THINGS I SHOULD OR SHOULD NOT HAVE SAID IN MY THESIS

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Abstract. We give a list of things I wish I had said in my thesis ([8]) and things I wish I hadn't said. References are to the single-spaced version of my thesis available on my home page.

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1. Concerning Chapter 1

"...a bounded domain in the plane in Chapter 3..." **p. 1**

I really should have pointed out that I am using Navier boundary conditions not the more usual no-slip boundary conditions in Chapter 3, in case someone were to read only my very brief overview and get a false impression. After all, if I had done this for no-slip boundary conditions, even in two dimensions, I would have solved one of the biggest open problems in mathematical fluid mechanics.

Convention on $\nabla v u$ in Section 1.3 **p. 5**

I should have just stuck with the one convention of using only $u \cdot \nabla v$, but the Navier boundary conditions are often written $D(v)\mathbf{n} \cdot \boldsymbol{\tau}$, so I started using $\nabla v u$ for boundary integrals. Having both notations was confusing.

2. Concerning Chapter 2

Definition of M **p. 11**

The $\sum$ in the definition of M should be sup (this was a LaTeX typo, using \sum in place of \sup), so the definition should read

$$M = \sup_{\nu > 0} \| \left| w_\nu \right|^2 \|_{L^\infty(\mathbb{R} \times \mathbb{R}^2)} = \sup_{\nu > 0} \| \left| v_\nu \right|^2 - 2 v_\nu v_\nu' + \left| v_\nu' \right|^2 \|_{L^\infty(\mathbb{R} \times \mathbb{R}^2)}.$$

Long string of inequalities in the middle of the page **p. 11**

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There are two obvious typos here. This string of inequalities should read:

$$\begin{aligned}\int_{\mathbb{R}^2} \left|\nabla v_\nu'(s,x)\right| \left|w_\nu(s,x)\right|^2 \, dx &= \int_{\mathbb{R}^2} AB = \int_{\mathbb{R}^2} A^\epsilon A^{1-\epsilon} B \le M^\epsilon \int_{\mathbb{R}^2} A^{1-\epsilon} B \\ &\le M^\epsilon \left\|A^{1-\epsilon}\right\|_{L^{1/(1-\epsilon)}} \left\|B\right\|_{L^{1/\epsilon}} = M^\epsilon \left\|A\right\|_{L^1}^{1-\epsilon} \left\|B\right\|_{L^{1/\epsilon}} \\ &= M^\epsilon L_\nu(s)^{1-\epsilon} \left\|\nabla v'\right\|_{L^{1/\epsilon}} \le C_0 M^\epsilon L_\nu(s)^{1-\epsilon} \frac{1}{\epsilon} \|\omega^0\|_{L^{1/\epsilon}} \\ &\le C_0 M^\epsilon L_\nu(s)^{1-\epsilon} \frac{1}{\epsilon} \theta(1/\epsilon).\end{aligned}$$

p. 14 Last paragraph in Section 2.4

The units do work out, but my comment on them here is totally off. The first observation is that the units of $\theta(1/\epsilon)$ is the same as the units of

$$\|\omega^0\|_{L^{1/\epsilon}} = \left(\int_{\mathbb{R}^2} \left|\omega^0\right|^{1/\epsilon}\right)^\epsilon$$

which is

$$(d^2)^\epsilon \operatorname{units}(\omega^0) = d^{2\epsilon} \operatorname{units}(\nabla \omega^0) = d^{2\epsilon}(v/d) = d^{2\epsilon-1} v,$$

where d is distance and v is velocity (and later t will be time).

From their defining relations, we also have

$$\begin{gathered}\operatorname{units}(R) = \operatorname{units}(\|\omega^0\|_{L^2}) = d^2(v/d)^2 = v^2, \\ \operatorname{units}(R\nu t) = v^2 d^2, \ \operatorname{units}(L_\nu) = v^2 d^2, \ \operatorname{units}(M) = v^2.\end{gathered}$$

These units result in units of $v^3 d$ for β in the inequalities on p. 11.

The units of s in $\beta(s)$ must be $v^2 d^2$ from Equation (2.4.1) p. 12 and we can see that both sides of this equation have units of time. Letting x have the units of $v^2 d^2$ in Equation (2.1.2) p. 7, the defining equation for β, returns units of $v^3 d$ for β, in agreement with the units derived from the inequalities on p. 11.

p. 18 Second paragraph in Section 2.6

There are three issues in this paragraph. First, in the second sentence, I should just have said (E) not (E) or (NS). Second, as I point out in my comment on the last paragraph in Chapter 2 (p. 25-26) below, one needs to assume that the initial vorticity is in L^a for some $a < 2$, though not necessarily for $a = 1$. Third, I should have included a proof of the conservation of the L^p-norms of the vorticity for solutions to (NS), which can be done using elementary means, though the argument is a little more delicate than one might expect (unless there is an easier way). I will probably write this up and include it in a later version of these comments.

p. 25-26 Last paragraph in Chapter 2

My comments in the last paragraph of Chapter 2 are not quite right. First, for (NS) there is another way (in fact, essentially the classical way) to derive the energy bound in Equation (2B.2) that avoids the L^2-norm of the vorticity (or gradient). However, the only way I know to prove the conservation of the L^p-norms of the vorticity for (NS) in $\mathbb{R}^2$ for $p > 2$ requires "bootstrapping" up from the L^2-bound on vorticity. Thus, I might be right in the end about (NS), though I am not sure.

As regards my comments about (E) in this paragraph, though, I am almost certainly wrong. One can have the initial vorticity in L^p for all p in $[a, \infty)$ with Yudovich vorticity and have conservation of the L^p-norms of vorticity only if $a < 2$, as far as I know. At least, the straightforward modification of Majda's proof of the existence of a solution to (E) (see p. 311-319 of [12] for instance) requires $a < 2$. Thus, it would appear that the requirements for (E) are stronger than those for (NS), not the other way around.

3. Concerning Chapter 3

Corollary 3.2.3 **p. 30**

The second inequality in this corollary is correct, but the first is not. Thus, each use of the first inequality needs to be replaced by a use of the second, as we explain in separate notes below. The corollary and its proof should read:

Corollary 3.2.3. *For all v in V,*

$$\|v\|_{L^2(\Gamma)} \le C(\Omega) \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} \le C(\Omega) \|v\|_V . \tag{3.1}$$

Proof. This follows from the following claim, since $\operatorname{div} u = 0$ and $u \cdot \mathbf{n} = 0$ on Γ give $\int_\Omega u = 0$. □

Claim. *Assume that Ω is a bounded domain in $\mathbb{R}^2$ with C^2-boundary Γ. Let f be a scalar-valued function in $H^1(\Omega)$ with $\int_\Omega f = 0$. Then*

$$\|f\|_{L^2(\Gamma)}^2 \le C \|f\|_{L^2(\Omega)} \|\nabla f\|_{L^2(\Omega)} .$$

Proof. We prove this for f in $C^\infty(\Omega)$ with $\int_\Omega f = 0$, the result then following from the density of the space of all such functions in the space of all functions f in $H^1(\Omega)$ with $\int_\Omega f = 0$.

Let Σ be a tubular neighborhood of Γ of uniform width δ, where δ is half of the maximum possible width. Place coordinates (s, r) on Σ where s is arc length along Γ and r is the distance of a point in Σ from Γ, with negative distances being inside of Ω. Then s is piecewise linear, being discontinuous across each component of Γ, and r ranges from $-\delta$ to δ, with points $(s, 0)$ lying on Γ. Also, because Σ is only half the maximum possible width, $|J|$ is bounded from below, where

$$J = \det \frac{\partial(x,y)}{\partial(s,r)}$$

is the Jacobian of the transformation from (x,y) coordinates to (s,r) coordinates.

Let φ in $C^\infty(\Omega)$ equal 1 on Γ and equal zero on $\Omega \setminus \Sigma$. Then if a is the arc length of Γ,

$$\begin{aligned}
\|f\|_{L^2(\Gamma)}^2 &= \int_0^a \int_{-\delta}^0 \frac{\partial}{\partial r}(\varphi f(s,r))^2 \, dr \, ds \le \int_0^a \int_{-\delta}^0 \left| \frac{\partial}{\partial r}(\varphi f(s,r))^2 \right| \, dr \, ds \\
&\le \int_0^a \int_{-\delta}^0 \left|\nabla(\varphi f(s,r))^2\right| \, dr \, ds \\
&= \frac{1}{\inf |J|} \int_0^a \int_{-\delta}^0 \left|\nabla(\varphi f(s,r))^2\right| \inf |J| \, dr \, ds \\
&\le \frac{1}{\inf |J|} \int_0^a \int_{-\delta}^0 \left|\nabla(\varphi f(s,r))^2\right| |J| \, dr \, ds \\
&= C \int_{\Sigma \cap \Omega} \left|\nabla(\varphi f(x,y))^2\right| \, dx \, dy = C \left\|\nabla(\varphi f)^2\right\|_{L^1(\Sigma\cap\Omega)} \\
&= C \left\|\nabla(\varphi f)^2\right\|_{L^1(\Omega)} = C \left\|\varphi f \nabla(\varphi f)\right\|_{L^1(\Omega)} \\
&\le C \left\|\varphi f\right\|_{L^2(\Omega)} \left\|\nabla(\varphi f)\right\|_{L^2(\Omega)} \\
&\le C \left\|f\right\|_{L^2(\Omega)} \left\|\varphi \nabla f + f \nabla \varphi\right\|_{L^2(\Omega)} \\
&\le C \left\|f\right\|_{L^2(\Omega)} \left(\left\|\varphi \nabla f\right\|_{L^2(\Omega)} + \left\|f \nabla \varphi\right\|_{L^2(\Omega)} \right) \\
&\le C \left\|f\right\|_{L^2(\Omega)} \left(\left\|\nabla f\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(\Omega)} \right) \\
&\le C \left\|f\right\|_{L^2(\Omega)} \left\|\nabla f\right\|_{L^2(\Omega)}.
\end{aligned}$$

In the last step we used Poincaré's inequality.

□

Remark 3.1. This is not a two-dimensional result; the claim holds for a bounded domain in $\mathbb{R}^d$ for $d \ge 1$, the proof being essentially unchanged.

p. 30 First equation on page 30

The first "=" should be a "≤" in this series of inequalities, not that this matters one bit.

p. 32 $H_c \subseteq C^\infty(\overline{\Omega})$ in discussion of basis for H_c

Actually, this is only $C^\infty(\Omega)$. What I should have pointed out is that the basis vectors are in V, which is all we need here (but see next comment). Also, instead of taking Temam's approach using the multi-valued functions

q_i, I would have been better off defining harmonic functions ψ_i that are equal to zero on all boundary components except for one then defining $h_i = \nabla^\perp \psi$, because it is easier to apply standard elliptic regularity results. This is the approach I take in [9].

Statement of Lemma 3.3.1 **p. 33**

This lemma should have been stated more generally, assuming that v is in H_0 with $\omega(v)$ in $L^p(\Omega)$. In fact, this is what we actually prove. The only place we need this added generality is on p. 50 (see comment below).

Statement of Corollary 3.3.2 **p. 33**

This corollary should have been stated more generally, assuming that v is in H with $\omega(v)$ in $L^p(\Omega)$. In fact, this is what we actually prove, though we do not need this little bit of extra generality.

Statement and proof of Corollary 3.3.2 **p. 33**

First, I need to add more regularity to the boundary here: I need to assume that the boundary is $C^{2,\epsilon}$ for some $\epsilon > 0$. Then, using the functions ψ_i described in the previous comment, I can apply elliptic regularity theory to conclude that that each ψ_i is in $C^{2,\epsilon}(\overline{\Omega})$ (applying, for instance, Theorem 6.14 p. 101 of [7])). Then each basis element $h_i = \nabla^\perp \psi_i$ for H_c is in $C^{1,\epsilon}(\overline{\Omega})$ and so ∇h_i is in $L^\infty(\Omega)$, which is all I need in the argument that follows, which is unchanged except for the next comment. This is how I do it in [9].

(The assumption of extra regularity is not a problem, since I only apply this theorem under that assumption the the boundary is $C^{2,1/2+\epsilon}$ anyway.)

Proof of Corollary 3.3.2 **p. 33**

In the proof of Corollary 3.3.2 I write, "But, $H_0 = H_c^\perp$, so $\|v\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega)} + \|v_c\|_{L^2(\Omega)}$ and thus $\|v_c\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$." This should have said, "But, $H_0 = H_c^\perp$, so $\|v\|^2_{L^2(\Omega)} = \|v_0\|^2_{L^2(\Omega)} + \|v_c\|^2_{L^2(\Omega)}$ and thus $\|v_c\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$."

The bound on $|\langle Au, v\rangle_{V,V'}|$. **p. 36**

Corollary 3.2.3 should have been applied as follows:

$$\begin{aligned} |\langle Au, v\rangle_{V,V'}| &\le \left|\int_\Omega \nabla u \cdot \nabla v\right| + \left|\int_\Gamma (\kappa - \alpha) u \cdot v\right| \\ &\le \|u\|_V \|v\|_V + C\|u \cdot v\|_{L^1(\Gamma)} \\ &\le \|u\|_V \|v\|_V + C\|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\ &\le \|u\|_V \|v\|_V + C\|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)}^{1/2} \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} \\ &\le C\|u\|_V \|v\|_V. \end{aligned}$$

p. 41 “This is to insure that u lying in $C^{1/2}([0,T];(H^1(\Omega))^2)$ implies that $(\kappa - \alpha/2)u \cdot \boldsymbol{\tau}$ lies in $C^{1/2}([0,T];H^1(\Omega))$.”

This should read: “This is to insure that u lying in $C^{1/2}([0,T];(H^1(\Omega))^2)$ implies that $(\kappa - \alpha/2)u \cdot \boldsymbol{\tau}$ lies in $C^{1/2}([0,T];H^{1/2}(\Gamma))$.”

p. 41 “...but with the extra regularity assumed on Γ (and the lower regularity assumed on α).”

The regularity of Γ is extra over that of Theorem 3.6.1, the regularity of α is lower than that of [5] and [10] (references [7] and [27] in my thesis). In [5], an unspecified regularity of Γ is assumed (they just say “sufficiently smooth”) and in [10] Γ is assumed to be smooth, which I take to mean C^∞.

p. 46 The bound in Equation (3.8.6).

This bound should read:

$$\begin{aligned} \left|\int_\Gamma (\kappa - \alpha) u_\nu \cdot w\right| &\le \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|u_\nu \cdot w\|_{L^1(\Gamma)} \\ &\le \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|u_\nu\|_{L^2(\Gamma)} \|w\|_{L^2(\Gamma)} \\ &\le C\|\kappa - \alpha\|_{L^\infty(\Gamma)} \|\nabla u_\nu\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T}. \end{aligned}$$

where we used (3.1) above.

p. 48 Theorem 3.9.2

It is clear from the proof of Theorem 3.9.2, that we also have convergence of $u_{\nu,\gamma}$ to $\widetilde{u}_\nu$ in $L^2([0,T];\dot{H}^1(\Omega))$.

p. 49 $\int_\Omega |\nabla w|^2$

There is a missing factor of ν on this term in the two equations in which it appears; this has no effect on the proof.

$\boxed{\text{Bound on } -\int_\Gamma (\nabla \widetilde{u}_\nu \mathbf{n}) \cdot u_{\nu,\gamma}}$ **p. 49**

It was totally unnecessary to bring in the vorticity using Equation (3.4.1); the bound is actually more direct just sticking with the gradient.

$\boxed{\text{Use of Lemma 3.3.1 in paragraph following the proof of Lemma 3A.1}}$ **p. 50**

The use of Lemma 3.3.1 to prove that v_0 is in $H^{1,p}(\Omega)$ and that K_Ω is continuous require, when $p < 2$, the slight strengthening of the statement of Lemma 3.3.1 that we described above.

4. Concerning Chapter 4

I do not have much to say on Chapter 4 yet, having had no reason to reexamine it since I defended. I will say, however, that the motivation for this chapter was twofold. First, it was my own curiosity about the best possible convergence rates one can obtain for circularly symmetric initial vorticities, which I could not find clearly stated in the literature (though they must be there somewhere, because for circularly symmetric initial vorticity solutions to the Navier-Stokes equations are just solutions to the heat equation). Theorem 4.2.1 was the first observation I made along these lines. I decided not to include my original proof, which is longer and references the proof of the energy argument from Chapter 2, but I regret this now. What I like about the proof is that it highlights the role of νt: the only fact it uses about the heat kernel is that it depends only upon νt and does not bring in any of the technicalities of what the heat kernel actually is. I include this proof in a comment below.

The second motivation, having read [11], was to establish the rate of convergence for a superposition of confined eddies. This turned out to be a lot easier than I had at first thought, because all the hard stuff was done in [11]. But I still think the results themselves are interesting, even if the difficulties involved are insufficient to try publishing the results.

As for convergence in Besov spaces, I came across Abidi's and Danchin's paper ([1]) in the middle of writing this chapter, and I couldn't resist specializing their paper to circularly symmetric initial vorticities.

$\boxed{\text{Section 4.2 title: } \textit{Bounds on convergence rate for circular symmetry}}$ **p. 54**

In earlier drafts I used "circular symmetry" where I should have used "radial symmetry," and this is a holdover that I overlooked while editing. It should read, "Bounds on convergence rate for radial symmetry."

$\boxed{\text{Bottom of page: } v^N = v^H}$ **p. 54**

I am really using that $v^N = v^H$, which I should perhaps have been more careful to demonstrate. One way to show this is as follows.

The solution $v^H = p_{\nu t} * v_0$ to (H) satisfies

$$\partial_t v^H = \nu \Delta v^H, \ \operatorname{div} v^H = p_{\nu t} * \operatorname{div} v_0 = 0.$$

Observe that

$$\omega(v^H \cdot \nabla v^H) = v^H \cdot \nabla \omega^H = 0$$

by the radial symmetry of $\omega^H = \omega(p_{\nu t} * v_0) = p_{\nu t} * \omega(v_0) = p_{\nu t} * \omega_0$, because v^H is perpendicular to $\nabla \omega^H$ by the Biot-Savart law. (See, for instance, p. 11 of [4].)

But for all $t > 0$,

$$\|v^H\|_{L^\infty} \le \|p_{\nu t}\|_{L^1} \|v_0\|_{L^\infty} < \infty,$$
$$\|\nabla v^H\|_{L^2} \le \|\nabla p_{\nu t}\|_{L^1} \|v_0\|_{L^2} < \infty,$$

so $v^H \cdot \nabla v^H$ is in L^2 for all positive time. By Theorem 2A.5, there is unique divergence-free vector field in L^2 whose vorticity is 0. But the zero vector field is in L^2 with zero vorticity, so the unique vector field is the zero vector field. Hence, $v^H \cdot \nabla v^H = \nabla h$ for some scalar field h by the Hodge decomposition.

Letting $p = -h$, we see that

$$\partial_t v^H + v^H \cdot \nabla v^H + \nabla p = \partial_t v^H + \nabla h - \nabla h = \partial_t v^H = \nu \Delta v^H.$$

Since we have already shown that $\operatorname{div} v^H = 0$, we see that v^H is, in fact, a solution to (NS) as well.

p. 54 $\boxed{\text{A slight twist on the proof of Theorem 4.2.1}}$

The following proof does not use the heat kernel at all. This makes the extension to a bounded domain as in the situation investigated in Remark (4.2) particularly transparent, since the boundary integrals vanish. [Bona and Wu, however, also show convergence of the vorticity, which is more of an issue.]

We have

$$\begin{aligned}(\partial_t(v^N - v^E)) \cdot (v^N - v^E) &= \partial_t v^N \cdot (v^N - v_0) \\ &= (\nu \Delta v^N - \nabla p^N) \cdot (v^N - v_0).\end{aligned}$$

Integrating over $\mathbb{R}^2$ and applying the divergence theorem gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^N - v^E\|_{L^2}^2 + \nu \|\nabla v^N\|_{L^2}^2 &= \nu \int_{\mathbb{R}^2} \nabla v^N \cdot \nabla v_0 \\ &\le \nu \|\nabla v^N\|_{L^2} \|\nabla v_0\|_{L^2} \le \frac{\nu}{2}\|\nabla v^N\|_{L^2}^2 + \frac{\nu}{2}\|\nabla v_0\|_{L^2}^2.\end{aligned}$$

It follows that

$$\|(v^N - v^E)(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla v^N\|_{L^2}^2 \le \nu \int_0^t \|\nabla v_0\|_{L^2}^2 = \nu t \|\nabla v_0\|_{L^2}^2,$$

from which Theorem 4.2.1 follows.

$\boxed{\text{My original proof of Theorem 4.2.1}}$ **p. 54**

As in the proof on p. 55 of my thesis, $v^E(t) = v_0$ is the steady state solution to (E), and $v^N = v^E$.

We assume first that ω_0 is in $L^2 \cap L^a$ for some a in $[2, \infty]$. We apply Theorem 2.3.1 p. 11 of my thesis, justifying its application below, giving

$$\int_{\mathbb{R}^2} \left|v^N(t,x) - v^E(t,x)\right|^2 \, dx = \int_{\mathbb{R}^2} \left|v^N(t,x) - v_0(x)\right|^2 \, dx$$
$$\leq R\nu t + 2\int_0^t \int_{\mathbb{R}^2} |\nabla v_0(x)| \left|v^N(s,x) - v_0(x)\right|^2 \, dx \, ds,$$

where $R = C\|\omega_0\|_{L^2}^2$ and C is an absolute constant. Using Cauchy-Schwarz's inequality,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left|v^N(t,x) - v^E(t,x)\right|^2 \, dx \\ &\quad \leq R\nu t + 2\int_0^t \|\nabla v_0\|_{L^2} \|v^N(s,\cdot) - v_0(\cdot)\|_{L^4}^2 \, ds, \\ &\quad \leq R\nu t + 2t\|\nabla v_0\|_{L^2} \|v^N - v_0\|_{L^\infty([0,t], L^4(\mathbb{R}^2))}^2. \end{aligned} \tag{4.1}$$

But, $\|\nabla v_0\|_{L^2} \leq C\|\omega_0\|_{L^2}$, and

$$\|v^N - v_0\|_{L^\infty([0,t],L^4(\mathbb{R}^2))} \leq \|v^N\|_{L^\infty([0,t],L^4(\mathbb{R}^2))} + \|v_0\|_{L^4} \leq 2\|v_0\|_{L^4},$$

since

$$\|v^N(s)\|_{L^4} = \|p_{\nu s} * v_0\|_{L^4} \leq \|p_{\nu s}\|_{L^1} \|v_0\|_{L^4} = \|v_0\|_{L^4}.$$

Also, by Lemma 2B.1 p. 23 of my thesis, since ω_0 is in L^a for $a > 2$ and v_0 is in E_m, we conclude that v_0 is in $L^p \cap L^\infty$ for all $p > 2$, so v_0 is in L^4. Thus, (4.1) gives

$$\|v^N(t) - v_0\|_{L^2}^2 \leq R\nu t + 8Ct\|\omega_0\|_{L^2} \|v_0\|_{L^4}^2 = R\nu t + Ct.$$

But $v^N(t)$ depends only upon νt (since $p_{\nu t}$ only depends upon νt), and the steady-state solution v_0 to the Euler equations does not depend upon t or upon ν, so the best bound for $\|v^N(t) - v_0\|_{L^2}$ is achieved in the limit as t goes to zero while νt remains fixed. This gives Equation (4.2.1) p. 54 of my thesis.

To remove the assumption that ω_0 is in L^a, we can use a sequence of vorticities in $L^2 \cap L^a$ approaching ω_0 in L^2, and use the fact that the bound in Equation (4.2.1) p. 54 of my thesis depends only upon the L^2-norm of ω_0.

It remains to justify the use of Theorem 2.3.1 of my thesis. If we examine its proof on p. 14-18 of my thesis, we can see that we can apply it if we can establish that:

(i) v^N and v_0 are in $L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$;

(ii) $v^N - v_0$ is in $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$;
(iii) ∇v^N and ∇v_0 are in $L^\infty_{loc}(\mathbb{R}; L^2)$.

[In an earlier version of my thesis I had a remark to this effect following the statement of the thesis.]

To establish these properties, let σ be a stationary solution to (E) with total vorticity m, as in Appendix 2A p. 19-23 of my thesis. Then $v_0 - \sigma$ is in L^2, as is $v^N - \sigma$, since

$$\begin{aligned}\left\|v^N - \sigma\right\|_{L^2} &= \|p_{\nu t} * v_0 - \sigma\|_{L^2} \le \|p_{\nu t} * (v_0 - \sigma)\|_{L^2} + \|p_{\nu t} * \sigma - \sigma\|_{L^2} \\ &\le \|p_{\nu t}\|_{L^1} \|v_0 - \sigma\|_{L^2} + \|p_{\nu t} * \sigma - \sigma\|_{L^2},\end{aligned}$$

which is finite by Lemma 2A.6 p. 22 of my thesis. (This shows that v^N remains in E_m.) Also, ω_0 is in L^a, as is ω^N, since

$$\|\omega^N\|_{L^a} = \|p_{\nu t} * \omega^0\| \le \|p_{\nu t}\|_{L^1} \|\omega_0\|_{L^a} = \|\omega_0\|_{L^a}.$$

Property (i) then follows from Lemma 2B.1 p. 23 of my thesis applied to $v_0 - \sigma$ and to $v^N - \sigma$, since σ is in L^∞. Similarly, ω_0 and ω^N are both in L^2, and property (iii) follows from the Calderon-Zygmund inequality. Finally, property (ii) holds because E_m is an affine space (see Appendix 2A p. 19-23 of my thesis).

□

Remark 4.1. This alternate proof of Theorem 4.2.1 does not depend upon the specific form of, or even existence of, the heat kernel, only that the solution to the homogeneous heat equation depends only upon the product of the viscosity and the time. (Except, that is, for the value of the constant in Equation (4.2.1) p. 54 of my thesis.)

Remark 4.2. In [2], Bona and Wu consider vanishing viscosity in the setting of radially symmetric initial vorticity in the unit disk D, with the very special additional condition that $\int_0^1 r\omega^0(r)\,dr = 0$. This condition is equivalent to assuming that $v^0 = v^E = 0$ on ∂D. This means that the integration by parts in Equation (4.2.2) p. 55 and in the slight variation of the proof of Theorem 4.2.1 given above—as well as the appeal to the proof of Theorem 2.3.1 in my original proof—are valid on D as well. Since the solution to the heat equation on D depends only upon νt, my original proof gives the vanishing viscosity result of [2] very easily. Or, if we use the fact that the heat kernel on D is bounded in L^2, the proof in my thesis gives this result more easily still. But the slight variation of the proof I gave above has the simplest translation to D.

Remark 4.3. In the comment following Proposition 2.3 p. 49 of [12], Majda and Bertozzi state a result with the same convergence as in Equation (4.2.1) p. 54 of my thesis under the much stronger assumption that the initial vorticity is Lipschitz. An implication of Theorem 1 of [6] (see also Theorem 1.5 of [3]) is that for a single compactly supported eddy with ω_0 in $L^2 \cap L^\infty$, $v^N \to v^E$ in $L^\infty([0,t]; L^2)$ at a rate bounded by $C(t)\sqrt{\nu t}$, but with

$C(t)$ increasing with t. In [1], Abidi and Danchin show that for a radially symmetric vortex patch in $\mathbb{R}^2$—meaning that $\omega_0 = \mathbf{1}_\Omega$, where Ω is a disk—$\left\|v^N(t) - v^E\right\|_{L^2} \le C(\nu t)^{3/4}$ for all $\nu t \le 1$.

$\boxed{\text{“circularly symmetric”}}$ **p. 65**

The phrase “circularly symmetric” in the first sentence in Section 4.5 should read “radially symmetric.” See comment on title of Section 4.2 p. 54.

5. Concerning Chapter 5

I gave a talk on the material in Chapter 5 on 14 September 2005 at Brown. This was the first such talk, and so I found a lot of things I should or should not have said!

$\boxed{\text{Expression for } \Gamma_t}$ **p. 70**

There is a missing constant in the expression

$$\int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta_{1,\phi}(r)} = t,$$

and a missing square. It should read

$$\int_{s^2/4}^{\Gamma_t(s)^2/4} \frac{dr}{\beta_{1,\phi}(r)} = Ct.$$

The constant C is a unitless absolute constant and ultimately derives from Equation (5.2.12) p. 77. The missing squares came from an error in a change of variables: see comment on p. 80, below.

$\boxed{\text{The bound on } L}$ **p. 74-76**

The bound on L needs to be done not for a fixed value of $\overline{p}$ but for a general value of p in $[p_0, \infty]$, because the bound in Equation (5.2.8) the term $C_3 a$ needs to include a factor of $\|\omega^0\|_{L^p}$. This is not too much of a problem, though. The bound on L on the top of p. 75 becomes

$$L \le 16\|\omega^0\|_{L^p} \left(\frac{\pi}{q-1}\right)^{1/q} \lambda^{2/q-2} a.$$

Using the relations between p and q on p. 75 and 76, we have

$$\begin{aligned}\left(\frac{\pi}{q-1}\right)^{1/q} \lambda^{2/q-2} &= \left(\frac{\pi}{1/(p-1)}\right)^{1-1/p} \lambda^{2/q-1-1} = (\pi(p-1))^{1-1/p}\lambda^{1-2/p-1} \\ &= (\pi(p-1))^{1-1/p}\lambda^{-2/p}.\end{aligned}$$

But,

$$(\pi(p-1))^{1-1/p} \le \pi(p-1) \le \pi p$$

and

$$\lambda^{-2/p} \le a^{-2/p}$$

since $\lambda \ge 1$ (in fact, $\lambda = \sqrt{2}$) and we have assumed that $|x - x'| = 2a \le 1$ (on the top of p. 72) so $\lambda > a$. Thus,

$$L \le 16\pi \|\omega^0\|_{L^p} p a^{1-2/p},$$

which is what we need in place of the term $C_3 a$ to make the second inequality in Equation (5.2.8) valid.

p. 78 Typo in the last equation on the page

I forgot the argument s in this equation. It should read:

$$x_{k+1}(t) = x_0 + \int_0^t v(s, x_k(s))\, ds.$$

p. 80 Last equation on page

In the change of variables, I forgot the square (though I included the division by 4!). This equation should read:

$$\int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = \int_{s^2/4}^{\Gamma_t(s)^2/4} \frac{dr}{\beta(r)} = t.$$

p. 81 The paragraph containing Equation (5.5.1)

This paragraph is stating something I intended to show, but only properly showed for the first two of Yudovich's examples. And even for the second example, this is a charitable view of things, since my constant decreases exponentially in time (which isn't such a big deal). For the higher examples, it may not even be quite true, as time enters in in a more complicated way. So I should have struck this paragraph.

Also, the more important question is what happens to the modulus of continuity of the flow, though it is admittedly driven, in the examples I construct, by the upper bound on the modulus of continuity of the velocity.

p. 83 First equation on the page and the sentence before it

There is a double sign error here that cancel each other out, and the argument is perhaps a little unclear.

First, I never stated the conclusion of the sentence preceding this equation, which is that $\omega(t, x_1, -x_2) = -\omega(t, x_1, x_2)$, which follow from the uniqueness of our solutions to (E).

Second, we are using the fact that $K^2(x_1, -x_2) = -K^2(x_1, x_2)$: I dropped the negative sign. In the third equality, this leads to no change in sign since both factors in the integrand change sign.

Third, the change of variables in the final equality introduces a negative sign. Thus, the equation should read:

$$\begin{aligned} v^2(t, x_1, -x_2) &= (K^2 * \omega)(t, x_1, -x_2) \\ &= \int_{\mathbb{R}^2} K^2(x_1 - y_1, -x_2 - y_2)\omega(t, y_1, y_2)\, dy \\ &= \int_{\mathbb{R}^2} K^2(x_1 - y_1, x_2 + y_2)\omega(t, y_1, -y_2)\, dy \\ &= \int_{\mathbb{R}^2} K^2(x_1 - y_1, x_2 - (-y_2))\omega(t, y_1, -y_2)\, dy \\ &= -v^2(t, x_1, x_2). \end{aligned}$$

Fourth, I am bringing in the issue of time here prematurely, as I should just be dealing with the symmetry of ω^0, then making this observation concerning the symmetry of ω at time t later.

The two equations between Equations (5.5.5) and (5.5.6) **p. 84**

In both of these equations, y_2 should be y_1.

The equation before Equation (5.5.14) **p. 87**

This equation would be better written as

$$\eta_\lambda(x_1) = \max_{k \in \mathbb{Z}^+} \{k : r_k \geq x_1^\lambda\}.$$

Equation (5.5.14) **p. 87**

The "1" in Equation (5.5.14) should be bolded.

"The difficulty in applying Theorem 5.5.1 lies in choosing a function ω_0 that both gives a bound in Equation (5.5.2) that is an admissible function in the sense of β in Definition 2.1.1,..." **p. 87**

I believe I meant to say, "The difficulty in applying Theorem 5.5.1 lies in choosing a function ω_0 that gives a bound in Equation (5.5.2) that is both an admissible function in the sense of β in Definition 2.1.1,..." That is, I put the word "both" in the wrong place.

This is still a somewhat silly statement, though. The sentence as a whole should just have read, "The difficulty in applying Theorem 5.5.1 lies in choosing a function ω_0 that is both an admissible function in the sense of β in Definition 2.1.1, and is such that we can obtain an asymptotic formula for its L^p-norms that obeys Yudovich bounds on the vorticity." The resulting bound in Equation (5.5.2) will then always be there (assuming ω^0 is chosen to have the right symmetries).

p. 92

"To achieve a lower bound on $\omega_0(t)$ and hence a lower bound on $v_0^1(x_1, 0)$, we assume that Equation (5.6.3) holds for all x and t. This will minimize the lower bound on $v_0^1(x_1, 0)$ in Equation (5.5.2), since ω_0 decreases with the distance from the origin."

First, the 0 subscripts here are a mistake.

Second, the phrase "minimizes the lower bound" is ridiculous; it should say "gives a lower bound." But the real problem is that this is not much of an explanation, and misses the point. What I produce by assuming that Equation (5.6.3) holds is a point-by-point lower bound on the magnitude of the vorticity. Call this lower bound $\overline{\omega}(t, x)$, and observe that it obeys all the symmetry conditions in Section 5.5. Now, the actual vorticity $\omega(t, x)$ we should not expect to obey all of the symmetry conditions of Section 5.5, but it will obey the first one—what we called symmetry by quadrant—because of the space-time symmetries of the Euler equations themselves.

Thus, $\omega - \overline{\omega}$ obeys symmetry by quadrant, where we note in particular that $\omega - \overline{\omega} \geq 0$ in the first quadrant because $\overline{\omega}$ is a point-by-point lower bound. If $u = K * (\omega - \overline{\omega})$, it follows from Equation (5.5.3) p. 82 of [8] that $u_1(x_1, t) \geq 0$, since symmetry by quadrant was all that was required to derive this equation. So, for the associated velocities $\overline{v} = K * \overline{\omega}$ and $v = K * \omega$, $v_1(x_1, t) \geq \overline{v}_1(x_1, t)$, which shows that assuming that Equation (5.6.3) holds gives a lower bound on $v_0^1(x_1, t)$ in Equation (5.5.2) (which is what I really meant by $v_0^1(x_1, 0)$, believe it or not).

p. 92

$\rho(t) = \psi(t, x)$

This should be $\rho(t) = |\psi(t, x)|$.

p. 94

Equation in middle of page

The first "$\geq$" is actually an "$=$," since $\psi^2(t, x1_1, 0) = 0$ (because $v^2(t, x_1, 0) = 0$ for all t and x_1. Thus, the equation could be written

$$\frac{|\psi(t, a, 0) - \psi(t, 0, 0)|}{a^\alpha} = \frac{\left|\psi^1(t, a, 0)\right|}{a^\alpha} \geq a^{\exp(-2(1-\lambda')t/\pi)-\alpha}.$$

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