Type II Almost-Homogeneous Manifolds of Cohomogeneity One

Daniel Guan

August 15, 2006

Abstract: This paper is one in a series generalizing our results in [GC, Gu4, 5, 8] on the existence of extremal metrics to the general almost-homogeneous manifolds of cohomogeneity one. In this paper, we deal with the affine and the type II cases with hypersurface ends. In particular, we study the existence of Kähler-Einstein metrics on these manifolds and obtain new Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metrics. As a consequence, we give a solution of the problem posted by Akhiezer on the nonhomogeneity of compact almost-homogeneous manifolds of cohomogeneity one; this clarifies the classification of these manifolds as complex manifolds. We also deal with Fano properties of affine and type II manifolds.

1 Introduction

The theory of simply connected compact Kähler homogeneous manifolds has applications in many branches of mathematics. These complex manifolds possess significant properties: they are projective, Fano, Kähler-Einstein, rational, etc..

One class of more general Kähler manifolds which would be useful is the class of almost compact Kähler manifolds with two orbits. Especially those manifolds of cohomogeneity one.

If we assume that they are simply connected, then they are automatically projective. One of many interesting questions of them is when they are Fano, Kähler-Einstein. Other questions might be: What is the biholomorphic

\footnote{Supported by DMS-0103282}
classification of them? What are the automorphism groups of them? When are they actually homogeneous?

This paper is the one of a series of papers in which we answer above questions and we finished the project of the existence of Calabi extremal metrics in any Kähler class on any compact almost-homogeneous manifolds of cohomogeneity one.

There are three types of these kind of manifolds. We refer the readers to the next section for the details. The type III compact complex almost homogeneous manifolds of real cohomogeneity one were dealt in [Gu2] more than ten years ago. There is no much stability involved there.

We shall deal with the type I case in [Gu9] and the type II case in this paper. This is the first class of manifolds for which the existence is completely understood and it is equivalent to the geodesic stability.

In this paper, we prove that there is a Kähler metric of constant scalar curvature on the affine (or type II) almost-homogeneous manifold of cohomogeneity one if the generalized Futaki invariant is positive. We shall prove the converse in [Gu6]. In [GC] and [Gu4,5,8] we dealt some examples.

We should mention that our concept of generalized Futaki invariant might not be the same as the one in [DT] although it might be the same for our case. The generalized Futaki invariant in our case comes from some kind of combination of the generalized Futaki invariants along the maximal geodesic rays in the moduli space of Kähler metric but does not necessarily come directly from any one of them as we have described and observed in [Gu5,8].

In [Gu8], we only deal with one manifold which is the example (3) in [Ak p.68]. In this separate paper, we deal with the other two cases there, which might cause some difficulties, since the manifolds there are quite unfamiliar. In the same time, we shall treat also the manifolds which are fiber bundles with typical fibers of the first and fifth cases in [Ak p.73] as one situation. Although the fiber of the last case is just $\mathbb{C}P^n \times (\mathbb{C}P^n)^*$, it is still in the case of affine type. Therefore, to finish the affine case we have to deal with that case also.

We should also notice the difference of the last case from the manifolds we treated in [Gu4,5]. For example, the isotropic group $U$ of last case is $GL(n)$ as that of the first manifold in the table 2 of [Ak p.67], while the isotropic groups of the manifolds in [Gu4,5] are not semisimple at all. Another point is that the manifolds in [Ak p.67] are actually all homogeneous, that is not true for the examples in [Gu4,5]. We shall come to some generalizations of those examples from [Gu4,5] in the last section (see Theorem 13).
What we have done in this paper take care of the type II case (see section 9). The only one we did not consider so far is the sixth and seventh cases in [Ak p.67]. But in those situations the groups are $G_2$ and they can not come from the semisimple parts of proper parabolic subgroups of any bigger group. There are only two of them and they are homogeneous.

For the type I cases, one only need to consider manifolds with three classes of fibers, i.e., (1) the second and third cases, (2) the fourth case, (3) the eighth and ninth cases in [Ak p.67]. We should deal with these in another paper [Gu9].

As in [Gu8], we take our original method in [Gu4,5]. From Lie group point of view our method can be regarded as a nilpotent path method, i.e., we consider a path, starting from the singular real orbit, generated by the action of a 1-parameter subgroup generated by a nilpotent element. One could also consider the path as a path generated by a semisimple element $H$, where $H$ is the root which generates the $\mathfrak{sl}(2)$ Lie algebra $\mathfrak{a}$.

In this paper, we first give a preliminary on compact almost homogeneous manifolds of cohomogeneity one in the second section, and look back to what we did in [GC], [Gu4, 5] from a Lie group point of view in the third section. Then apply the same argument in the third section of [Gu8] to the affine case. We found that the same method works for the complex structure of both the affine and the type II cases. We deal with three cases we mentioned above.

In the fourth section, we found that the same argument works for the Kähler structure. This is a section in which we deal with many different possibilities of the pairs of groups $(S, G)$. This also shows that the affine and type II classes are very big and are not extraordinary at all (see also the proof of the Lemma 6 and the appendix for a huge amount of this kind of manifolds). A new ingredient is the appearing of the 3-strings. It is quite different from the situation in [Gu8]. Fortunately, the determinants of 3-strings are linear functions of the energy norm function $U$.

The fifth section is one of our major input in this research. To calculate the Ricci curvature we apply a modified Koszul’s trick which was motivated by [Ks p.567–570] as we did in [Gu8]. This is a difficult part and was missing in [Si]. It turns out that both our earlier works in holomorphic symplectic manifolds [Gu3] and homogeneous spaces [Gu7], [DG1,2] help us go through this research. The formula we used from [DG1 4.11] is due to Professor Dorfmeister.

We calculate the scalar curvature in the sixth section and setting up the
equations in the seventh section. The pattern of these equations make it possible to reduce a fourth order ODE to a second order ODE.

We finally prove our Theorem in the eighth section.

We then treat the type II case in the ninth section and the Kähler Einstein case in the tenth section. The pattern of the examples seems quite bizarre in the tenth section if the asymptotic Mumford weakly stable is the same as geodesically stable or weakly K-stable.

In the last section, we obtain some new results on these manifolds. First, we solve a problem on the nonhomogeneous property of compact almost-homogeneous manifolds of cohomogeneity one and with a hypersurface end. This is a question raised by Akhiezer. I later found that he also obtained a solution but with a different proof (in Russian only). In our proof we actually prove that if $M$ is not homogeneous, then the group is actually the identity component of the automorphism group and the manifolds are different from each other. This gives a complete classification of compact almost homogeneous manifolds of cohomogeneity one and with a hypersurface end. They are either homogeneous or nonhomogeneous completions of $\mathbb{C}^*$ bundles, or nonhomogeneous almost-homogeneous manifolds of cohomogeneity one with semisimple group action and a hypersurface end. The first and the second classes in this classification are well understood before. Our new result clarify the third class. Second, we explain how to check the Fano property and nef property of the anticannical line bundle with our calculation. It turns out that not every one of them is Fano nor even nef. However, they are very close to nef. We also generalize our results in [Gu4,5].

In this paper, as in [Gu8] we also have three natural variables: $t$ the nipotent time, $\theta$ the phase angle, $\tau$ the micro time. They help us understand the equation very much. The choice $\theta = \frac{t}{2\pi} + \frac{s}{2\pi}$ make the equation much simpler. We avoided another natural variable $s$ the semisimple time which was in [Gu5], but it will eventually appear in [Gu6]. As in [Gu8], the energy norm function $U$ and the Ricci mixed energy norm function $U_\rho$ in the sections 4 and 6 are seemly God given, which are the reasons that we can solve this problem.

We put some tedious calculation of $F_4$ and $E_8$ in the Appendix. It also shows the complication and numerous properties of the classification. The calculation of $a_{\rho,s}$ will be crucial if we really try to determine that the given manifolds admit extremal metrics or not. We also checked the Fano properties of these manifolds. By taking the advantage of the solution for higher codimensional ends in [Gu10], we also checked the possibility of
blowing down of our manifolds. In all our calculations we also need to take care carefully of the change of the invariant inner products when we restrict our calculation to a typical subgroup $S$ in $G$.

2 Preliminaries

Here, we summarize some known results about compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we only consider manifolds with a Kähler structure. For earlier results on this subject, we refer the readers to [Ak] and [HS].

We call a compact complex manifold an almost-homogeneous manifold if its complex automorphism group has an open orbit. We say that a manifold is of cohomogeneity one if the maximal compact subgroup has a (real) hypersurface orbit. In [GC] and [Gu5], we reduced compact complex almost homogeneous manifolds of cohomogeneity one into three types of manifolds.

We denote the manifold by $M$ and let $G$ be a complex subgroup of its automorphism group which has an open orbit on $M$.

Let us assume first that $M$ is simply connected. Let the open orbit be $G/H$, $K$ be the maximal connected compact subgroup of $G$, $L$ be the generic isotropic subgroup of $K$, i.e., $K/L$ be a generic $K$ orbit. We have that (see [GC Theorem 1]):

**Proposition 1.** If $G$ is not semisimple, then $M$ is a completion of a $C^*$ bundle over a projective rational homogeneous space.

For the structure of the projective rational homogeneous spaces, we refer the reader for the detailed discussion in [Gu7]. Here, we just recall some results which will be used in this paper.

A projective rational homogeneous space is a quotient of a complex semisimple Lie group $G$ over a parabolic subgroup $P$. Let $\Delta$ be a root system of $G$. A subgroup $P$ is a parabolic subgroup if its Lie algebra contains all the roots and the positive root vectors.

If a compact almost-homogeneous Kähler manifold is a completion of a $C^*$ bundle over a product of a torus and a projective rational homogeneous space, we call it a manifold of type III. We have dealt with this kind of manifolds in our dissertation [Gu1,2]. There always exists an extremal metric in any Kähler class. Recently, we generalized this existence result to a family of metrics, which connects the extremal metric [Gu2] and the generalized quasi-einstein metric [Gu10], called the extremal-soliton metrics in [Gu11].
The existence of the extremal-soliton is the same as the geodesic stability with respect a generalized Mabuchi functional.

In general, if $M$ is a compact almost homogeneous Kähler manifold and $O$ is the open orbit, then $D = M - O$ is a proper closed submanifold. Moreover, $D$ has at most two components. We call each component of $D$ an end. If $D$ has two components (or one component), we say $M$ is an almost homogeneous manifold with two ends (or one end). We have (see [HS Theorem 3.2]):

**Proposition 2.** If $M$ is a compact almost-homogeneous Kähler manifolds with two ends, then $M$ is a manifold of type III.

Therefore, we only need to deal with the case with one end. Again, in the case of $M$ being simply connected, we only need to take care of the case in which $G$ is semisimple. If $G$ is semisimple and $M$ has two $G$ orbits, one open and one closed, and moreover if the closed orbit is a complex hypersurface, there are two possibilities. Let $k, l$ be the Lie algebras of $K, L$. Then the centralizer of $l$ in $k$ is a direct sum of $l$ and a Lie subalgebra $\mathcal{A}$ which is either one dimensional or the 3-dimensional Lie algebra $su(2)$. If $\mathcal{A}$ is one dimensional, we call $M$ a manifold of type I. If $\mathcal{A}$ is $su(2)$, we call $M$ a manifold of type II.

In general, if the closed orbit has a higher codimension, we can always blow up the closed orbit to obtain a manifold $\tilde{M}$ with a hypersurface end. We call the manifold $M$ a manifold of type I (or II) if $\tilde{M}$ is of type I (or II).

There is a special case of the type II manifolds. If the open orbit is a $C^k$ bundle over a projective rational homogeneous manifold, we call $M$ an affine type manifold (not to be confused with the closed complex submanifolds of $C^m$).

Then, we have (see [Gu5 section 12]):

**Proposition 3.** Any compact almost-homogeneous Kähler manifold $M$ of cohomogeneity one is an $Aut_0(M)$ equivariant fibration over a product of a rational projective homogeneous manifold $Q$ and a complex torus $T$ with a fiber $F$. Therefore, $M$ can be regarded as a fiber bundle over $T$ with a simply connected fiber $M_1$. One of following holds:

(i) $M$ is a manifold of type III.
(ii) $M_1$ is of type II but not affine.
(iii) $M_1$ is affine.
(iv) $M_1$ is of type I.

We say that $M$ is a manifold of type I (or type II, affine) if $M_1$ is a manifold of type I (or type II, affine).
We actually can also obtain a structure of the $M_1$ bundle over $T$ from [HS]. We only need to understand the bundle structure for the open orbit. By [HS Corollary 4.4], we have that the bundle structure is a product unless when we apply Proposition 3 to $M$ we have that $F = Q^k$. In the latter case, there is an unbranched double covering $\tilde{M}$ of $M$ such that the bundle structure is a product. We have that:

**Proposition 4.** The $M_1$ bundle over $T$ is a product except in the case with which the open orbit is a $F_0$ bundle over $Q \times T$ such that $F_0$ is in either the second, or the sixth, or the eighth case in [Ak p.67]. In the latter cases the $M_1$ bundle has an unbranched double covering which is a product of $M_1$ and $T$.

In [Gu8] and this paper, we dealt with the simply connected affine and the type II cases with a hypersurface end. In [Gu9], we shall deal with the simply connected type I cases with a hypersurface end. Then, we shall deal with the simply connected cases with a higher codimensional end in [Gu9 section 11], and the general case in [Gu9 section 12].

### 3 The complex structures of the affine almost homogeneous manifolds

In this section we will deal with the complex structure of the affine almost homogeneous manifolds. Let us recall some basic notations of the Lie algebras.

To make the things simpler we look at two special cases [Ak p.68] first. We let $G$ and $U$ be the corresponding complex Lie groups and $O = G/U$ be the open orbit, then:

1. $G = B_2$, roots of $U$ are $\pm(e_1 - e_2), e_1 + e_2$.
2. $G = C_3$, roots of $U$ are $\pm(e_2 \pm e_3), \pm2e_2, \pm2e_3, 2e_1$.

In the case (1), the roots of the affine space are $e_1$ and $e_2$. The long root $\alpha_1 = e_1 - e_2$ and the short root $\alpha_2 = e_2$ consist a fundamental root system of this Lie algebra. $B_2$ has other positive roots $\alpha_1 + \alpha_2 = e_1, \alpha_1 + 2\alpha_2 = e_1 + e_2$. $B_2$ has Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & -a_1i & 0 & 0 & 0 \\ a_1i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2i & 0 \\ 0 & 0 & a_2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{C} \right\}.$$
The vector $e_1$ corresponds to $(a_1, a_2) = (1, 0)$ and $e_2$ corresponds to $(a_1, a_2) = (0, 1)$. The open orbit is generated by the combine action of $B_2$ on

$$A = [0, 0, 0, 0, 1]^T$$

which represents a 4 dimensional complex subspace $\pi = \ker A^T$ of $\mathbb{C}^5$ and

$$B = \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T$$

which represents a 2 dimensional complex subspace $l \subset \pi$ generated by the column vectors of $B$. We let

$$E_{\pm e_1} = \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times 2} & B^T \\ 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 1} \\ -B & 0_{1 \times 2} & 0 \end{bmatrix}$$

with $B = \frac{1}{\sqrt{2}}[\pm 1, i]$, 

$$E_{e_1 \pm e_2} = \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times 2} & A & 0_{2 \times 1} \\ -A^T & 0_{2 \times 2} & 0_{2 \times 1} \\ 0_{1 \times 2} & 0_{1 \times 2} & 0 \end{bmatrix}$$

for

$$A = \frac{1}{2} \begin{bmatrix} 1 & \pm i \\ i & \mp 1 \end{bmatrix},$$

$$E_{-\alpha} = E_{\alpha}^T.$$ 

$$F_{\alpha} = E_{\alpha} - E_{-\alpha}, G_{\alpha} = i(E_{\alpha} + E_{-\alpha}),$$

then

$$[F_{\alpha}, G_{\alpha}] = 2H_{\alpha}$$

and

$$[H_{\alpha}, F_{\alpha}] = i(H_{\alpha}, H_{\alpha})_0 E_{\alpha},$$

where $(\ , \ )_0$ is the standard inner product such that $(e_i, e_i)_0 = 1$. We also have

$$[E_{\pm e_i}, E_{\pm (e_j - e_i)}] = \mp E_{\pm e_j},$$

$$[E_{e_i}, E_{\pm e_j}] = \mp E_{e_i \pm e_j},$$

$$[E_{- e_i}, E_{\pm e_j}] = \mp E_{- e_i \pm e_j}.$$
and

\[ [E_{\pm e_i}, E_{\pm(e_i+e_j)}] = \pm E_{\mp e_j}. \]

The tangent space is generated by \( E_{\alpha} \)'s with

\[ \alpha = \pm (\alpha_1 + \alpha_2), -(\alpha_1 + 2\alpha_2), \pm(\alpha_2). \]

The affine space \( \mathbb{C}^2 \) is generated by

\[ \alpha_1 + \alpha_2, \alpha_2. \]

As in the case of [Gu4], we consider the nilponent orbit generated by \( E_{\alpha_1+\alpha_2} \).

Now,

\[
\begin{align*}
p_t &= \exp(tE_{\alpha_1+\alpha_2}) \left( [0,0,0,1]^T \times \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T \right) \\
&= \left( \frac{t}{\sqrt{2}}, \frac{it}{\sqrt{2}}, 0, 0, 1 \right)^T \times \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T, \\
p_\infty &= [1, i, 0, 0, 0]^T \times \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T.
\end{align*}
\]

Let

\[ F = E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2}, G = i(E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2}), H = H_{\alpha_1+\alpha_2}, \]

then as before

\[ JF = -G + \frac{2H}{t}. \]

Let \( T \) be the tangent vector of the curve \( p_t \), then

\[ JH = -tT. \]

Similarly, \( JF_{\alpha_1} = -G_{\alpha_1}, JF_{\alpha_1+\alpha_2} = -G_{\alpha_1+\alpha_2}, JF_{\alpha_2} = -G_{\alpha_2} + \frac{2G_{\alpha_1}}{t}. \)

In particular, at \( P_\infty \) we have \( JF_\alpha = -G_\alpha. \)

Similarly, we consider the case (1) in [Ak p.68] with \( G = B_n \), then the roots of \( U \) are

\[ \pm(e_i - e_j), e_i + e_j. \]
with $1 \leq i < j \leq n$. The open orbit is a combination of the $B_n$ action on

$$[0, \cdots, 0, 1]^T_{1 \times (2n+1)} \times \begin{bmatrix} 1 & i & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & i & \cdots & 0 & 0 & 0 \\ & & & & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & i & 0 \end{bmatrix}^T.$$ 

For the complex Lie group $B_n$, we have $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_n$. Therefore, $e_i = \sum_{j=i}^{n} \alpha_j$, $e_i - e_k = \sum_{j=i}^{k-1} \alpha_j$, $e_i + e_k = \sum_{j=i}^{k-1} \alpha_j + 2 \sum_{j=k}^{n} \alpha_j$. In particular $e_1 = \sum_{i=1}^{n} \alpha_j$. Therefore, similarly we have that

$$JF_{e_1} = -G_{e_1} + \frac{2H}{t},$$
$$JF_{e_i+e_k} = -G_{e_i+e_k},$$
$$JF_{e_i} = -G_{e_i} + \frac{2G_{e_1-e_i}}{t}$$

and

$$JF_{e_1-e_i} = -G_{e_1-e_i}.$$

We also have

$$F_{e_i-e_k} = G_{e_i-e_k} = 0$$

for $i > 1$. In particular, at $p_\infty$, $JF_\alpha = -G_\alpha$ for $\alpha \neq e_i - e_k$ with $1 < i < k$.

In the case of (2), the roots of the affine space are $e_1 \pm e_2$ and $e_1 \pm e_3$. The two short roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$ and the long root $\alpha_3 = 2e_3$ consist a fundamental root system of the Lie algebra. $C_3$ has other positive roots $\alpha_1 + \alpha_2 = e_1 - e_3, \alpha_1 + \alpha_2 + \alpha_3 = e_1 + e_3, \alpha_1 + 2\alpha_2 + \alpha_3 = e_1 + e_2, \alpha_2 + \alpha_3 = e_2 + e_3, 2\alpha_2 + \alpha_3 = 2e_2$ and $2\alpha_1 + 2\alpha_2 + \alpha_3 = 2e_1$.

The complex Lie group $C_3$ has Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_3 \end{bmatrix} \right\}_{a_1,a_2,a_3 \in \mathbb{C}}.$$ 

The vector $e_1$ corresponds to $(a_1, a_2, a_3) = (1, 0, 0), e_2$ to $(0, 1, 0), e_3$ to $(0, 0, 1)$. The open orbit is generated by the combine $C_3$ action on $A =
\([1,0,0,0,0,0]^T\) which represents a complex 1 dimensional subspace \(l\) of \(\mathbb{C}^6\) generated by \(A\) and

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}^T
\]

which represents the complex 2 dimensional column space \(\pi\) of \(B\). We have \(l \subseteq \pi\).

We let

\[
E_{\alpha} = \begin{bmatrix}
A_{\alpha} & 0 \\
0 & -A_{\alpha}^T
\end{bmatrix}
\]

with

\[
A_{e_1-e_2} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
A_{-e_1+e_2} = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We let

\[
E_{\alpha} = \begin{bmatrix}
0 & B_{\beta} \\
0 & 0
\end{bmatrix}
\]

with

\[
B_{2e_1} = \begin{bmatrix}
\sqrt{2} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
B_{e_1+e_2} = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We also let \(E_{-\beta} = E_{\beta}^T\).

We have that \([F_{\alpha},G_{\alpha}] = 2H_{\alpha}\) and \([H_{\alpha},E_{\alpha}] = i(H_{\alpha},H_{\alpha})_0E_{\alpha}\), where \((,)_0\) is the standard inner product such that \((e_1-e_2,e_1-e_2)_0 = 2\),

\[
[E_{\pm 2e_1},E_{\mp (e_i+e_j)}] = \pm \sqrt{2}E_{\pm (e_i-e_j)},
\]

\[
[E_{\pm (e_i-e_j)},E_{\pm (e_i+e_j)}] = \pm \sqrt{2}E_{\pm (e_i+e_j)},
\]

\[
[E_{\pm (e_i-e_j)},E_{\pm (e_i+e_j)}] = \pm \sqrt{2}E_{\pm 2e_1},
\]

\[
[E_{e_i-e_j},E_{e_i+e_j}] = E_{e_i-e_k},
\]

11
\[ E_{\pm(e_i-e_j)}, E_{\pm(e_j+e_k)} = \pm E_{\pm(e_i+e_k)}. \]

The tangent space is generated by \( E_\alpha \)'s with \( \alpha = \pm(e_1 \pm e_j), -2e_1. \)

The affine space \( \mathbb{C}^4 \) is generated by \( \pm(e_1 \pm e_j). \)

As above we consider the nilponent orbit generated by \( E_{\alpha_1}. \)

Now,
\[
p_t = \exp(tE_{\alpha_1}) \left( [1,0,0,0,0,0]^T \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \right) = [1,0,0,0,0,0]^T \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},
\]
\[
p_\infty = [1,0,0,0,0,0]^T \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.
\]

Let \( F = E_{\alpha_1} - E_{-\alpha_1}, G = i(E_{\alpha_1} + E_{-\alpha_1}), \) then as above we have \( JF = -G + \frac{H}{t}. \)

Let \( T \) be the tangent vector of the curve \( p_t, \) then \( JF = -2tT. \)

Similarly,
\[
JF_{\alpha_2} = G_{\alpha_2}, F_{\alpha_3} = G_{\alpha_3} = 0,
\]
\[
JF_{\alpha_2+\alpha_3} = G_{\alpha_2+\alpha_3}, JF_{2\alpha_2+\alpha_3} = G_{2\alpha_2+\alpha_3},
\]
\[
JF_{2\alpha_1+2\alpha_2+\alpha_3} = -G_{2\alpha_1+2\alpha_2+\alpha_3},
\]
\[
JF_{\alpha_1+2\alpha_2+\alpha_3} = -G_{\alpha_1+2\alpha_2+\alpha_3} - \frac{\sqrt{2}G_{2\alpha_2+\alpha_3}}{t},
\]
\[
JF_{\alpha_1+\alpha_2+\alpha_3} = -G_{\alpha_1+\alpha_2+\alpha_3} - \frac{2G_{\alpha_2+\alpha_3}}{t}
\]
and
\[ JF_{\alpha_1 + \alpha_2} = -G_{\alpha_1 + \alpha_2} - \frac{2G_{\alpha_2}}{t}. \]
At \( p_\infty \), \( F_{2e_3} = G_{2e_3} = 0 \), \( JF_{2e_2} = G_{2e_2}, JF_{e_2 \pm e_3} = G_{e_2 \pm e_3} \). For other roots \( \alpha \) we have that \( JF_\alpha = -G_\alpha \).

Similarly, we consider the case (2) in [Ak p.68] with \( G = C_n \), then the roots of \( U \) are
\[ \pm (e_i \pm e_j) \]
with \( 1 < i < j \leq n \) and \( \pm 2e_1, 2e_1 \).

The open orbit is a combination of the \( C_n \) action on
\[ [1, 0, \ldots, 0; 0, 0, \ldots, 0]^T \times \left[ \begin{array}{cccccccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \end{array} \right]^T. \]

For \( C_n \) we have \( \alpha_i = e_i - e_{i+1} \) for \( 1 \leq i < n \) and \( \alpha_n = 2e_n \). Therefore,
\[ e_i - e_k = \sum_{j=i}^{k-1} \alpha_j, \]
\[ e_i + e_k = \sum_{j=i}^{k-1} \alpha_j + 2 \sum_{j=k}^{n-1} \alpha_j + \alpha_n, \]
\[ 2e_i = 2 \sum_{j=i}^{n-1} \alpha_j + \alpha_n. \]

Therefore, similarly we have that
\[ JF_{\alpha_1} = -G_{\alpha_1} + \frac{H}{t}, \]
\[ JF_{2e_1} = -G_{2e_1}, \]
\[ JF_{2e_2} = G_{2e_2}. \]

We also have that \( F_\alpha = G_\alpha = 0 \)
for
\[ \alpha = e_i - e_k, 2e_i, e_i + e_k \]
with \( i > 2. \)
And

\[ JF_{e_1+e_2} = -G_{e_1+e_2} - \frac{\sqrt{2}G_{e_2}}{t}, \]

\[ JF_{e_2+e_k} = G_{e_2+e_k}, \]

\[ JF_{e_1+e_k} = -G_{e_1+e_k} - \frac{2G_{e_2+e_k}}{t} \]

for \( k > 2 \).

Moreover

\[ JF_{e_2-e_k} = G_{e_2-e_k}, \]

\[ JF_{e_1-e_k} = -G_{e_1-e_k} - \frac{2G_{e_2-e_k}}{t}. \]

At \( p_\infty \) we have that \( F_\alpha = G_\alpha = 0 \) if \( \alpha = 2e_i, e_i \pm e_k, \; i > 2 \); \( JF_{2e_2} = G_{2e_2}, JF_{e_2 \pm e_k} = G_{e_2 \pm e_k} \). For other roots \( \alpha \) we have that \( JF_\alpha = -G_\alpha \).

In general, as in [Ak] \( G \) is semisimple, \( U_G \) is the 1-subgroup. There is a parabolic subgroup \( P = SS_1R \) with \( S, S_1 \) semisimple and \( R \) solvable such that \( U_G = US_1R \) where \( U \) is a 1-subgroup of \( S \). The manifold is a fibration over \( G/P \) with the completion of \( P/U_G = S/U \) as the affine almost homogeneous fiber. In this case, the root system of \( S \) is a subsystem of the root system of \( G \). In the Lie algebra of \( G \), we also have \( F_\alpha, G_\alpha \) for those roots of \( G \) which are not in \( S \). The tangent space of \( G/U_G \) along \( p_t \) is decomposed into irreducible \( \mathcal{A} \) representations. \( F_\alpha, G_\alpha \) are in the complement representation of \( S \). But \( JF_\alpha = -G_\alpha \) (mod \( S \)) as it is in the tangent space of \( G/P \). Therefore, we have \( JF_\alpha = -G_\alpha \) for any \( \alpha \) which is not in the root system of \( S \). This discussion is corresponding to the discussion in the last paragraph of the second section of [Gu8].

If \( S \) is \( B_2 \), the bigger complex Lie group \( G \) can be \( B_n, C_n, F_4 \). If \( S \) is \( B_3 \), \( G \) can be \( B_n, F_4 \). If \( S \) is \( C_3 \), \( G \) can be \( C_n, F_4 \). If \( S \) is \( B_n \) with \( n > 3 \), \( G \) can only be \( B_{m+n} \). If \( S \) is \( C_n \) with \( n > 3 \), then \( G \) can be \( C_{m+n} \).

We now treat the case of the \( SL(n+1) \) action of [Ak p.73], which includes both the first case and the fifth case there.

Let us look at the case for \( n = 1 \) first. The action is

\[ A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times [1, 0]A^{-1} \]
where \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}, [1.0]
\]
represent the points in \( \mathbb{CP}^1 \). We have

\[
E_{\alpha_1} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

And

\[
\exp(tE_{\alpha_1}) \begin{bmatrix}
1 \\
0
\end{bmatrix} \times [1.0] \exp(-tE_{\alpha_1}) = \begin{bmatrix}
1 & -t \\
0 & 0
\end{bmatrix} = p_t,
\]

\[
p_{\infty} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix} \times [0,1].
\]

As before we have \( JF = -G + \frac{H}{t} \) and \( JH = -2tT \).

In general, if \( S = SL(n + 1) = A_n \), \( S \) has simple roots \( \alpha_i = e_i - e_{i+1} \).
\( C^n \) is generated by \( e_i - e_j, 1 < j \leq n + 1 \). The action is

\[
A[1,0,\ldots,0]^T \times [1,0,\ldots,0]A^{-1}.
\]

As above \( F = F_{\alpha_1}, G = G_{\alpha_1} \) and \( H = H_{\alpha_1} \).

\[
p_t = \exp(tE_{\alpha_1})[1,0,\ldots,0]^T \times [1,0,\ldots,0] \exp(-tE_{\alpha_1})
= [1,0,0,\ldots,0]^T[1,-t,0,\ldots,0].
\]

\[
JF = -G + \frac{H}{t} \text{ and } JH = -2tT.
\]

\[
JF_{e_2-e_j} = G_{e_2-e_j}, JF_{e_1-e_j} = -G_{e_1-e_j} - \frac{2G_{e_2-e_j}}{t} \quad 2 < j,
\]

\[
F_{e_k-e_j} = G_{e_k-e_j} = 0 \quad 2 < k < j.
\]

At \( p_{\infty} \), \( JF_{e_1-e_k} = -G_{e_1-e_k}, JF_{e_2-e_k} = G_{e_2-e_k}, F_{e_1-e_k} = G_{e_1-e_k} = 0 \).

However, in the case of \( S = A_n \), the bigger complex Lie group \( G \) can be any complex semisimple Lie group.

## 4 More

This is only an announcement. If you are interested in the whole paper, please send me an email.

### References


Author’s Addresses:
Zhuang-Dan Guan
Department of Mathematics
University of California at Riverside
Riverside, CA 92521 U. S. A.