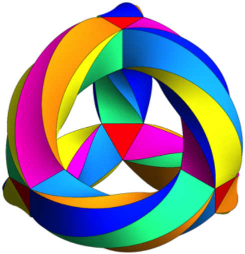
THE ANSWER TO THE ULTIMATE QUESTION OF LIFE, THE UNIVERSE, AND EVERYTHING



John Baez

December 4, 2015
Canadian Mathematical Society Winter Meeting

"You're really not going to like it," observed Deep Thought.

"Tell us!"

"All right," said Deep Thought. "The Answer to the Great Question..."

"Yes...!"

"Of Life, the Universe and Everything..." said Deep Thought.

"Yes...!"

"Is..." said Deep Thought, and paused.

"Yes...!"

"Is..."

"Yes...!!!...?"

"Forty-two," said Deep Thought, with infinite majesty and calm.

— Douglas Adams, *The Hitchhiker's Guide to the Galaxy*

If 42 is the answer, what's the question?

When asked for The Ultimate Question, Deep Thought says it doesn't know. But it offers to help design an even more powerful computer that can figure that out. It will take ten million years.

This computer turns out to be the planet Earth.

Unfortunately, five minutes before the computation is done, the Earth is destroyed to make way for a Hyperspace Bypass.

After Adams' death, his friend Stephen Fry said:

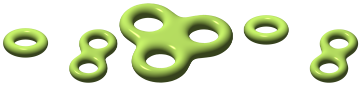
> *Douglas told me in the strictest confidence exactly why 42. The answer is fascinating, extraordinary and, when you think hard about it, completely obvious. Nonetheless amazing for that. Remarkable really. But sadly I cannot share it with anyone and the secret must go with me to the grave. Pity, because it explains so much beyond the books. It really does explain the secret of life, the universe, and everything.*

Adams himself said:

> *The answer to this is very simple. It was a joke. It had to be a number, an ordinary, smallish number, and I chose that one.*

Nonetheless, the number 42 has some remarkable properties.

If string theory is correct, the universe is full of ‘vacuum bubbles’: strings that pop in and out of existence.



The most symmetrical of these vacuum bubbles have a number of symmetries *that is always a multiple of 42*.

Let’s see why.

This equation has ten positive integer solutions:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$

The biggest number that appears is 42:

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \qquad \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \qquad \frac{1}{5} + \frac{1}{5} + \frac{1}{10} = \frac{1}{2}$$

$$\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2} \qquad \frac{1}{3} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2} \qquad \frac{1}{3} + \frac{1}{10} + \frac{1}{15} = \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{1}{2} \qquad \frac{1}{4} + \frac{1}{5} + \frac{1}{20} = \frac{1}{2} \qquad \frac{1}{3} + \frac{1}{8} + \frac{1}{24} = \frac{1}{2}$$

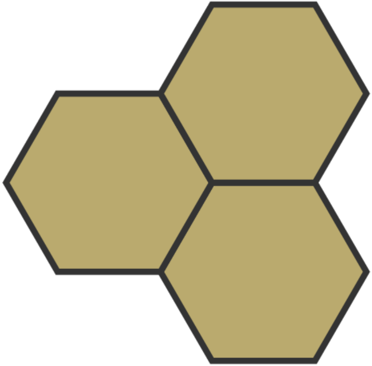
$$\frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{1}{2}$$

So what???

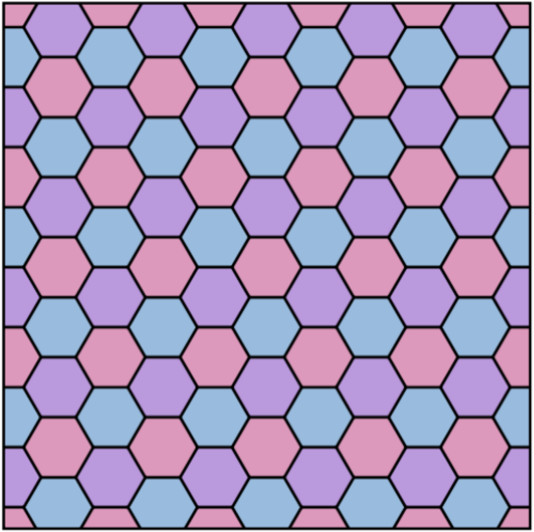
This solution:

$$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$$

gives this:



We can use this pattern to tile the plane!

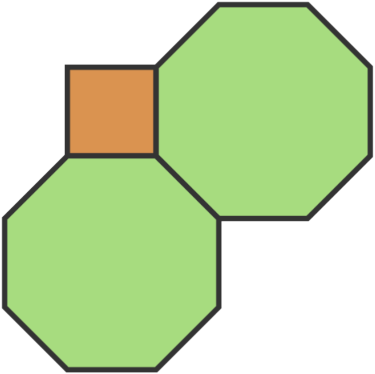


$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

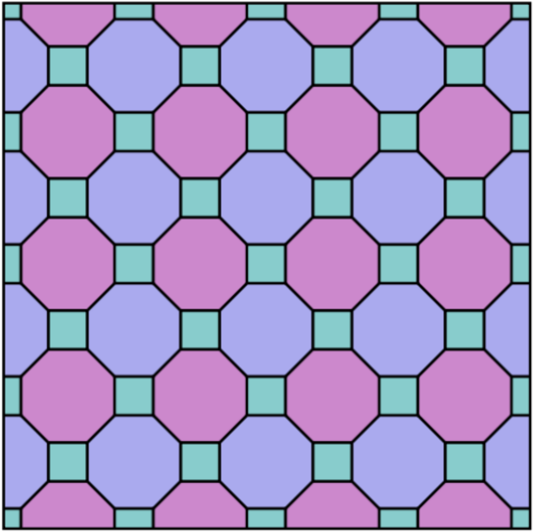
This solution:

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

gives this:



We can also use this one to tile the plane!

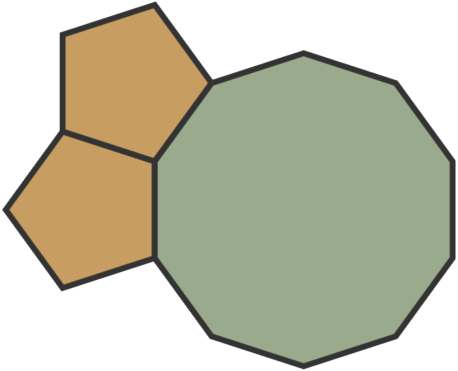


$$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

This solution:

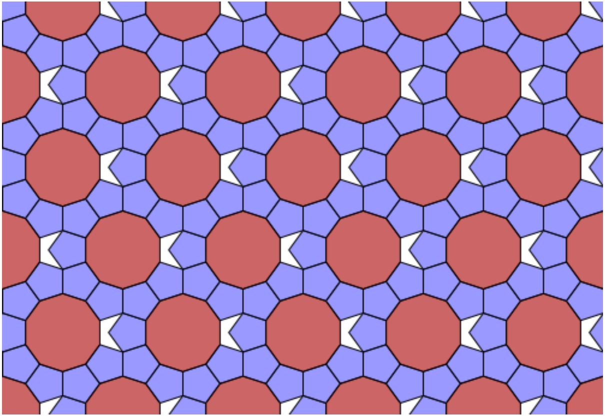
$$\frac{1}{5} + \frac{1}{5} + \frac{1}{10} = \frac{1}{2}$$

gives this:



This does not tile the plane: it's "too odd".

We can't tile the plane with regular pentagons and decagons, but Kepler tried and came close!

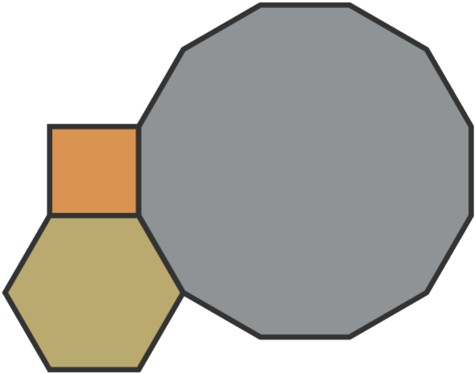


My friend Greg Egan showed how close we can get.

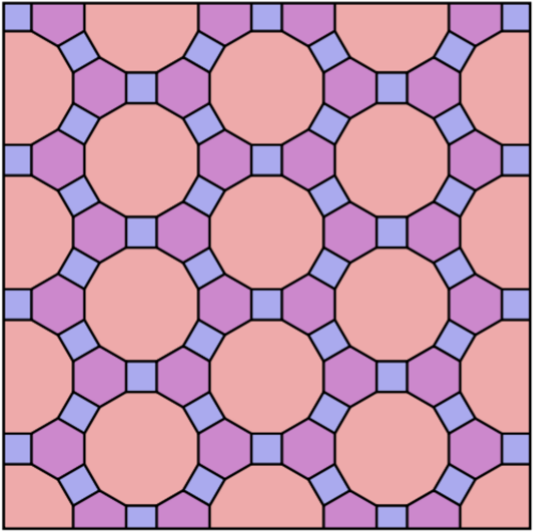
This solution:

$$\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$$

gives this:



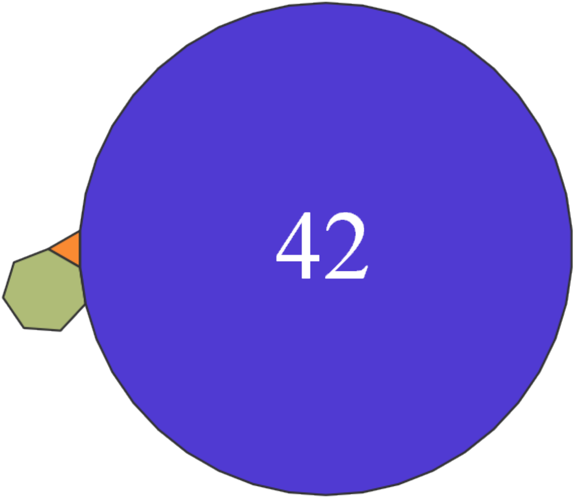
We can use this one to tile the plane!



$$\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=\frac{1}{2}$$

And so on. And the winner is:

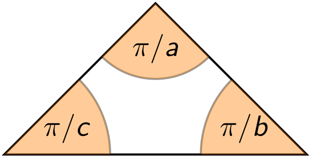
$$\frac{1}{3} + \frac{1}{7} + \frac{1}{\color{red}{42}} = \frac{1}{2}$$



But what's so great about being the winner of this size contest?
If you choose positive integers a, b, c with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

then you get a triangle:



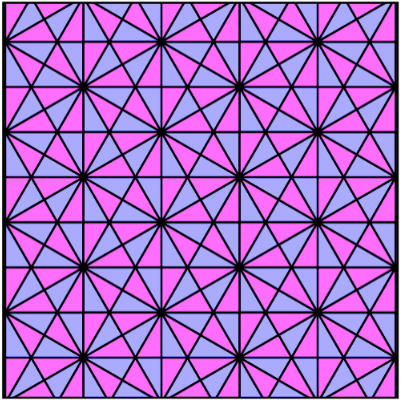
since the angles $\frac{\pi}{a}$, $\frac{\pi}{b}$ and $\frac{\pi}{c}$ add up to π radians, or 180°.

And this triangle tiles the plane!

For example:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

gives us the 30°-60°-90° triangle, which tiles the plane:



How many more solutions of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ can you find?

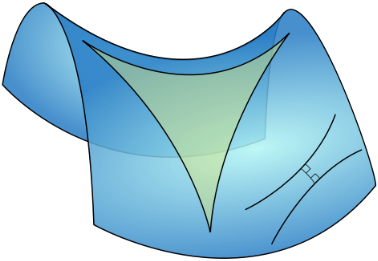
If

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

then the angles $\frac{\pi}{a}$, $\frac{\pi}{b}$ and $\frac{\pi}{c}$ add up to less than π.

So, they aren't the angles of an ordinary triangle.

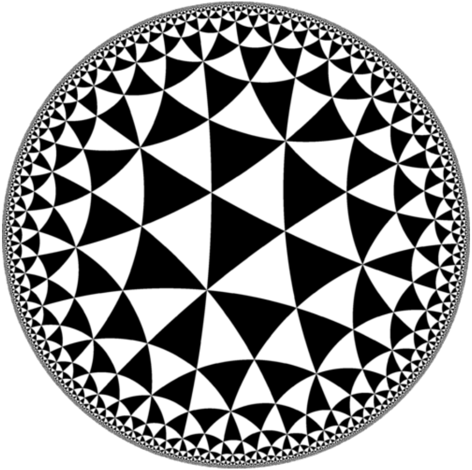
Instead, we get a triangle in a curved surface called the "hyperbolic plane":



For example:

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{4}=\frac{11}{12}<1$$

gives a 60°-60°-45° triangle, which tiles the hyperbolic plane:



In fact *any* positive integers a, b, c with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

give a triangle that tiles the hyperbolic plane!

The closer

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

is to 1, the flatter this triangle will be.

How close can we get?

Remember our previous prize-winner:

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=\frac{1}{2}$$

Add $\frac{1}{2}$ to both sides:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1$$

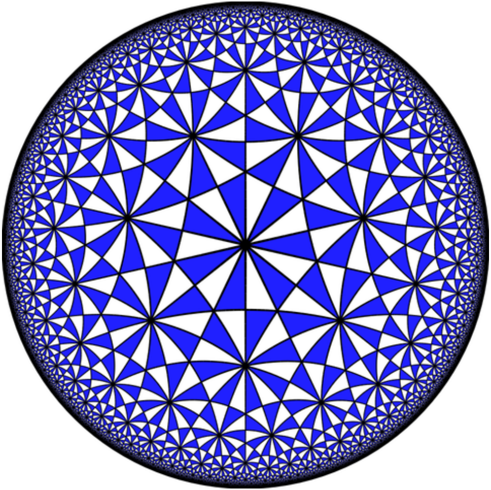
Subtract $\frac{1}{42}$:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}=\frac{41}{42}$$

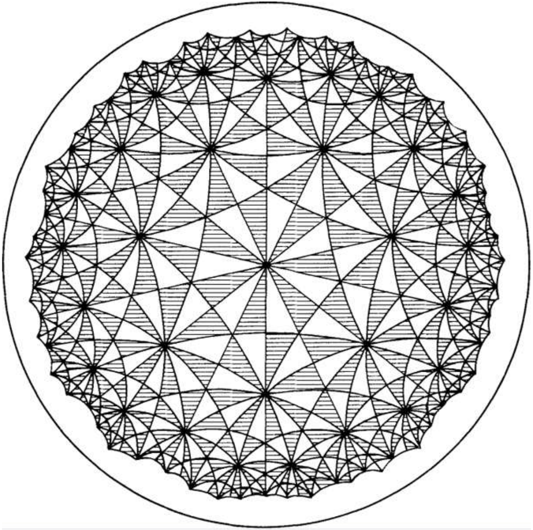
This is as close to 1 as $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ can get while still being less than 1! It's not obvious that the winner of our previous contest should give the winner to this contest, but it's true.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}=\frac{41}{42}<1$$

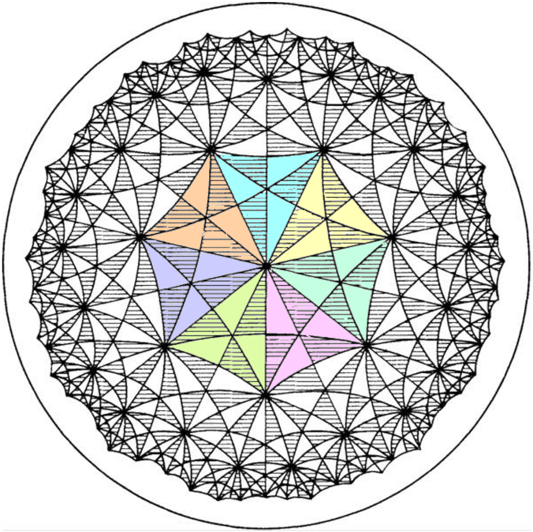
gives a right triangle that tiles the hyperbolic plane:



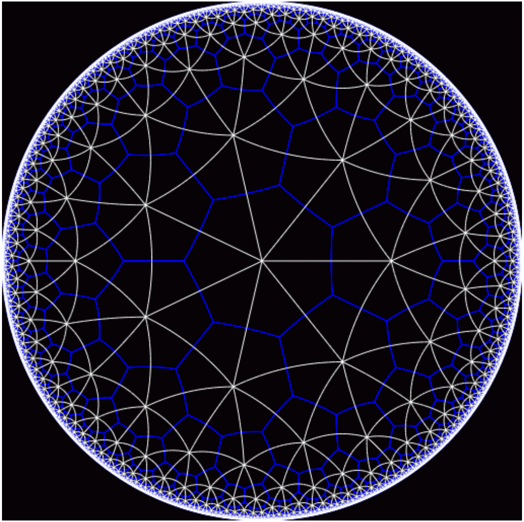
In the 1870's, Felix Klein fell in love with this tiling.



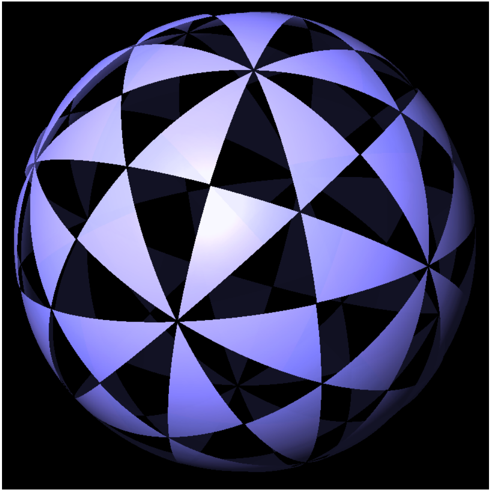
He noticed that the right triangles form groups of 6, which are larger equilateral triangles.



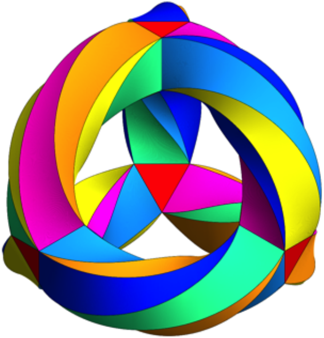
These equilateral triangles tile the hyperbolic plane, with 7 meeting at each corner:



He knew that equilateral triangles with 5 meeting at each corner can tile a sphere:

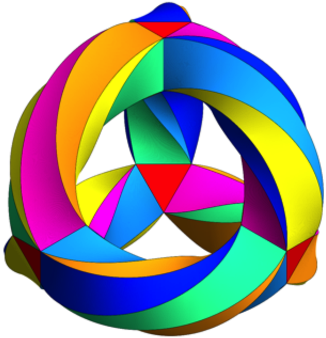


He discovered that equilateral triangles with 7 meeting at each corner can tile a 3-holed torus:

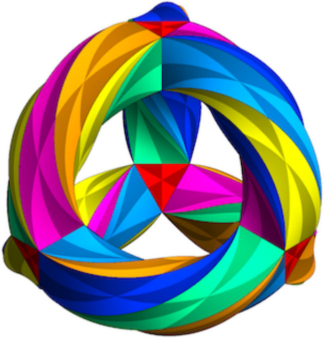


Unfortunately the equilateral triangles are distorted in this picture, because Klein's surface doesn't fit inside 3d Euclidean space!

Klein's 3-holed torus is tiled by 56 equilateral triangles:

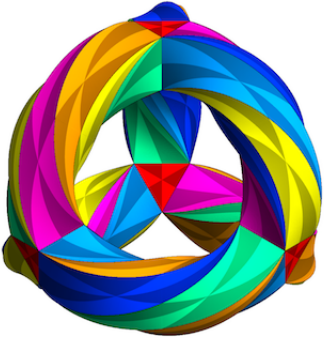


Each of the 56 equilateral triangles is made of 6 right triangles.

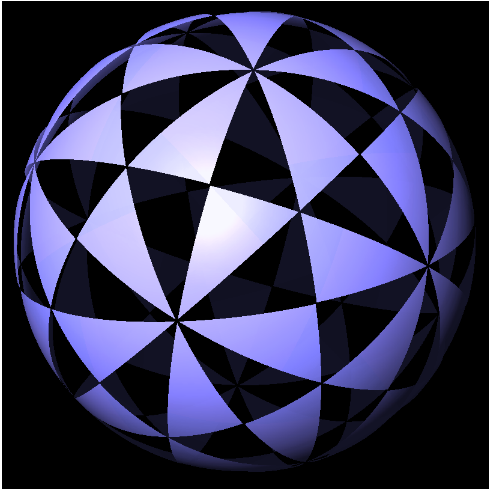


So, this surface can be tiled by $56 \times 6 = 336$ right triangles.

Though it's distorted in this picture, in reality this surface is so symmetrical that it has 336 symmetries: just enough to carry any right triangle to any other!



It’s similar to how this surface has enough symmetries to carry any right triangle to any other:



In fact, Klein's tiling has *the largest number of symmetries* of any tiling of the 3-holed torus by triangles that are all the same shape!

In 1893, Hurwitz proved that if you can tile a g-holed surface by triangles whose angles are

$$\frac{\pi}{a}, \ \frac{\pi}{b}, \ \frac{\pi}{c}$$

then the surface has at most

$$\frac{4(g-1)}{1-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}}$$

symmetries. To make this as big as possible, we need

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$$

to be *as close to 1 as possible!*

So, we should use:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;=\; \frac{1}{2}+\frac{1}{3}+\frac{1}{7} \;=\; \frac{41}{42}$$

That gives this number of symmetries:

$$\frac{4(g-1)}{1-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}} \;=\; \frac{4(g-1)}{1-\frac{41}{42}} \;=\; 42 \times 4(g-1)$$

For a 3-holed torus $g=3$, so the most symmetries we can have is

$$42 \times 8 = 336$$

Klein's surface is the most symmetrical 3-holed torus, thanks to the power of the number 42!

All this began as pure mathematics — beautiful geometry for its own sake.

It's called the theory of 'Riemann surfaces' and 'Coxeter groups'.

But if string theory is true, the world is made of strings tracing out Riemann surfaces with the passage of time, and then the number 42 is woven into the fabric of the universe.

Of course, so are all the other numbers.

To learn more

You can learn more about Riemann surfaces and Coxeter groups on Wikipedia. The theorem by Hurwitz is called Hurwitz's automorphism theorem, and click the link for a precise statement: mine was rather sloppy, since I didn't want to explain Riemann surfaces.

You can also learn more by clicking on the links in the following pages!

Illustrations

Click on pictures in these slides for more details, or click here:

- Klein's quartic curve tiled by 56 equilateral triangles, by Greg Egan.
- Quote of Stephen Fry on the BBC News.
- Douglas Adams' explanation of why he chose 42, on alt.fan.douglas-adams.
- Riemann surfaces of genus one, genus two and genus three by Oleg Alexandrov, put into the public domain on Wikicommons.
- Pictures of 3 regular polygons meeting at a vertex, by Dllu, put into the public domain on Wikicommons.
- Semiregular tilings, by R. A. Nonemacher, put on Wikicommons under a Creative Commons Attribution-Share Alike 4.0 International license.

- Pentagon-decagon packing, by Greg Egan, on my blog *Visual Insight*. See also the Pentagon-decagon branched covering by Greg Egan.
- Tiling by 30°-60°-90° triangles, by R. A. Nonemacher, put on Wikicommons under a Creative Commons Attribution-Share Alike 4.0 International license.
- Hyperbolic plane with triangle, by LucasVB, put into the public domain on Wikicommons. For more, read the article Hyperbolic geometry.
- (3,3,4) triangle tiling of the hyperbolic plane by Anton Sherwood, aka Tamfang, put into the public domain on Wikicommons. For more, read the articles Triangle group.

- (2,3,7) triangle tiling of the hyperbolic plane by Claudio Rocchini, put on Wikicommons under a Creative Commons Attribution-Share Alike 3.0 Unported license. For more, read the article (2,3,7) triangle group.
- (2,3,7) triangle tiling of the hyperbolic plane from Felix Klein and Robert Fricke's 1890 paper 'Vorlesungen über die Theorie der Elliptischen Modulfunctionen', reproduced from Vladimir Bulatov's presentation 'Tilings of the hyperbolic space and their visualization'.
- (2,3,7) triangle tiling of the hyperbolic plane with equilateral triangles emphasized, by Don Hatch.
- (2,3,5) triangle tiling of the sphere by Tom Ruen, put into the public domain on Wikicommons. For more, read the article icosahedral symmetry.
- Klein's quartic curve tiled by 56 equilateral triangles or 336 right triangles, by Greg Egan.