $0 < K < 1$, the function is called a contraction.

Note that Lipschitz continuity is a stronger condition than regular continuity.

Definition 47 (Topological manifold). A topological $n$-manifold is a second countable Hausdorff space for which each point has a neighborhood homeomorphic to euclidean $n$-space.

Definition 48 (Smooth manifold). An $n$-dimensional smooth manifold is a second countable Hausdorff space $M^n$ together with a collection of maps called charts such that:

1. each chart $(U, \phi)$ is a homeomorphism $\phi: U \to \phi(U) \subset \mathbb{R}^n$, where $U \subset M^n$ and $\phi(U) \subset \mathbb{R}^n$ are open;
2. each point $x \in M^n$ is in the domain of some chart (i.e. the charts cover $M$);
3. for charts $(U, \phi)$ and $(V, \psi)$, the transition maps (also called “change of coordinates”),

$$\psi \circ \phi^{-1}: \phi(U \cap V) \to \psi(U \cap V),$$

are smooth (as maps from $\mathbb{R}^n$ to $\mathbb{R}^n$);
4. the collection of charts is maximal with the above properties.

A set of charts satisfying properties (1)-(3) is often called an atlas.

Also, note that by a theorem this definition implies that a smooth manifold is paracompact and its one-point compactification is metrizable.

Definition 49 (Functional structure). Let $X$ be a topological space. A functional structure on $X$ is a function $F_X$ defined on the collection of open sets $U$ in $X$, such that:
Definition 50. A morphism of functionally structured spaces $(X, F_X) \to (Y, F_Y)$ is a continuous function $\phi : X \to Y$ such that composition $f \mapsto f \circ \phi$ carries $F_Y(U)$ into $F_X(\phi^{-1}(U))$. An isomorphism is a morphism $\phi$ such that $\phi^{-1}$ exists as a morphism.

Definition 51 (Smooth manifold). An $n$-dimensional smooth manifold is a second countable functionally structured Hausdorff space $(M^n, F)$ which is locally isomorphic to $(\mathbb{R}^n, C^\infty)$. That is, each point in $M$ has a neighborhood $U$ such that $(U, F_U) \cong (V, C^\infty_V)$ for some open $V \subset \mathbb{R}^n$.

Definition 52 (Smooth map). A map $\Phi : M \to N$ between smooth manifolds $M$ and $N$ is called smooth if, for any charts $(U, \phi)$ on $M$ and $(V, \psi)$ on $N$, the function $\psi \circ \Phi \circ \phi^{-1}$ is smooth where it is defined.

Definition 53 (Covering map). If $X$ and $Y$ are Hausdorff, path connected, and locally path connected topological spaces, then a continuous function $p : X \to Y$ is called a covering map if each point $y \in Y$ has a path connected neighborhood $U$ such that $p^{-1}(U)$ is a nonempty disjoint union of sets $U_\alpha$ (which are the path components of $p^{-1}(U)$) on which $p|_{U_\alpha}$ is a homeomorphism. Such sets $U$ are called elementary or evenly covered.

Definition 54 (Properly discontinuous). An action of a group $G$ on a space $X$ is called properly discontinuous if each point $x \in X$ has a neighborhood $U$ such that $g(U) \cap U \neq \emptyset \implies g = e$, the identity element in $G$.

Definition 55 (Directional derivative). Let $M$ be a smooth manifold and $\gamma : \mathbb{R} \to M$ a smooth curve with $\gamma(0) = p$. Let $f : U \to \mathbb{R}$ be smooth, where $U$ is an open neighborhood of $p$. The directional derivative of $f$ along $\gamma$ at $p$ is defined as,

$$D_\gamma(f) = \left. \frac{df}{dt} \right|_{t=0} (f \circ \gamma)(t).$$

The operator $D_\gamma$ is called the tangent vector to $\gamma$ at $p$.

Denote by $T_p(M)$ the set of all tangent vectors to $M$ at $p$. This set in fact forms a vector space, called the tangent space to $M$ at $p$.

Definition 56 (Germ). A germ of a smooth real valued function $f$ at $p \in M$ on a smooth manifold $M$ is the equivalence class of $f$ under the equivalence relation $f_1 \sim f_2 \iff f_1(x) = f_2(x)$ for all $x$ in some neighborhood of $p$.

Note that $D_\gamma(f)$ is well-defined on the germ of $f$.

Definition 57 (Differential). If $\phi : M \to N$ is a smooth map of smooth manifolds, then define the differential (or pushforward) of $\phi$ at $p \in M$ as the function,

$$\phi_* : T_p(M) \to T_{\phi(p)}(N), \quad \text{by} \quad \phi_*(D_\gamma) = D_{\phi \circ \gamma}.$$

This definition is well-defined and satisfies the following properties:

If $F : M \to N$ and $G : N \to P$ are smooth maps, and $p \in M$, then

1. $F_* : T_p(M) \to T_{F(p)}(N)$ is linear.
2. $F_*(D_\gamma)(g) = D_{F \circ \gamma}(g) = D_\gamma(g \circ F)$.
3. $(G \circ F)_* = G_* \circ F_* : T_p(M) \to T_{G(F(p))}(P)$.
4. $(\text{Id}_M)_* = \text{Id}_{T_p(M)} : T_p(M) \to T_p(M)$.
5. If $F$ is a diffeomorphism, then $F_*$ is an isomorphism.
Definition 58 (Submanifold). Let \( \phi: M \to N \) be a smooth map between smooth manifolds. Then:

1. if \( \phi_* \) is an injective at all points, then \( \phi \) is called an immersion;
2. if \( \phi_* \) is an surjective at all points, then \( \phi \) is called a submersion;
3. if \( \phi \) is an injective immersion, then \( (M, \phi) \) is called a submanifold; and
4. if \( (M, \phi) \) is a submanifold and \( \phi: M \to \phi(M) \) is a homeomorphism for the relative topology on \( \phi(M) \), then \( \phi \) is called an embedding, and \( \phi(M) \) is called an embedded submanifold of \( N \).

Definition 59 (Special points). If \( \phi: M^m \to N^n \) is a smooth map, then a point \( p \in M \) is called a critical point of \( \phi \) if \( \phi_* \) has rank \( < n \). The image in \( N \) of a critical point is called a critical value. A point of \( N \) which is not a critical value is called a regular value.

Definition 60 (Transverse submanifolds). Suppose that \( N_1 \) and \( N_2 \) are embedded submanifolds of \( M \). We say that \( N_1 \) intersects \( N_2 \) transversely (symbolically \( N_1 \cap N_2 \) if, whenever \( p \in N_1 \cap N_2 \), we have \( T_p(N_1) + T_p(N_2) = T_p(M) \).

The sum is not direct, just the set of sums of vectors, one from each of the two subspaces of \( T_p(M) \).

Definition 61 (Vector field). A vector field on a smooth manifold \( M^n \) is a function \( \xi \) on \( M^n \) such that \( \xi(p) \in T_p(M) \), which is smooth in the following sense: Given local coordinates \( x_1, \ldots, x_n \) near \( p \in M \), we can write

\[
\xi(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},
\]

and smoothness of \( \xi \) means that the \( a_i \) are smooth functions.

Definition 62 (Smooth flow). A (smooth) flow on a smooth manifold \( M^n \) is a smooth map \( \theta: \mathbb{R} \times M \to M \) such that

1. \( \theta(0, x) = x \) for all \( x \in M \); and
2. \( \theta(s + t, x) = \theta(s, \theta(t, x)) \) for all \( x \in M \) and \( s, t \in \mathbb{R} \).

Note that this definition defines flow as an action of the additive topological group \( \mathbb{R} \) on the manifold \( M^n \). Moreover, a flow determines a vector field by assigning to each point \( p \in M \), the vector \( \xi(p) \) which is tangent to the curve \( \gamma(t) = \theta(t, p) \) at \( t = 0 \). That is,

\[
\xi(p) = \theta_* \left( \frac{d}{dt} \big|_{t=0} \right) \in T_p(M).
\]

This vector field \( \xi \) is called the tangent field of the flow \( \theta \).

Definition 63 (Tangent bundle). Let \( M^n \) be a smooth \( n \)-manifold. Define the tangent bundle as the set,

\[
T(M) = \bigcup_{p \in M} T_p(M) = \{(p, v) \mid v \in T_p(M)\},
\]

endowed with the smooth structure obtained by taking charts \((\pi^{-1}(U), (\varphi \circ \pi) \times \varphi_*p)\) on \( T(M) \) for each chart \((U, \varphi) \) on \( M \), where \( \pi: T(M) \to M \) is the natural projection onto the points of \( M \). Then for \( p \in M \) and \( v \in T_p(M) \), we have \((\varphi(\pi(v)), \varphi_*(v)) = (\varphi(p), \varphi_*(v))\). This makes \( T(M) \) into a smooth \( 2n \)-manifold.

A vector field \( \xi \) on \( M \) is then a smooth section of this bundle, i.e., a smooth map \( \xi: M \to T(M) \) such that \( \pi \circ \xi = 1_M \).

A manifold \( M^n \) is called parallelizable if there is a diffeomorphism \( \theta: T(M) \to M \times \mathbb{R}^n \) such that every \( T_p(M) \) is isomorphic (as vector spaces) to \( \{p\} \times \mathbb{R}^n \).

Definition 64 (Normal bundle). Let \( M^m \) be a smooth \( m \)-manifold embedded in \( \mathbb{R}^n \). At any \( p \in M \), the embedding provides an identification that allows us to view \( T_pM \) as a subspace of \( T_p\mathbb{R}^n \), which inherits a Euclidean dot product courtesy of its canonical identification with \( \mathbb{R}^n \). We define the normal space to \( M \) at \( p \) to be the subspace \( N_pM \subseteq T_p\mathbb{R}^n \) consisting of all vectors that are orthogonal to \( T_pM \) with respect to the Euclidean dot product. That is,

\[
N_pM = \{ v \in T_p\mathbb{R}^n \mid v \perp T_pM \}.
\]
Then we define the normal bundle of $M$ as the subset $NM \subset T\mathbb{R}^n$ given by,

$$NM = \bigsqcup_{p \in M} N_pM = \{(p, v) \in T\mathbb{R}^n : p \in M, v \in N_pM\}.$$ 

There is a natural projection $\pi_{NM}: NM \to M$ given by the restriction to $NM$ of $\pi: T\mathbb{R}^n \to \mathbb{R}^n$, i.e., $\pi(p, v) = p$. Each fiber $N_pM$ is a vector space of dimension $n - m$.

**Definition 65** (Tubular neighborhood). A tubular neighborhood of $M$ is a neighborhood $U$ of $M$ in $\mathbb{R}^n$ that is the diffeomorphic image under the map $(x, v) \mapsto x + v$ of an open subset $V \subset NM$ of the form,

$$V = \{(x, v) \in NM : |v| < \delta(x)\},$$

for some positive continuous function $\delta: M \to \mathbb{R}$.

It is a theorem that in fact every embedded submanifold of $\mathbb{R}^n$ has a tubular neighborhood.
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