Crash Course in Set Theory

AMP Lecture 1

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The realm of Set Theory and abstract math in general is weird. It might help to forget everything you know about math. Set thoery is about building math from the ground up using only Logic™.

SETS

Formally defining a set can actually be difficult, so I'll defer to my undergrad textbook:

> The central concept of this book, that of a **set**, is, at least on the surface, extremely simple. A set is any collection, group, or conglomerate. So we have the set of all students registered at the City University of New York in February 1998, the set of all even natural numbers, the set of all the points in the plane π exactly 2 inches distant from a given point P, the set of all pink elephants.
>
> Sets are not objects of the real world, like tables or stars; they are created by our mind, not by our hands. A heap of potatoes is not a set of potatoes, the set of all molecules in a drop of water is not the same object as that drop of water. The human mind possesses the ability to abstract, to think of a variety of different objects as being bound together by some common property, and thus form a set of objects having that property.
>
> *Introduction to Set Theory* by Karel Hrbacek and Thomas Jech

So, we'll run with this:

Definition

A **set** is a collection of (mathematical) objects bound by some property. The objects are called **elements** of the set. Defining a set usually happens between curly braces and looks something like:

$$\{\text{element}, \text{element}, \text{element}, \text{element}\} \text{ or } \{\ \text{elements} \ : \ \text{properties}\ \}$$

If a is an element of the set A, we write $a \in A$.

Notice, we usually denote sets with capital letters and elements with lowercase. We also try to be suggestive with our naming. If we have two sets running around A and B, and we want to talk about elements of these sets we'll name these elements $a \in A$ and $b \in B$. It's not always easy or possible to keep this up, but it can make our proofs easier to read.

Example

Here are some sets!

- $X = \{1, 2, 3\}$
- $A = \{a, b, c\}$
- $\emptyset = \{\}$

The last one here is called the **empty set**. It's special because it has no elements! Here are some other popular sets that you should know if you're going to talk to other mathematicians:

- The **Natural Numbers***: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.
- The **Integers**: $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$
- The **Rational Numbers**: $\mathbb{Q} = \left\{\frac{m}{n} \;:\; m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\right\}$
- The **Real Numbers**: $\mathbb{R} = \{\text{Any decimal expansion.}\}$
- The **Complex Numbers**: $\mathbb{C} = \{x + iy \;:\; x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

*There is some contention with whether or not $0 \in \mathbb{N}$.

We like to think of sets as "living" in some ambient set so we need to define what it means for a set to live in another:

Definition

A set A is a **subset** of B, denoted $A \subseteq B$, if any element $a \in A$ is also an element of B. We say two sets A and B are **equal**, denoted $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Example

Some toy examples:

- $\{a, b\} \subseteq \{a, b, c\}$
- $\{1, 2, 3\} \subseteq \mathbb{N}$
- $\emptyset \subseteq A$ for *any* set A

Regarding our "popular" sets:

$$\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Now, let's talk about the two basic operations we have for sets:

Definition

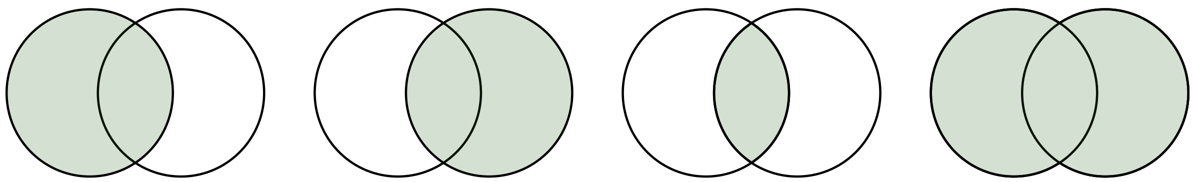
Given two sets $A \subseteq X$ and $B \subseteq X$, define the **intersection** of the sets as

$$A \cap B := \{x \in X \, : \, x \in A \text{ and } x \in B\}$$

and the **union** of the sets as

$$A \cup B := \{x \in X \, : \, x \in A \text{ or } x \in B\}$$

We usually think of these using **Venn diagrams**:



A $\qquad$ B $\qquad$ $A \cap B$ $\qquad$ $A \cup B$

Note that logical “or” is different from the way normal people use the word “or.” The logical statement “this or that” includes the case of “this and that.”

Exercise Prove that $(A \cup B) \cup C = A \cup (B \cup C)$.

> We want to prove that
> $$(A \cup B) \cup C \subseteq A \cup (B \cup C)$$
> and
> $$A \cup (B \cup C) \subseteq (A \cup B) \cup C$$
> Starting with the former, let $x \in (A \cup B) \cup C$. By definition, then $x \in A \cup B$ or $x \in C$. Let’s take it case by case.
>
> Case I: If $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cup C)$. If $x \in B$, then $x \in B \cup C$ and hence $x \in A \cup (B \cup C)$.
>
> Case II: If $x \in C$, then $x \in B \cup C$ and hence $x \in A \cup (B \cup C)$.
>
> This proves the first inclusion. To prove the other inclusion, let $x \in A \cup (B \cup C)$. Again we have cases:
>
> Case I: If $x \in A$, the $x \in A \cup B$ and hence $x \in (A \cup B) \cup C$.
>
> Case II: If $x \in B \cup C$, then $x \in B$ or $x \in C$. If $x \in B$, the $x \in A \cup B$ and hence $x \in (A \cup B) \cup C$. If $x \in C$, then $x \in (A \cup B) \cup C$.

We call this property **associativity**. This property means there’s no ambiguity in writing $A \cup B \cup C$. Moreover, we get the following defintion:

Definition

Let $\mathcal{A}$ denote an arbitrary set of sets, that is a set whose elements are sets. The we can define the **arbitrary union** over $\mathcal{A}$ as

$$\bigcup_{A \in \mathcal{A}} A := \{x : x \in A \text{ for some } A \in \mathcal{A}\}$$

and the **arbitrary intersection** over $\mathcal{A}$ as

$$\bigcap_{A \in \mathcal{A}} A := \{x : x \in A \text{ for all } A \in \mathcal{A}\}$$

FUNCTIONS

Technically, a good set theorist would say there is only one thing—sets. You can actually define a function as a set! For our sake, we'll think of them as a different object.

Definition

A **function** f from a set A to a set B, denoted $f : A \to B$, is an assignment of an element $f(a) \in B$ for each element of $a \in A$. Moreover, this assignment must be **well-defined**, is that if $a_1 = a_2$, then $f(a_1) = f(a_2)$. We call the set A the **domain** of f and B the **codomain** of f.

A nice real-life example of a function is going to a store and asking for the price of something. The assignment of a price to product is well-defined, one product can't have two prices, but it might be the case that two products are assigned the same price.

Mathematically, a place you've seen well-definedness before is with maps $f : \mathbb{R} \to \mathbb{R}$. Such a map is well-defined when its graph on the plane satisfies the vertical line test.

Definition

A function $f : A \to B$ is said to be **injective** if for any distinct elements $a_1 \neq a_2 \in A$, $f(a_1) \neq f(a_2)$. This property is sometimes referred to as **one-to-one**.

A function $f : A \to B$ is said to be **surjective** if for every element $b \in B$, there exists an element $a \in A$ such that $f(a) = b$. This property is sometimes referred to as **onto**.

Example

Let $A \subseteq X$. Then we can define a map $\iota : A \to X$ by $\iota(a) = a$. Such a map is called the **inclusion map**. The inclusion map is injective since, if $a \neq b \in A$, then $f(\iota) = a \neq b = \iota(b)$.

A particular example of this example: $\iota : \{a, b\} \to \{a, b, c\}$ defined by $\iota(a) = a$ and $\iota(b) = b$.

Definition
A function which is both surjective and injective is said to be **bijective**.

EQUIVALENCE RELATIONS & QUOTIENT SETS

Definition
A **relation** is a way of comparing two elements in a set. An **equivalence relation** is a relation $\sim$ that "acts like" equality. Specifically, $\sim$ has the following properties:

- (Reflexive) $a \sim a$
- (Symmetric) If $a \sim b$, then $b \sim a$
- (Transitive) If $a \sim b$ and $b \sim c$, then $a \sim c$.

Definition
Given an equivalence relation $\sim$ on a set A and en element $a \in A$, we can define the **equivalence class** of a by

$$[a] := \{b \in A \ : \ a \sim b\}$$

We can further define the **quotient set** of A by $\sim$ as the set of equivalence relations. That is

$$A/\sim := \{[a] \ : \ a \in A\}$$