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Higher Gauge Theory II: 2-Connections

John Baez

Department of Mathematics
University of California
Riverside, CA 92521, USA
Email: `baez@math.ucr.edu`

Urs Schreiber

Fachbereich Mathematik
Universität Hamburg
Hamburg, 20146, Germany
Email: `urs.schreiber@math.uni-hamburg.de`

Abstract: Connections and curvings on gerbes are beginning to play a vital role in differential geometry and theoretical physics — first abelian gerbes, and more recently nonabelian gerbes and the twisted nonabelian gerbes introduced by Aschieri and Jurčo in their study of M-theory. These concepts can be elegantly understood using the concept of '2-bundle' recently introduced by Bartels. A 2-bundle is a generalization of a bundle in which the fibers are categories rather than sets. Here we introduce the concept of a '2-connection' on a principal 2-bundle. We describe principal 2-bundles with connection in terms of local data, and show that under certain conditions this reduces to the cocycle data for twisted nonabelian gerbes with connection and curving subject to a certain constraint — namely, the vanishing of the 'fake curvature', as defined by Breen and Messing. This constraint also turns out to guarantee the existence of '2-holonomies': that is, parallel transport over both curves and surfaces, fitting together to define a 2-functor from the 'path 2-groupoid' of the base space to the structure 2-group. We give a general theory of 2-holonomies and show how they are related to ordinary parallel transport on the path space of the base manifold.

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Contents

1. Introduction

The concept of 'connection' lies at the very heart of modern physics, and it is central to much of modern mathematics. A connection describes parallel transport along *curves*. Motivated both by questions concerning M-theory (see §1.1) and ideas from higher category theory (§1.2), we seek to describe parallel transport along *surfaces*. After describing these motivations, we outline our results (§1.3) and sketch the structure of this paper (§1.4). Readers intimidated by the length of this paper may prefer to read a summary of results [5].

1.1 Nonabelian Surface Holonomies in Physics

In the context of M-theory (the only partially understood expected completion of string theory), 2- and 5-dimensional surfaces in 10-dimensional space are believed to play a fundamental role. These are called '2-branes' and '5-branes', respectively. The general configuration of 2-branes and 5-branes involves 2-branes having boundaries that are attached to 5-branes. This is a higher-dimensional analogue of how open strings may end on various types of branes in string theory. So, let us recall what happens in that simpler case.

When an open string ends on a single brane, its end acts like a point particle coupled to a 1-form, or more precisely, to a U(1) connection. However, it is also possible for the end of a string to mimic a point particle coupled to a nonabelian gauge field. This happens when the string ends on a number of branes that are coincident, or 'stacked'. When an open string ends on a stack of n branes, its end acts like a point particle coupled to a U(n) connection, or more generally a connection on some bundle whose structure group consists of $n \times n$ matrices. The reason is that the action for the string involves the holonomy of this connection along the curve traced out by the motion of the string's endpoint as time passes.

It is natural to hope that something similar happens when a 2-brane ends on a 5-brane or a stack of 5-branes. Indeed, it is already known that when a 2-brane ends on a single 5-brane, it acts like a string coupled to a 2-form — or more precisely, to a connection on a U(1) gerbe. (Just as a connection on a U(1) bundle can be locally identified with a 1-form, but not globally, so a connection on a U(1) gerbe can be locally but not globally identified with a 2-form.)

Similarly, we expect that a 2-brane ending on a stack of coinciding 5-branes should behave like a string coupled to a Lie-algebra-valued 2-form — or more precisely, to a connection on a nonabelian gerbe. The theory of such connections is under development, and its application to this problem have already been considered [13, 14, 15]. However, for this application to work, the action for a 2-brane ending on a stack of 5-branes should involve some sort of 'holonomy' of this kind of connection over the 2-dimensional surface traced out by the motion of the 2-brane's boundary. For this, we need a concept of 'nonabelian surface holonomy'.

It is well understood how connections on *abelian* gerbes give rise to a notion of *abelian* surface holonomy, but for nonabelian gerbes so far no concept of surface holonomy has been defined. In fact, using 2-groups it is straightforward to come up with a local version of this notion. This has already been done in a discretized setting — a categorified version of lattice gauge theory [9]. However, to construct a well-defined action for a 2-brane ending on a stack of 5-branes, it is crucial to take care of global issues.

For these reasons, a globally defined notion of nonabelian surface holonomy seems to be a necessary prerequisite for fully understanding the fundamental objects of M-theory. In this paper we present such a notion, which we call '2-holonomy'.

In fact, this concept is interesting for ordinary field theory as well, quite independently of whether strings really exist. Configurations of membranes ending on a stack of 5-branes can alternatively be described in terms of certain (super)conformally invariant field theories involving Lie-algebra-valued 2-form fields defined on the six-dimensional manifold traced out by the 5-branes. When these field theories are compactified on a torus they give rise to (super)Yang-Mills theory in four dimensions. In this context, the famous Montonen-Olive $\mathrm{SL}(2,\mathbb{Z})$ duality exhibited by this 4-dimensional gauge theory should arise simply from the modular transformations on the internal torus, which act as symmetries of the conformally invariant six-dimensional theory. From this point of view, nonabelian 2-form gauge theory in six dimensions appears as a tool for understanding ordinary gauge theory in four dimensions.

It is interesting to note that these six-dimensional theories require the curvature 3-form of the 2-form field to be *self-dual* with respect to the Hodge star operator. In the nonabelian case this is subtle, because this 3-form should obey the local transition laws for nonabelian gerbes, which are not — at least not in any obvious way — compatible with self-duality in general, since they involve corrections to a covariant transformation of the 3-form. The only *obvious* solution to this compatibility problem is to require the so-called '*fake curvature*' of the gerbe to vanish. If this is the case, the 3-form field strength transforms covariantly and can hence consistently be chosen to be self-dual.

This solution has a certain appeal, because the constraint of vanishing fake curvature also shows up naturally in the study of nonabelian surface holonomy. Indeed, a major theme of the present paper is to show that what we call 2-connections have well-defined surface holonomies only when their fake curvature vanishes.

For these reasons, 2-connections may also be suited to play a role in relating nonabelian gauge theories in four dimensions to nonabelian 2-form theories in six dimensions. We do not work out the application of our 2-connections to either M-theory or this relation between four- and six-dimensional theories. But, as we explain in the next section, our theory of 2-connections is developed from the intrinsic logic of the problem of surface holonomy. So, either it or something very similar should be relevant.

An overview of the relation between four- and six-dimensional conformal field theories is given in [20]. A list of references on the physics of 5-branes is given in [11].

1.2 Higher Gauge Theory

The search for a theory of surface holonomies predates the recent interest in M-theory, because it seems like a natural generalization of ordinary gauge theory. However, until recently this search has been held back by trying to use familiar algebraic structures —

groups and Lie algebras — which are appropriate for ordinary gauge theory but not for this generalization.

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a group element to each path:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

The reason is that composition of paths then corresponds to multiplication in the group:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \end{tikzcd}$$

while reversing the direction of a path corresponds to taking inverses:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g^{-1}"'] \end{tikzcd}$$

and the associative law makes the holonomy along a triple composite unambiguous:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \arrow[r, bend left, "g''"] & \bullet \end{tikzcd}$$

In short, the topology dictates the algebra.

Now suppose we wish to do something similar for 1-dimensional 'strings' that trace out 2-dimensional surfaces as they move. Naively we might wish our holonomy to assign a group element to each surface like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "g"] \end{tikzcd}$$

There are two obvious ways to compose surfaces of this sort, vertically:

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=70, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=M, "g"] \arrow[Rightarrow, from=M2, to=D, "g'"] \end{tikzcd}$$

and horizontally:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=U1, below}] \arrow[r, bend right=50, ""{name=D1}] & \bullet \arrow[r, bend left=50, ""{name=U2, below}] \arrow[r, bend right=50, ""{name=D2}] & \bullet \arrow[Rightarrow, from=U1, to=D1, "g"] \arrow[Rightarrow, from=U2, to=D2, "g'"] \end{tikzcd}$$

Suppose that both of these correspond to multiplication in the group G. Then to obtain a well-defined holonomy for this surface regardless of whether we do vertical or horizontal composition first:

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U1, below}] \arrow[r, ""{name=M1, above}, ""{name=N1, below}] \arrow[r, bend right=70, ""{name=D1}] & \bullet \arrow[r, bend left=70, ""{name=U2, below}] \arrow[r, ""{name=M2, above}, ""{name=N2, below}] \arrow[r, bend right=70, ""{name=D2}] & \bullet \arrow[Rightarrow, from=U1, to=M1, "g_1"] \arrow[Rightarrow, from=N1, to=D1, "g_1'"] \arrow[Rightarrow, from=U2, to=M2, "g_2"] \arrow[Rightarrow, from=N2, to=D2, "g_2'"] \end{tikzcd}$$

we must have

$$(g_1 g_2)(g_1' g_2') = (g_1 g_1')(g_2 g_2').$$

This forces G to be abelian!

In fact, this argument goes back to a classic paper by Eckmann and Hilton [21]. Moreover, they showed that even if we allow G to be equipped with two products, say $g \circ g'$ for vertical composition and gg' for horizontal, so long as both products share the same unit and satisfy this '**interchange law**':

$$(g_1 \circ g_1')(g_2 \circ g_2') = (g_1 g_2) \circ (g_1' g_2')$$

then in fact they must agree — so by the previous argument, both are abelian. The proof is very easy:

$$gg' = (g \circ 1)(1 \circ g') = (g1) \circ (1g') = g \circ g'.$$

Pursuing this approach, we would ultimately reach the theory of connections on 'abelian gerbes'. If $G = \mathrm{U}(1)$, such a connections looks locally like a 2-form — and it shows up naturally in string theory, satisfying equations very much like those of electromagnetism.

To go beyond this and obtain *nonabelian* higher gauge fields, we must let the topology dictate the algebra. Readers familiar with higher categories will already have noticed that 1-dimensional pictures above resemble diagrams in category theory, while the 2-dimensional pictures resemble diagrams in 2-category theory. This suggests that instead of a Lie group, the holonomies in higher gauge theory should take values in some sort of categorified analogue, which we could call a 'Lie 2-group'.

In fact, even without knowing about higher categories, we can be led to the definition of a Lie 2-group by considering a kind of connection that gives holonomies *both for paths and for surfaces.*

So, let us assume that for each path we have a holonomy taking values in some Lie group G, where composition of paths corresponds to multiplication in G. Assume also that for each 1-parameter family of paths with fixed endpoints we have a holonomy taking values in some other Lie group H, where vertical composition corresponds to multiplication in H:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=U, to=M, "h"] \arrow[Rightarrow, from=M, to=D, "h'"]
\end{tikzcd}$$

Next, assume that we can parallel transport an element $g \in G$ along a 1-parameter family of paths to get a new element $g' \in G$:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=40, "g", ""{name=U, below}] \arrow[r, bend right=40, "g'"', ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=U, to=D, "h"]
\end{tikzcd}$$

This picture suggests that we should think of h as a kind of 'arrow' or 'morphism' going from g to g'. We can use category theory to formalize this. However, in category theory, when a morphism goes from an object x to an object y, we think of the morphism as determining both its source x and its target y. The group element h does not determine g or g'. However, the pair (g, h) does.

For this reason, it is useful to create a category $\mathcal{G}$ where the set of objects, say $\mathrm{Ob}(\mathcal{G})$, is just G, while the set of morphisms, say $\mathrm{Mor}(\mathcal{G})$, consists of ordered pairs $f = (g, h) \in G \times H$. Switching our notation to reflect this, we rewrite the above picture as

$$\begin{tikzcd}
\bullet \arrow[r, bend left=40, "g"] \arrow[r, bend right=40, "g'"'] \arrow[r, phantom, "\Downarrow f"] & \bullet
\end{tikzcd}$$

and write $f \colon g \to g'$ for short.

In this new notation, we can vertically compose $f \colon g \to g'$ and $f' \colon g' \to g''$ to get $f \circ f' \colon g \to g''$, as follows:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, "g"] \arrow[r, "g'" near start] \arrow[r, bend right=70, "g''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f"] \arrow[r, phantom, shift right=3ex, "\Downarrow f'"] & \bullet
\end{tikzcd}$$

This is just composition of morphisms in the category $\mathcal{G}$. However, we can also horizontally compose $f_1 \colon g_1 \to g_1'$ and $f_2 \colon g_2 \to g_2'$ to get $f_1 f_2 \colon g_1 g_2 \to g_1' g_2'$, as follows:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=40, "g_1"] \arrow[r, bend right=40, "g_1'"'] \arrow[r, phantom, "f_1 \Downarrow"] & \bullet \arrow[r, bend left=40, "g_2"] \arrow[r, bend right=40, "g_2'"'] \arrow[r, phantom, "f_2 \Downarrow"] & \bullet
\end{tikzcd}$$

We assume this operation makes $\mathrm{Mor}(\mathcal{G})$ into a group with the pair $(1, 1) \in G \times H$ as its multiplicative unit.

The good news is that now we can assume an interchange law saying this holonomy is well-defined:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, "g_1"] \arrow[r, "g_2" near start] \arrow[r, bend right=70, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_1'"] & \bullet \arrow[r, bend left=70, "g_1'"] \arrow[r, "g_2'" near start] \arrow[r, bend right=70, "g_3'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_2"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_2'"] & \bullet
\end{tikzcd}$$

namely:

$$(f_1 \circ f_1')(f_2 \circ f_2') = (f_1 f_2) \circ (f_1' f_2') \tag{1.1}$$

without forcing either G or H to be abelian! Instead, the group $\mathrm{Mor}(\mathcal{G})$ is forced to be a semidirect product of G and H.

The structure we are rather roughly describing here is in fact already known to mathematicians under the name of a 'categorical group' [3, 22, 23]. The reason is that $\mathcal{G}$ turns out to be a category living in the world of groups: that is, a category where the set of objects is a group, the set of morphisms is a group, and all the usual category operations are group homomorphisms. To keep the terminology succinct and to hint at generalizations to still higher-dimensional holonomies, we prefer to call this sort of structure a '2-group'. Moreover, we shall focus our attention on 'Lie 2-groups', where the objects and morphisms form Lie groups, and all the operations are smooth.

In fact, one can develop a full-fledged theory of bundles, connections, curvature, and so on with a Lie 2-group taking the place of a Lie group. To do this, one must systematically engage in a process of 'categorification', replacing set-based concepts by their category-based analogues. For example, just as 2-groups are categorified groups, we can define 'Lie 2-algebras', which are categorified Lie algebras [4].

So far most work on categorified gauge theory has focused on the special case when G is trivial and $H = \mathrm{U}(1)$, using the language of 'U(1) gerbes' [18, 24, 25, 26, 27, 28]. Here, however, we really want H to be nonabelian, and for this we need G to be nontrivial. Some important progress in this direction can be found in Breen and Messing's paper on the differential geometry of 'nonabelian gerbes' [13]. While they use different terminology, their work basically develops the theory of connections and curvature for Lie 2-groups where H is an arbitrary Lie group, $G = \mathrm{Aut}(H)$ is its group of automorphisms, t sends each element of H to the corresponding inner automorphism, and the action of G on H is the obvious one. We call this sort of Lie 2-group the 'automorphism 2-group' of H. Luckily, it is easy to extrapolate the whole theory from this case.

In particular, for any Lie 2-group $\mathcal{G}$ one can define the notion of a 'principal 2-bundle' having $\mathcal{G}$ as its gauge 2-group; this has recently been done by Bartels [6]. The first goal of this paper is to define a concept of '2-connection' for these principal 2-bundles The second is to show that given a 2-connection, one can define holonomies for paths and surfaces which behave just as one would hope:

- composing paths corresponds to multiplying their holonomies in the group $\mathrm{Ob}(\mathcal{G})$:

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet
\end{tikzcd}$$

- reversing the direction of a path corresponds to taking the inverse of its holonomy in the group $\mathrm{Ob}(\mathcal{G})$:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "g^{-1}"']
\end{tikzcd}$$

- horizontally composing surfaces corresponds to multiplying their holonomies in the group $\mathrm{Mor}(\mathcal{G})$:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g_1", ""{name=U1, below}] \arrow[r, bend right=50, "g_1'"', ""{name=D1, above}] \arrow[Rightarrow, from=U1, to=D1, "f_1"'] & \bullet \arrow[r, bend left=50, "g_2", ""{name=U2, below}] \arrow[r, bend right=50, "g_2'"', ""{name=D2, above}] \arrow[Rightarrow, from=U2, to=D2, "f_2"'] & \bullet
\end{tikzcd}$$

- horizontally reversing a surface corresponds to taking the inverse of its holonomy in the group $\mathrm{Mor}(\mathcal{G})$:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, bend right=40, "g^{-1}"', ""{name=U, below}] \arrow[l, bend left=40, "g'^{-1}", ""{name=D, above}] \arrow[from=D, to=U, Rightarrow, "f^{-1}"']
\end{tikzcd}$$

- vertically composing surfaces corresponds to composing their holonomies as morphisms in the category $\mathcal{G}$:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g", ""{name=A, below}] \arrow[r, "g'", ""{name=B}] \arrow[r, bend right=60, "g''"', ""{name=C, above}] \arrow[from=A, to=B, Rightarrow, "f"] \arrow[from=B, to=C, Rightarrow, "f'"] & \bullet
\end{tikzcd}$$

- vertically reversing a surface corresponds to taking the inverse of its holonomy as a morphism in the category $\mathcal{G}$:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=40, "g", ""{name=U, below}] \arrow[r, bend right=40, "g'"', ""{name=D, above}] \arrow[from=D, to=U, Rightarrow, "\overline{f}"'] & \bullet
\end{tikzcd}$$

A third goal of this paper is to relate such 2-connections to connections on the space $P(M)$, whose points are paths in M. A connection on $P(M)$ assigns a holonomy to any path in $P(M)$, and a path traces out a surface in M. Such a connection thus assigns a holonomy to a surface — but this will depend on the *parameterization* of the surface unless we impose extra conditions.

Intuitively it is clear that these two concepts should be closely related, but little is known about the details of this relation. Motivated by the recent discovery [11] that a certain consistency condition for surface holonomy appearing in the loop space approach is discussed also in the literature on 2-groups [9], while other such consistency conditions have exclusively been discussed in the loop space context [10], we seek to clarify this issue.

Our fourth goal is to relate Bartels' theory of 2-bundles to the theory of nonabelian gerbes, incuding the 'twisted' nonabelian gerbes introduced by Aschieri and Jurčo in their work on M-theory [15], and our final goal is to give a precise description of surface holonomy as a 2-functor. We discuss these ideas in more detail in the next section.

1.3 Outline of Results

In this paper we categorify the concept of *principal bundle with connection*, replacing the structure group by a 2-group, defining *principal 2-bundles with 2-connection*. We show how to describe these in terms of local data and show that under certain conditions this is equivalent to the cocycle description of nonabelian gerbes satisfying a certain constraint — the vanishing of the 'fake curvature'. We show that this constraint is also sufficient to guarantee the existence of *2-holonomies*, i.e., parallel transport over surfaces. We examine these 2-holonomies in detail using 2-functors into 2-groups on the one hand, and connections on path space on the other hand.

Several aspects of this have been studied before. Categorification is described in [1] and its application to groups and Lie algebras, which yields 2-groups and Lie 2-algebras, is discussed in [3, 4, 2]. The concept of 2-group was incorporated in the definition of principal 2-bundles (without connection) in [6]. A description of 2-connections as 2-functors was introduced in [7, 8, 9], but only in a discretized context, which makes it a bit tricky to treat global issues. Connections on path space were discussed in [10, 11], and reparametrization invariance for a special case was investigated by [12]. Cocycle data for nonabelian gerbes with connection and curving were obtained first by Breen and Messing [13] using algebraic geometry, and later by Aschieri and Jurčo [14, 15] using a nonabelian generalization of bundle gerbes [16].

Here we extend this work by:

- defining the concept of a principal 2-bundle with 2-connection,
- showing that a 2-connection on a trivial principal 2-bundle has 2-holonomies defining a 2-functor into the structure 2-group when the 2-connection has vanishing 'fake curvature' (a concept already defined for nonabelian gerbes by Breen and Messing [13]),
- clarifying the relation between connections on a trivial principal bundle over the path space of a manifold and 2-connections on a trivial principal 2-bundle over the manifold itself, showing that a connection on the path space whose holonomies are invariant under arbitrary surface reparameterizations defines a 2-connection on the original manifold,
- deriving the local 'gluing data' that describe how a nontrivial 2-bundle with 2-connection can be built from trivial 2-bundles with 2-connection on open sets that cover the base manifold,
- demonstrating that these gluing data for 2-bundles with 2-connection coincide with the cocycle description of nonabelian gerbes, subject to the constraint of vanishing fake curvature.

The starting point for all these considerations is the ordinary concept of a principal fiber bundle. Such a bundle can be specified using the following 'gluing data':

- a base manifold M,
- a cover of M by open sets $\{U_i\}_{i\in I}$,
- a Lie group G (the 'gauge group' or 'structure group'),
- on each double overlap $U_{ij} = U_i \cap U_j$ a G-valued function g_{ij},
- such that on triple overlaps the following transition law holds:

$$g_{ij}g_{jk} = g_{ik}.$$

Such a bundle is augmented with a connection by specifying:

- in each open set U_i a smooth functor $\mathrm{hol}_i\colon \mathcal{P}_1(U_i) \to G$ from the path groupoid of U_i to the gauge group,
- such that for all paths γ in double overlaps U_{ij} the following transition law holds:

$$\mathrm{hol}_i(\gamma) = g_{ij}\,\mathrm{hol}_j(\gamma)\; g_{ij}^{-1}.$$

Here the 'path groupoid' $\mathcal{P}_1(M)$ of a manifold M has points of M as objects and certain equivalence classes of smooth paths in M as morphisms. There are various ways to work out the technical details and make $\mathcal{P}_1(M)$ into a 'smooth groupoid'; see [18] for the approach we adopt, which uses 'thin homotopy classes' of smooth paths. Technical details aside, the basic idea is that a connection on a trivial G-bundle gives a well-behaved map assigning to each path γ in the base space the holonomy $\mathrm{hol}(\gamma) \in G$ of the connection along that path. Saying this map is a 'smooth functor' means that these holonomies compose when we compose paths, and that the holonomy $\mathrm{hol}(\gamma)$ depends smoothly on the path γ.

Our task shall be to categorify all of this and to work out the consequences. The basic tool will be *internalization*: given a mathematical concept X defined solely in terms of sets, functions and commutative diagrams involving these, and given some category K, one obtains the concept of an 'X in K' by replacing all these sets, functions and commutative diagrams by corresponding objects, morphisms, and commutative diagrams in K.

For example, take X to be the concept of 'group'. A group in Diff (the category with smooth manifolds as objects and smooth maps as morphisms) is nothing but a *Lie group*. In other words, a Lie group is a group that is a manifold, for which all the group operations are smooth maps. Similarly, a group in Top (the category with topological spaces as objects and continuous maps as morphisms) is a *topological group*.

These examples are standard, but we will need some slightly less familiar ones. In particular, we will need the concept of '2-group', which is a group in Cat (the category with categories as objects and functors as morphisms). By a charming principle called 'commutativity of internalization', 2-groups can also be thought of as categories in Grp (the category with groups as objects and homomorphisms as morphisms). We will also need the concept of a '2-space', which is a category in Diff, or more generally in some category of smooth spaces that allows for infinite-dimensional examples. A specially nice sort of 2-space is a 'smooth groupoid', a concept already mentioned above without explanation: this is a groupoid in the category of smooth spaces. Finally, a 'Lie 2-group' is a 2-group in Diff.

To arrive at the definition of a 2-bundle $E \to M$, the first steps are to replace the total space E and base space M by 2-spaces, and to replace the structure group by a Lie 2-group. In this paper we will actually keep the base space an ordinary space, which can be regarded as a 2-space with only identity morphisms. This is sufficiently general for many purposes. However, for applications to string theory we may ultimately need 2-bundles where the base space has points in some manifold as objects and paths or loops in this manifold as morphisms [19]. This sort of application also requires that we consider smooth spaces that are more general than finite-dimensional manifolds.

Just as a connection on a trivial principal bundle over M gives a functor hol from the path groupoid of M to the structure group, one might hope that a 2-connection on a trivial principal 2-bundle would define a 2-functor from some sort of 'path 2-groupoid' to the structure 2-group. This has already been studied in the context of higher lattice gauge theory [8, 9]. Thus, the main issues not yet addressed are those involving differentiability.

To address these issues, we define for any smooth space M a smooth 2-groupoid $\mathcal{P}_2(M)$ such that:

- the objects of $\mathcal{P}_2(M)$ are points of M: $\bullet x$
- the morphisms of $\mathcal{P}_2(M)$ are smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant in a neighborhood of $t = 0$ and $t = 1$: $$\begin{tikzcd} x\bullet \arrow[r, bend left, "\gamma"] & \bullet y \end{tikzcd}$$
- the 2-morphisms of $\mathcal{P}_2(M)$ are thin homotopy classes of smooth maps $\Sigma\colon [0,1]^2 \to M$ such that $\Sigma(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$: $$\begin{tikzcd} x\bullet \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_2"'{name=D}] & \bullet y \arrow[Rightarrow, from=U, to=D, "\Sigma"] \end{tikzcd}$$

We call the 2-morphisms in $\mathcal{P}_2(M)$ 'bigons'. The 'thin homotopy' equivalence relation guarantees that two maps differing only by a reparametrization define the same bigon. This is important because we seek a *reparametrization-invariant notion of surface holonomy.*

We show that any 2-connection on a trivial principal 2-bundle over M yields a smooth 2-functor $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$, where $\mathcal{G}$ is the structure 2-group. We call this 2-functor the *2-holonomy* of the 2-connection. In simple terms, the existence of this smooth 2-functor means that the 2-connection has well-defined holonomies both for paths and surfaces, independent of their parametrization, compatible with the standard operations of composing paths and surfaces, and depending smoothly on the path or surface in question.

To expand on this slightly, one must recall [3] that a Lie 2-group $\mathcal{G}$ amounts to the same thing as a 'crossed module' of Lie groups (G, H, t, α), where:

- G is the group of objects of G, $\mathrm{Ob}(\mathcal{G})$:
- H is the subgroup of $\mathrm{Mor}(\mathcal{G})$ consisting of morphisms with source equal to $1 \in G$:
- $t\colon H \to G$ is the homomorphism sending each morphism in H to its target,
- α is the action of G as automorphisms of H defined using conjugation in $\mathrm{Mor}(\mathcal{G})$ as follows: $\alpha(g)h = 1_g h {1_g}^{-1}$.

In these terms, a 2-connection on a trivial principal 2-bundle over M with structure 2-group $\mathcal{G}$ consists of a $\mathfrak{g}$-valued 1-form A together with an $\mathfrak{h}$-valued 2-form B on M. Translated into this framework, Breen and Messing's 'fake curvature' is the $\mathfrak{g}$-valued 2-form

$$dt(B) + F_A,$$

where $F_A = dA + A \wedge A$ is the usual curvature of A. We show that *if and only if the fake curvature vanishes*, there is a well-defined 2-holonomy $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$.

The importance of vanishing fake curvature in the framework of lattice gauge theory was already emphasized in [9]. The special case where also $F_A = 0$ was studied in [10], while a discussion of this constraint in terms of loop space in the case $G = H$ was given in [11]. Our result subsumes these cases in a common framework.

This framework for 2-connections on trivial 2-bundles is sufficient for local considerations. Thus, all that remains is to turn it into a global notion by categorifying the *transition laws* for a principal bundle with connection, which in terms of local data read:

$$\begin{aligned} g_{ij} g_{jk} &= g_{ik} \\ \mathrm{hol}_i(\gamma) &= g_{ij}\,\mathrm{hol}_j(\gamma)\; g_{ij}^{-1} \,. \end{aligned}$$

The basic idea is to replace these equations by specified isomorphisms, using the fact that a 2-group $\mathcal{G}$ has not only objects (forming the group G) but also morphisms (described with the help of the group H). These isomorphisms should in turn satisfy certain coherence laws of their own. These coherence laws have already been worked out for 2-bundles without connection [6] and for twisted nonabelian gerbes with connection and curving [13, 14, 15]; here we put these ideas together. We show that the local data describing such 2-bundles with 2-connection matches the cocycle data describing nonabelian bundle gerbes with connection and curving, subject to the constraint of vanishing fake curvature.

In summary, we find that categorifying the notion of a principal bundle with connection gives a structure that includes as a special case nonabelian bundle gerbes with connection and curving, with vanishing fake curvature.

1.4 Structure of the Paper

We begin in §2 with a review of internalization, Lie 2-groups and Lie 2-algebras, and nonabelian gerbes. This prepares us to explain 2-bundles in §3, and to show how they can be described using local gluing data.

After 2-bundles have been described in this way, we define the concept of 2-connection in §4. Here we also state our main result, Theorem 39, which describes 2-connections in terms of Lie-algebra-valued differential forms. We begin the proof of this result in §5. Finally, in §6 we relate 2-connections on a manifold to connections on its path space, and use this to conclude the proof.

2. Preliminaries

To develop the theory of 2-connections on 2-bundles, we need some mathematical preliminaries on internalization (§2.1), with special emphasis on 2-spaces (§2.2), Lie 2-groups (§2.3), and Lie 2-algebras (§2.4). We also review the theory of nonabelian gerbes (§2.5).

2.1 Internalization

To categorify concepts from differential geometry, we will use a procedure called 'internalization'. Developed by Lawvere [29], Ehresmann [30] and others, internalization is a method for generalizing concepts from ordinary set-based mathematics to other contexts — or more precisely, to other *categories*. This method is simple and elegant. To internalize a concept, we merely have to describe it using commutative diagrams in the category of sets, and then interpret these diagrams in some other category K. For example, if we internalize the concept of 'group' in the category of topological spaces, we obtain the concept of 'topological group'.

For categorification, the main concept we need to internalize is that of a category itself! To do this, we start by writing down the definition of category using commutative diagrams. We do this in terms of the functions s and t assigning to any morphism $f\colon x \to y$ its source and target:

$$s(f) = x, \qquad t(f) = y,$$

the function i assigning to any object its identity morphism:

$$i(x) = 1_x,$$

and the function $\circ$ assigning to any composable pair of morphisms their composite:

$$\circ(f, g) = f \circ g$$

If we write $\mathrm{Ob}(C)$ for the set of objects and $\mathrm{Mor}(C)$ for the set of morphisms of a category C, the set of composable pairs of morphisms is denoted $\mathrm{Mor}(C)_t{\times}_s\mathrm{Mor}(C)$, since it consists of pairs (f, g) with $t(f) = s(g)$.

In these terms, the definition of category looks like this:

> A **small category**, say C, has a <u>set</u> of objects $\mathrm{Ob}(C)$, a <u>set</u> of morphisms $\mathrm{Mor}(C)$, source and target <u>functions</u>:
>
> $$s, t\colon \mathrm{Mor}(C) \to \mathrm{Ob}(C),$$
>
> an identity-assigning <u>function</u>:
>
> $$i\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$$
>
> and a composition <u>function</u>:
>
> $$\circ\colon \mathrm{Mor}(C)_t{\times}_s\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$
>
> making diagrams commute that express associativity of composition, the left and right unit laws for identity morphisms, and the behaviour of source and target under composition.

We omit the actual diagrams because they are not very enlightening: the reader can find them elsewhere [4, 31] or reinvent them. The main point here is not so much what they are, as that they *can* be written down.

To internalize this definition, we replace the word 'set' by 'object of K' and replace the word 'function' by 'morphism of K':

> A **category in** K, say C, has an object $\mathrm{Ob}(C) \in K$, an object $\mathrm{Mor}(C) \in K$, source and target morphisms:
>
> $$s, t\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$
>
> an identity-assigning morphism:
>
> $$i\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$
>
> and a composition morphism:
>
> $$\circ\colon \mathrm{Mor}(C)_t{\times}_s\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$
>
> making diagrams commute that express associativity of composition, the left and right unit laws for identity morphisms, and the behaviour of source and target under composition.

Here we must define $\mathrm{Mor}(C)_t{\times}_s\mathrm{Mor}(C)$ using a category-theoretic notion called a 'pullback' [23]. Luckily, in examples it is usually obvious what this pullback should be, since it consists of composable pairs of morphisms in C.

Using this method, we can instantly categorify various concepts used in gauge theory:

Definition 1. *A* **Lie 2-group** *is a category in* LieGrp*, the category whose objects are Lie groups and whose morphisms are smooth group homomorphisms.*

Definition 2. *A* **Lie 2-algebra** *is a category in* LieAlg*, the category whose objects are Lie algebras and whose morphisms are Lie algebra homomorphisms.*

(For the benefit of experts, we should admit that we are only defining 'strict' Lie 2-groups and Lie 2-algebras here. We will not need any other kind in this paper.)

We could also define a 'smooth 2-space' to be a category in Diff, the category whose objects are finite-dimensional smooth manifolds and whose morphisms are smooth maps. However, this notion is slightly awkward for two reasons. First, unlike LieGrp and LieAlg, Diff does not have pullbacks in general. So, the subset of $\mathrm{Mor}(C) \times \mathrm{Mor}(C)$ consisting of composable pairs of morphisms may not be a submanifold. Second, and more importantly, we will also be interested in infinite-dimensional examples.

To solve these problems, we need a category of 'smooth spaces' that has pullbacks and includes a sufficiently large class of infinite-dimensional manifolds. Various categories of this sort have been proposed. It is unclear which one is best, but we shall use a slight variant of an idea proposed by Chen [32]. We describe this category of smooth spaces and smooth maps in the Appendix (§7). We call this category C^∞. For the present purposes, all that really matters about this category is that it has many nice features, including:

- Every finite-dimensional smooth manifold (possibly with boundary) is a smooth space, with smooth maps between these being precisely those that are smooth in the usual sense.
- Every smooth space has a topology, and all smooth maps between smooth spaces are continuous.
- Every subset of a smooth space is a smooth space.
- We can form a 'quotient' of a smooth space X by any equivalence relation, which is again smooth space.
- If $\{X_\alpha\}_{\alpha \in A}$ are smooth spaces, so is their product $\prod_{\alpha \in A} X_\alpha$.
- If $\{X_\alpha\}_{\alpha \in A}$ are smooth spaces, so is their disjoint union $\coprod_{\alpha \in A} X_\alpha$.
- If X and Y are smooth spaces, so is the set $C^\infty(X, Y)$ consisting of smooth maps from X to Y.
- There is an isomorphism of smooth spaces
  $$C^\infty(A \times X, Y) \cong C^\infty(A, C^\infty(X, Y))$$
  sending any function $\phi\colon A \times X \to Y$ to the function $\hat{\phi}\colon A \to C^\infty(X, Y)$ given by
  $$\hat{\phi}(x)(a) = \phi(x, a).$$
- We can define vector fields and differential forms on smooth spaces, with many of the usual properties.

With the notion of smooth space in hand, we can make the following definition:

Definition 3. *A* **(smooth) 2-space** *is a category in* C^∞*, the category whose objects are smooth spaces and whose morphisms are smooth maps.*

Not only can we categorify Lie groups, Lie algebras and smooth spaces, we can also categorify the maps between them. The right sort of map between categories is a functor: a pair of functions sending objects to objects and morphisms to morphisms, preserving source, target, identities and composition. If we internalize this concept, we get the definition of a 'functor in K'. We then say:

Definition 4. *A* **homomorphism** *between Lie 2-groups is a functor in* LieGrp*.*

Definition 5. *A* **homomorphism** *between Lie 2-algebras is a functor in* LieAlg*.*

Definition 6. *A* **(smooth) map** *between 2-spaces is a functor in* C^∞*.*

There are also natural transformations between functors, and internalizing this notion we can make the following definitions:

Definition 7. *A* **2-homomorphism** *between homomorphisms between Lie 2-groups is a natural transformation in* LieGrp*.*

Definition 8. *A* **2-homomorphism** *between homomorphisms between Lie 2-algebras is a natural transformation in* LieAlg*.*

Definition 9. *A* **(smooth) 2-map** *between maps between 2-spaces is a natural transformation in* C^∞*.*

Writing down these definitions is quick and easy. It takes longer to understand them and apply them to higher gauge theory. For this we must unpack them and look at examples. We do this in the next two sections.

2.2 2-Spaces

Unraveling Def. 3, a smooth 2-space, or **2-space** for short, is a category X where:

- The set of objects, $\mathrm{Ob}(X)$, is a smooth space.
- The set of morphisms, $\mathrm{Mor}(X)$, is a smooth space.
- The functions mapping any morphism to its source and target, $s, t \colon \mathrm{Mor}(X) \to \mathrm{Ob}(X)$, are smooth maps.
- The function mapping any object to its identity morphism, $i \colon \mathrm{Ob}(X) \to \mathrm{Mor}(X)$, is a smooth map
- The function mapping any composable pair of morphisms to their composite, $\circ \colon \mathrm{Mor}(X)_s {\times}_t \mathrm{Mor}(X) \to \mathrm{Mor}(X)$, is a smooth map.

2-spaces are more common than one might at first guess. One just needs to know where to look. First of all, every ordinary smooth space is a 2-space with only identity morphisms. More interesting examples arise naturally in string theory: the path groupoid and the loop groupoid of a manifold. In the next section we consider another large class of examples: Lie 2-groups.

Definition 10. *A 2-space with only identity morphisms is called* **trivial***.*

Example 11. Any smooth space M gives a trivial 2-space X with $\mathrm{Ob}(X) = M$. This 2-space has $\mathrm{Mor}(X) = M$, with $s, t, i, \circ$ all being the identity map from M to itself. Every trivial 2-space is of this form.

Example 12. Given a smooth space M, there is a smooth 2-space $\mathcal{P}_1(M)$, the **path groupoid of** M, such that:

- the objects of $\mathcal{P}_1(M)$ are points of M,
- the morphisms of $\mathcal{P}_1(M)$ are thin homotopy classes of smooth paths $\gamma \colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0$ and $t = 1$.

Here a **thin homotopy** between smooth paths $\gamma_0, \gamma_1 \colon [0,1] \to M$ is a smooth map $F \colon [0,1]^2 \to M$ such that:

- $F(0,t) = \gamma_0(t)$ and $F(1,t) = \gamma_1(t)$,
- $F(s,t)$ is constant for t near 0 and constant for t near 1,
- $F(s,t)$ is independent of s for s near 0 and for s near 1,
- the rank of the differential $dF(s,t)$ is < 2 for all $(s,t) \in [0,1]^2$.

The last condition is what makes the homotopy 'thin': it guarantees that the homotopy sweeps out a surface of vanishing area.

To see how $\mathcal{P}_1(M)$ becomes a 2-space, first note that the space of smooth maps $\gamma \colon [0,1] \to M$ becomes a smooth space in a natural way, as does the subspace satisfying the constancy conditions near $t = 0, 1$, and finally the quotient of this subspace consisting of thin homotopy classes. This makes $\mathrm{Mor}(\mathcal{P}_1(M))$ into a smooth space. For short, we call this smooth space $P(M)$, the **path space of** M. $\mathrm{Ob}(\mathcal{P}_1(M)) = M$ is obviously a smooth space as well. The source and target maps

$$s, t \colon \mathrm{Mor}(\mathcal{P}_1(M)) \to \mathrm{Ob}(\mathcal{P}_1(M))$$

send any equivalence class of paths to its endpoints:

$$s([\gamma]) = \gamma(0), \qquad t([\gamma]) = \gamma(1).$$

The identity-assigning map sends any point $x \in M$ to the constant path at this point. The composition map $\circ$ sends any composable pair of morphisms $[\gamma], [\gamma']$ to $[\gamma \circ \gamma']$, where

$$\gamma \circ \gamma'(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

One can check that $\gamma \circ \gamma'$ is a smooth path and that $[\gamma \circ \gamma']$ is well-defined and independent of the choice of representatives for $[\gamma]$ and $[\gamma']$. One can also check that the maps $s, t, i, \circ$ are smooth and that the usual rules of a category hold. It follows that $\mathcal{P}_1(M)$ is a 2-space.

In fact, $\mathcal{P}_1(M)$ is not just a category: it is also a **groupoid**: a category where every morphism has an inverse. The inverse of $[\gamma]$ is just $[\overline{\gamma}]$, where $\overline{\gamma}$ is obtained by reversing the orientation of the path γ:

$$\overline{\gamma}(t) = \gamma(1-t).$$

Moreover, the map sending any morphism to its inverse is smooth. Thus $\mathcal{P}_1(M)$ is a **smooth groupoid**: a 2-space where every morphism is invertible and the map sending every morphism to its inverse is smooth.

Example 13. Given a 2-space X, any subcategory of X becomes a 2-space in its own right. Here a **subcategory** is a category Y with $\mathrm{Ob}(Y) \subseteq \mathrm{Ob}(X)$ and $\mathrm{Mor}(Y) \subseteq \mathrm{Mor}(X)$, where the source, target, identity-assigning and composition maps of Y are restrictions of those for X. The reason Y becomes a 2-space is that any subspace of a smooth space becomes a smooth space in a natural way (see §7) and restrictions of smooth maps to subspaces are smooth. We call Y a **sub-2-space** of X.

Example 14. Given a smooth space M, the path groupoid $\mathcal{P}_1(M)$ has a sub-2-space $\mathcal{L}M$ whose objects are all the points of M and whose morphisms are those equivalence classes $[\gamma]$ where γ is a loop: that is, a path with $\gamma(0) = \gamma(1)$. We call $\mathcal{L}M$ the **loop groupoid** of M. Like the path groupoid, the loop groupoid of M is not just a 2-space, but a smooth groupoid.

For 2-spaces, and indeed for all categorified concepts, the usual notion of 'isomorphism' is less useful than that of 'equivalence'. For example, in categorified gauge theory what matters is not 2-bundles whose fibers are all isomorphic to some standard fiber, but those whose fibers are all *equivalent* to some standard fiber. We recall the concept of equivalence here:

Definition 15. *Given 2-spaces X and Y, an* **isomorphism** *$f\colon X \to Y$ is a map equipped with a map $\bar{f}\colon Y \to X$ called its* **inverse** *such that $\bar{f}f = 1_X$ and $f\bar{f} = 1_Y$. An* **equivalence** *$f\colon X \to Y$ is a map equipped with a map $\bar{f}\colon Y \to X$ called its* **weak inverse** *together with invertible 2-maps $\phi\colon \bar{f}f \overset{\sim}{\Longrightarrow} 1_X$ and $\bar{\phi}\colon f\bar{f} \overset{\sim}{\Longrightarrow} 1_Y$.*

2.3 Lie 2-Groups

Unravelling Def. 1, we see that a Lie 2-group $\mathcal{G}$ is a category where:

- The set of objects, $\mathrm{Ob}(\mathcal{G})$, is a Lie group.
- The set of morphisms, $\mathrm{Mor}(\mathcal{G})$, is a Lie group.
- The functions mapping any morphism to its source and target, $s, t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$, are homomorphisms.
- The function mapping any object to its identity morphism, $i\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, is a homomorphism.
- The function mapping any composable pair of morphisms to their composite, $\circ\colon \mathrm{Mor}(\mathcal{G})_s\times_t\mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, is a homomorphism.

For applications to higher gauge theory it is suggestive to draw objects of $\mathcal{G}$ as arrows:

$$\begin{tikzcd}\bullet \arrow[r, bend left, "g"] & \bullet\end{tikzcd}$$

and morphisms $f\colon g \to g'$ as surfaces of this sort:

$$\begin{tikzcd}\bullet \arrow[r, bend left=40, "g"{name=U}] \arrow[r, bend right=40, "g'"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"]\end{tikzcd}$$

This lets us can draw multiplication in $\mathrm{Ob}(\mathcal{G})$ as composition of arrows, multiplication in $\mathrm{Mor}(\mathcal{G})$ as 'horizontal composition' of surfaces, and composition of morphisms $f\colon g \to g'$ and $f'\colon g' \to g''$ as 'vertical composition' of surfaces, as explained in §1.2.

In this notation, the fact that composition is a homomorphism says that the 'exchange law'

$$(f_1 \circ f_1')(f_2 \circ f_2') = (f_1 f_2) \circ (f_1' f_2')$$

holds whenever we have a situation of this sort:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2"] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_1'"] & \bullet \arrow[r, bend left=60, "g_1'"] \arrow[r, "g_2'"] \arrow[r, bend right=60, "g_3'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_2"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_2'"] & \bullet
\end{tikzcd}$$

In other words, we can interpret this picture either as a horizontal composite of vertical composites or a vertical composite of horizontal composites, without any ambiguity.

A Lie 2-group with only identity morphisms is the same thing as a Lie group. To get more interesting examples it is handy to think of a Lie 2-group as special sort of 'crossed module'. To do this, start with a Lie 2-group $\mathcal{G}$ and form the pair of Lie groups

$$G = \mathrm{Ob}(\mathcal{G}), \qquad H = \ker s \subseteq \mathrm{Mor}(\mathcal{G}).$$

The target map restricts to a homomorphism

$$t\colon H \to G.$$

Besides the usual action of G on itself by conjugation, there is also an action of G on H,

$$\alpha\colon G \to \mathrm{Aut}(H),$$

given by

$$\begin{aligned}
\alpha(g)(h) &= 1_g\, h\, 1_{g^{-1}} \\
&= \begin{tikzcd}
\bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left, "g^{-1}"] \arrow[r, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow 1_{g^{-1}}"] & \bullet
\end{tikzcd} .
\end{aligned}$$

The target map is equivariant with respect to this action:

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and satisfies the so-called 'Peiffer identity':

$$\alpha(t(h))(h') = hh'h^{-1}.$$

A setup like this with groups rather than Lie groups is called a 'crossed module', so here we are getting a 'Lie crossed module':

Definition 16. *A* **Lie crossed module** *is a quadruple* (G, H, t, α) *consisting of Lie groups* G *and* H*, a homomorphism* $t\colon H \to G$*, and an action of* G *on* H *(that is, a homomorphism* $\alpha\colon G \to \mathrm{Aut}(H)$*) such that* t *is equivariant:*

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and satisfies the Peiffer identity:

$$\alpha(t(h))(h') = hh'h^{-1}$$

for all $g \in G$ and $h, h' \in H$.

This definition becomes a bit more memorable if we abuse language and write $\alpha(g)(h)$ as ghg^{-1}; then the equations above become

$$t(ghg^{-1}) = g\, t(h)\, g^{-1}$$

and

$$t(h)\, h'\, t(h)^{-1} = hh'h^{-1}.$$

As we shall see, Lie 2-groups are essentially the same as Lie crossed modules. The same is true for the homomorphisms between them. We have already defined a homomorphism of Lie 2-groups as a functor in LieGrp. We can also define a homomorphism of Lie crossed modules:

Definition 17. *A* **homomorphism** *from the Lie crossed module (G, H, t, α) to the Lie crossed module (G', H', t', α') is a pair of homomorphisms $f\colon G \to G'$, $\tilde{f}\colon H \to H'$ such that*

$$t(\tilde{f}(h)) = f(t'(h))$$

and

$$\tilde{f}(\alpha(g)(h)) = \alpha'(f(g))(\tilde{f}(h))$$

for all $g \in G$, $h \in H$.

Not only does every Lie 2-group give a Lie crossed module; every Lie crossed module gives a Lie 2-group. In fact:

Proposition 18. *The category of Lie 2-groups is equivalent to the category of Lie crossed modules.*

Proof. This follows easily from the well-known equivalence between crossed modules and 2-groups [35]; details can also be found in [3]. For the convenience of the reader, we sketch how to recover a Lie 2-group from a Lie crossed module.

Suppose we have a Lie crossed module (G, H, t, α). Let

$$\mathrm{Ob}(\mathcal{G}) = G, \qquad \mathrm{Mor}(\mathcal{G}) = H \rtimes G$$

where the semidirect product is formed using the action of G on H, so that multiplication in $\mathrm{Mor}(\mathcal{G})$ is given by:

$$(h, g)(h', g') = (h\alpha(g)(h'), gg') \tag{2.1}$$

In the 2-group $\mathcal{G}$, the ordered pair (h, g) represents the horizontal composite $h\, 1_g$.

The inverse of an element of the group $\mathrm{Mor}(\mathcal{G})$ is given by:

$$(h,g)^{-1} = \left(\alpha\!\left(g^{-1}\right)\!\left(h^{-1}\right), g^{-1}\right).$$

We make $\mathcal{G}$ into a Lie 2-group where the source and target maps $s, t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ are given by:

$$s(h,g) = g, \qquad t(h,g) = t(h)g, \tag{2.2}$$

the identity-assigning map $i\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$ is given by:

$$i(g) = (g,1), \tag{2.3}$$

and the composite of the morphisms

$$(h,g)\colon g \to g', \qquad (h',g')\colon g' \to g'',$$

is

$$(h,g) \circ (h',g') = (hh',g)\colon g \to g''. \tag{2.4}$$

It is also worth noting that every morphism has an inverse with respect to composition, which we denote by:

$$\overline{(h,g)} := \left(h^{-1}, t(h)g\right).$$

One can check that this construction indeed gives a Lie 2-group, and that together with the previous construction it sets up an equivalence between the categories of Lie 2-groups and Lie crossed modules. $\square$

The result of horizontally composing h and 1_g as in the above proof is usually drawn as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] \arrow[Rightarrow, from=U, to=D, "h"'] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd} \qquad := \qquad h\,1_g$$

For obvious reasons, this operation is called **right whiskering**. One can also define **left whiskering**:

$$\begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] \arrow[Rightarrow, from=U, to=D, "h"'] & \bullet \end{tikzcd} \qquad := \qquad 1_g\,h$$

When we think of morphisms of $\mathcal{G}$ as elements of $H \rtimes G$, the construction in Prop. 18 defines h right whiskered by g to be (h,g). Similarly, one can show that h left whiskered by g is $(\alpha(g)h, g)$.

Crossed modules are important in homotopy theory [36], and the reader who is fonder of crossed modules than categories is free to think of Lie 2-groups as a way of talking about Lie crossed modules. Both perspectives are useful, but one advantage of Lie crossed modules is that they allow us to quickly describe some examples:

Example 19. Given any abelian group H, there is a Lie crossed module where G is the trivial group and t, α are trivial. This gives a Lie 2-group $\mathcal{G}$ with one object and H as the group of morphisms. Lie 2-groups of this sort are important in the theory of *abelian* gerbes.

Example 20. More generally, given any Lie group G, abelian Lie group H, and action α of G as automorphisms of H, there is a Lie crossed module with $t\colon G \to H$ the trivial homomorphism. For example, we can take H to be a finite-dimensional vector space and α to be a representation of G on this vector space.

In particular, if G is the Lorentz group and α is the defining representation of this group on Minkowski spacetime, this construction gives a Lie 2-group called the **Poincaré 2-group**, because its group of morphisms is the Poincaré group. After its introduction in work on higher gauge theory [2], this 2-group was used in in some recent work on quantum gravity by Crane, Sheppeard and Yetter [37, 38].

Example 21. Given any Lie group H, there is a Lie crossed module with $G = \mathrm{Aut}(H)$, $t\colon H \to G$ the homomorphism assigning to each element of H the corresponding inner automorphism, and the obvious action of G as automorphisms of H. We call the corresponding Lie 2-group the **automorphism 2-group** of H, and denote it by $\mathcal{AUT}(H)$. This sort of 2-group is important in the theory of *nonabelian* gerbes.

In particular, if we take H to be the multiplicative group of nonzero quaternions, then $G = \mathrm{SU}(2)$ and we obtain a 2-group that plays a basic role in Thompson's theory of quaternionic gerbes [39].

We use the term 'automorphism 2-group' because $\mathcal{AUT}(H)$ really is a 2-group of symmetries of H. An object of $\mathcal{AUT}(H)$ is a symmetry of the group H in the usual sense: that is, an automorphism $f\colon H \to H$. On the other hand, a morphism $\theta\colon f \to f'$ in $\mathcal{AUT}(H)$ is a 'symmetry between symmetries': that is, an element $h \in H$ that sends f to f' in the following sense: $hf(x)h^{-1} = f'(x)$ for all $x \in H$.

Example 22. Suppose that $1 \to A \hookrightarrow H \xrightarrow{t} G \to 1$ is a central extension of the Lie group G by the Lie group H. Then there is a Lie crossed module with this choice of $t\colon G \to H$. To construct α we pick any section s, that is, any function $s\colon G \to H$ with $t(s(g)) = g$, and define

$$\alpha(g)h = s(g)hs(g)^{-1}.$$

Since A lies in the center of H, α independent of the choice of s. We do not need a global smooth section s to show $\alpha(g)$ depends smoothly on g; it suffices that there exist a local smooth section in a neighborhood of each $g \in G$.

It is easy to generalize this idea to infinite-dimensional cases if we work not with Lie groups but **smooth groups**: that is, groups in the category of smooth spaces. The basic theory of smooth groups, smooth 2-groups and smooth crossed modules works just like the finite-dimensional case, but with the category of smooth spaces replacing Diff. In particular, every smooth group G has a Lie algebra $\mathfrak{g}$.

Given a connected and simply-connected compact simple Lie group G, the loop group ΩG is a smooth group. For each 'level' $k \in \mathbb{Z}$, this group has a central extension

$$1 \to \mathrm{U}(1) \hookrightarrow \widehat{\Omega_k G} \xrightarrow{t} \Omega G \to 1$$

as explained by Pressley and Segal [40]. The above diagram lives in the category of smooth groups, and there exist local smooth sections for $t\colon \widehat{\Omega_k G} \to \Omega G$, so we obtain a smooth crossed module $(\Omega G, \widehat{\Omega_k G}, t, \alpha)$ with α given as above. This in turn gives an smooth 2-group which we call the **level-k loop 2-group** of G, $\mathcal{L}_k G$.

It has recently been shown [41] that $\mathcal{L}_k G$ fits into an exact sequence of smooth 2-groups:

$$1 \to \mathcal{L}_k G \hookrightarrow \mathcal{P}_k G \longrightarrow G \to 1$$

where the middle term, the **level-k path 2-group** of G, has very interesting properties. In particular, when $k = \pm 1$, the geometric realization of the nerve of $\mathcal{P}_k G$ is a topological group that can also be obtained by killing the 3rd homotopy group of G. When $G = \mathrm{Spin}(n)$, this topological group goes by the name of $\mathrm{String}(n)$, since it plays a role in defining spinors on loop space [42]. The group $\mathrm{String}(n)$ also shows up in Stolz and Teichner's work on elliptic cohomology, which involves a notion of parallel transport over surfaces [43]. So, we expect that $\mathcal{P}_k G$ will be an especially interesting structure 2-group for applications of 2-bundles to string theory.

To define the holonomy of a connection, we need smooth groups with an extra property: namely, that for every smooth function $f\colon [0,1] \to \mathfrak{g}$ there is a unique smooth function $g\colon [0,1] \to G$ solving the differential equation

$$\frac{d}{dt} g(t) = f(t) g(t)$$

with $g(0) = 1$. We call such smooth groups **exponentiable**. Similarly, we call a smooth 2-group **exponentiable** if its crossed module (G, H, t, α) has both G and H exponentiable. In particular, every Lie group and thus every Lie 2-group is exponentiable. The smooth groups ΩG and $\widehat{\Omega_k G}$ are also exponentiable, as are the 2-groups $\mathcal{L}_k G$ and $\mathcal{P}_k G$. So, for the convenience of stating theorems in a simple way, *we henceforth implicitly assume all smooth groups and 2-groups under discussion are exponentiable.*

2.4 Lie 2-Algebras

Just as Lie groups give rise to Lie algebras, Lie 2-groups give rise to Lie 2-algebras. These can also be described using a differential version of crossed modules. Recall that a Lie 2-algebra is a category $\mathcal{L}$ where:

- The set of objects, $\mathrm{Ob}(\mathcal{L})$, is a Lie algebra.
- The set of morphisms, $\mathrm{Mor}(\mathcal{L})$, is a Lie algebra.
- The functions mapping any morphism to its source and target, $s, t\colon \mathrm{Mor}(\mathcal{L}) \to \mathrm{Ob}(\mathcal{L})$, are Lie algebra homomorphisms,
- The function mapping any object to its identity morphism, $i\colon \mathrm{Ob}(\mathcal{L}) \to \mathrm{Mor}(\mathcal{L})$, is a Lie algebra homomorphism.
- The function mapping any composable pair of morphisms to their composite, $\circ\colon \mathrm{Mor}(\mathcal{L})_s{\times}_t \mathrm{Mor}(\mathcal{L}) \to \mathrm{Mor}(\mathcal{L})$, is a Lie algebra homomorphism.

We can get a Lie 2-algebra by differentiating all the data in a Lie 2-group. Similarly, we can get a 'differential crossed module' by differentiating all the data in a Lie crossed module:

Definition 23. *A* **differential crossed module** *is a quadruple $\mathcal{C} = (\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ consisting of Lie algebras $\mathfrak{g}, \mathfrak{h}$, a homomorphism $dt\colon \mathfrak{h} \to \mathfrak{g}$, and an action α of $\mathfrak{g}$ as derivations of $\mathfrak{h}$ (that is, a homomorphism $\alpha\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$) satisfying*

$$dt(d\alpha(x)(y)) = [x, dt(y)] \tag{2.5}$$

and

$$d\alpha(dt(y))(y') = [y, y'] \tag{2.6}$$

for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$.

This definition becomes easier to remember if we allow ourselves to write $d\alpha(x)(y)$ as $[x, y]$. Then the fact that $d\alpha$ is an action of $\mathfrak{g}$ as derivations of $\mathfrak{h}$ simply means that $[x, y]$ is linear in each argument and the following 'Jacobi identities' hold:

$$[[x, x'], y] = [x, [x', y]] - [x', [x, y]], \tag{2.7}$$

$$[x, [y, y']] = [[x, y], y'] - [[x, y'], y] \tag{2.8}$$

for all $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$. Furthermore, the two equations in the above definition become

$$t([x, y]) = [x, t(y)] \tag{2.9}$$

and

$$[t(y), y'] = [y, y']. \tag{2.10}$$

Proposition 24. *The category of Lie 2-algebras is equivalent to the category of differential crossed modules.*

Proof. The proof is just like that of Prop. 18. □

Since every Lie 2-group gives a Lie 2-algebra and a differential crossed module, there are plenty of examples of the latter concepts. Here is another interesting class of examples:

Example 25. Just as every Lie 2-group gives rise to a Lie 2-algebra, so does every smooth 2-group. The reason is that not only smooth manifolds but also smooth spaces have tangent spaces (see §7), and the usual construction of Lie algebras from Lie groups generalizes to smooth groups. So, any smooth 2-group $\mathcal{G}$ gives a Lie 2-algebra $\mathcal{L}$ for which $\mathrm{Ob}(\mathcal{L})$ is the Lie algebra of $\mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}(\mathcal{L})$ is the Lie algebra of $\mathrm{Mor}(\mathcal{G})$, and the maps $s, t, i, \circ$ for $\mathcal{L}$ are obtained by differentiating the corresponding maps for $\mathcal{G}$.

In particular, suppose G is a simply-connected compact simple Lie group with Lie algebra $\mathfrak{g}$. Then the loop 2-group of G, as defined in Example 22, has a Lie 2-algebra.

This Lie 2-algebra has $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ as its Lie algebra of objects, and a certain central extension $\tilde{L}\mathfrak{g}$ of $L\mathfrak{g}$ as its Lie algebra of morphisms. We call this Lie 2-algebra the **level-*k* loop Lie 2-algebra of $\mathfrak{g}$**. It is another way of organizing the data in the affine Lie algebra corresponding to $\mathfrak{g}$. Moreover [41], this Lie 2-algebra fits into an exact sequence of Lie 2-algebras:

$$0 \to \mathcal{L}_k\mathfrak{g} \hookrightarrow \mathcal{P}_k\mathfrak{g} \longrightarrow G \to 0$$

where the middle term is called the **level-*k* path Lie 2-algebra** of $\mathfrak{g}$. One can construct $\mathcal{P}_k\mathfrak{g}$ and this exact sequence by taking the Lie 2-algebras of the Lie 2-groups in the exact sequence described in Example 22.

We conclude our preliminaries with a brief review of nonabelian gerbes.

2.5 Nonabelian Gerbes

Given a bundle $E \xrightarrow{p} M$, the sections of E defined on all possible open sets of B are naturally organized into a structure called a 'sheaf'. This codifies the fact that we can restrict a section from an open set $U \subseteq B$ to a smaller open set $U' \subseteq U$, and also piece together sections on open sets U_i covering U to obtain a unique section on U, as long as the sections agree on the intersections $U_i \cap U_j$. So, sheaves can be thought of as a tool for studying bundles — but there are also sheaves that do not arise from bundles.

This paper approaches higher gauge theory by categorifying the concept of bundle' to obtain the concept of '2-bundle'. However, most previous work on this subject starts by categorifying the concept of 'sheaf' to obtain the concept of 'stack' — with 'gerbes' as a key special case. We suspect that just as most mathematical physicists prefer bundles to sheaves, they will eventually prefer 2-bundles to gerbes. At present, however, it is crucial to clarify the relation between 2-bundles and gerbes. So, one of the goals of this paper is to relate 2-connections on 2-bundles to the already established notion of connections on gerbes. We begin here by recalling the history of stacks and gerbes, and the concept of a gerbe with connection.

The idea of a stack goes back to Grothendieck [44]. Just as a sheaf over a space M assigns a *set* of sections to any open set $U \subseteq M$, a stack assigns a *category* of sections to any open set $U \subseteq M$. Indeed, one may crudely define a stack as a 'sheaf of categories'. However, all the usual sheaf axioms need to be 'weakened', meaning that instead of equations between objects, we must use isomorphisms satisfying suitable equations of their own. For example, in a sheaf we can obtain a section s over U from sections s_i over open sets U_i covering U when these sections are *equal* on double intersections:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

For a stack, on the other hand, we can obtain a section s over U when the sections s_i are *isomorphic* over double intersections:

$$h_{ij} \colon s_i|_{U_i \cap U_j} \xrightarrow{\sim} s_j|_{U_i \cap U_j},$$

as long as the isomorphisms satisfy the familiar 'cocycle condition' on triple intersections: $h_{ij}h_{jk} = h_{ik}$ on $U_i \cap U_j \cap U_k$.

A good example is the stack of principal H-bundles over M, where H is any fixed Lie group. This associates to each open set $U \subseteq M$ the category whose objects are principal H-bundles over U and whose morphisms are H-bundle isomorphisms. The above cocycle condition is very familiar in this case: it says when we can build a H-bundle s over U by gluing together H-bundles s_i over open sets covering U, using H-valued transition functions h_{ij} defined on double intersections.

This example also motivates the notion of a 'gerbe', which is a special sort of stack introduced by Giraud [45, 46]. For a stack over M to be a gerbe, it must satisfy three properties:

- Its category of sections over any open set must be a *groupoid*: that is, a category where all the morphisms are invertible.
- Each point of M must have a neighborhood over which the groupoid of sections is nonempty.
- Given two sections s, s' over an open set $U \subseteq M$, each point of U must have a neighborhood $V \subseteq U$ such that $s|_V \cong s'|_V$.

It is easy to see that the stack of principal H-bundles satisfies all these conditions. It satisfies another condition as well: for any section s over an open set $U \subseteq M$, each point of U has a neighborhood V such that the automorphisms of $s|_V$ form a group isomorphic to the group of smooth H-valued functions on V. A gerbe of this sort is called an 'H-gerbe'. Sometimes these are called 'nonabelian gerbes', to distinguish them from another class of gerbes that only make sense when the group H is abelian.

There is a precise sense in which the gerbe of principal H-bundles is the 'trivial' H-gerbe. Every H-gerbe is *locally* equivalent to this one, but not globally. So, we can think of a H-gerbe as a thing whose sections look locally like principal H-bundles, but not globally. This viewpoint is emphasized by the concept of 'bundle gerbe', defined first in the abelian case by Murray [16, 17] and more recently in the nonabelian case that concerns us here by Aschieri, Cantini and Jurčo [14].

However, the most concrete way of getting our hands on H-gerbes over M is by gluing together trivial H-gerbes defined on open sets U_i that cover M. This leads to a simple description of H-gerbes in terms of transition functions satisfying cocycle conditions. Now the transition functions defined on double intersections take values not in H but in $G = \mathrm{Aut}(H)$:

$$g_{ij} \colon U_i \cap U_j \to G$$

Moreover, they need not satisfy the usual cocycle condition for triple intersections 'on the nose', but only up to conjugation by certain functions

$$h_{ijk} \colon U_i \cap U_j \cap U_k \to H.$$

In other words, we demand:

$$t(h_{ijk})\, g_{ij} g_{jk} = g_{ik}$$

where $t\colon H \to G$ sends $h \in H$ to the operation of conjugating by h. Finally, the functions h_{ijk} should satisfy an cocycle condition on quadruple intersections:

$$\alpha(g_{ij})(h_{jkl})\ h_{ijl} = h_{ijk}h_{ikl}$$

where α is the natural action of $G = \mathrm{Aut}(H)$ on H. All this can be formalized most clearly using the automorphism 2-group $\mathcal{AUT}(H)$ described in Example 21, since this Lie 2-group has (G, H, t, α) as its corresponding Lie crossed module. Indeed, one way that 2-bundles generalize gerbes is by letting an arbitrary Lie 2-group play the role that $\mathcal{AUT}(H)$ plays here; we call this 2-group the 'structure 2-group' of the 2-bundle.

Given an H-gerbe, we can specify a 'connection' on it by means of some additional local data. We begin by choosing $\mathfrak{g}$-valued 1-forms A_i on the open sets U_i, which describe parallel transport along paths. But these 1-forms need not satisfy the usual consistency condition on double intersections! Instead, they satisfy it only up to $\mathfrak{h}$-valued 1-forms a_{ij}:

$$A_i + dt(a_{ij}) = g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1}.$$

These, in turn, must satisfy a consistency condition on triple intersections:

$$a_{ij} + \alpha(g_{ij})(a_{jk}) = h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + \mathbf{d}h_{ijk}\, h_{ijk}^{-1} + d\alpha(A_i)(h_{ijk})\, h_{ijk}^{-1}.$$

Next, we choose $\mathfrak{h}$-valued 2-forms B_i describing parallel transport along surfaces. These satisfy a consistency condition on double intersections:

$$\alpha(g_{ij})(B_j) = B_i - k_{ij} + b_{ij}\ ,$$

where the $\mathfrak{h}$-valued 2-forms b_{ij} and

$$k_{ij} \equiv \mathbf{d}a_{ij} + a_{ij} \wedge a_{ij} - d\alpha(A_i) \wedge a_{ij}$$

measure the failure of B_i to transform covariantly. The 2-form k_{ij} is essentially the curvature of a_{ij}, while b_{ij} is a new object which turns out to have a transition law of its own, this time on triple intersections:

$$b_{ij} + \alpha(g_{ij})(b_{jk}) = h_{ijk}\, b_{ik}\, h_{ijk}^{-1} + h_{ijk}\, d\alpha(dt(B_i) + F_{A_i})\, (h_{ijk}^{-1})\,.$$

This description of connections on nonabelian gerbes was first given by Breen and Messing [13]. Aschieri, Cantini and Jurčo then gave a similar treatment using bundle gerbes [14].

To summarize, we list the local data for a nonabelian gerbe with connection. For maximum generality, we start with an arbitrary Lie 2-group $\mathcal{G}$ and form its Lie crossed module (G, H, α, t). The definition below reduces to that of Breen, Messing, Aschieri and Jurčo when $\mathcal{G} = \mathcal{AUT}(H)$.

Definition 26. *A* **nonabelian gerbe** *consists of:*

- *a base space M,*

- *an open cover U of M, with $U^{[n]}$ denoting the union of all n-fold intersections of patches in U,*
- *a Lie 2-group $\mathcal{G}$ with Lie crossed module (G, H, α, t) and differential crossed module $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$,*
- **transition functions***:*

$$\begin{aligned} g \colon U^{[2]} &\to G \\ (x,i,j) &\mapsto g_{ij}(x) \in G \end{aligned} \tag{2.11}$$

- **2-transition functions***:*

$$\begin{aligned} h \colon U^{[3]} &\to H \\ (x,i,j,k) &\mapsto h_{ijk}(x) \in H \end{aligned} \tag{2.12}$$

such that the following equations hold:

- *cocycle condition for the transition functions g_{ij}:*

$$g_{ij} g_{jk} = t(h_{ijk})\, g_{ik} \tag{2.13}$$

- *cocycle condition for the 2-transition functions h_{ijk}:*

$$\alpha(g_{ij})(h_{jkl})\; h_{ijl} = h_{ijk}\, h_{ikl} \tag{2.14}$$

Definition 27. *A* **connection** *on a nonabelian gerbe consists of*

- **connection 1-forms***:*

$$\begin{aligned} A &\in \Omega^1(U^{[1]}, \mathfrak{g}) \\ (x,i) &\mapsto A_i(x) \in \mathfrak{g} \end{aligned} \tag{2.15}$$

- **curving 2-forms***:*

$$\begin{aligned} B &\in \Omega^2(U^{[1]}, \mathfrak{h}) \\ (x,i) &\mapsto B_i(x) \in \mathfrak{h} \end{aligned} \tag{2.16}$$

- **connection transformation 1-forms***:*

$$\begin{aligned} a &\in \Omega^1(U^{[1]}, \mathfrak{h}) \\ (x,i,j) &\mapsto a_{ij}(x) \end{aligned} \tag{2.17}$$

- **curving transformation 2-forms***:*

$$\begin{aligned} d &\in \Omega^2(U^{[2]}, \mathfrak{h}) \\ (x,i,j) &\mapsto b_{ij}(x) \end{aligned} \tag{2.18}$$

such that the following equations hold:

- *cocycle condition for the connection 1-forms* A_i*:*

$$A_i + dt(a_{ij}) \;=\; g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1} \tag{2.19}$$

- *cocycle condition for the curving 2-forms* B_i*:*

$$B_i \;=\; \alpha(g_{ij})(B_j) + k_{ij} - b_{ij}\,. \tag{2.20}$$

  where

$$k_{ij} \;\equiv\; \mathbf{d} a_{ij} + a_{ij} \wedge a_{ij} - d\alpha(A_i) \wedge a_{ij} \tag{2.21}$$

- *cocycle condition for the connection transformation 1-forms* a_{ij}*:*

$$a_{ij} + \alpha(g_{ij})(a_{jk}) = h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + \mathbf{d} h_{ijk}\, h_{ijk}^{-1} + d\alpha(A_i)(h_{ijk})\, h_{ijk}^{-1}. \tag{2.22}$$

- *cocycle condition for the curving transformation 2-forms* b_{ij}*:*

$$b_{ij} + \alpha(g_{ij})(b_{jk}) \;=\; h_{ijk}\, b_{ik}\, h_{ijk}^{-1} + h_{ijk}\, d\alpha(dt(B_i) + F_{A_i})\,(h_{ijk}^{-1}) \tag{2.23}$$

Given a nonabelian gerbe with connection, the quantities

$$\tilde{F}_i := dt(B_i) + F_{A_i}$$

which appear in equation (2.23) above are called the **fake curvature 2-forms**. These will play a major role in our work.

3. 2-Bundles

Bartels [6] obtained a concept of '2-bundle' by categorifying Steenrod's definition of bundles in terms of local gluing data [49]. Here we take an initially different but ultimately equivalent approach. First, in §3.1, we define 2-bundles in terms of local trivializations. From this, we work out their description in terms of local gluing data in §3.2. This allows us to define 'principal' 2-bundles.

3.1 Locally Trivial 2-Bundles

In differential geometry an ordinary bundle consists of two smooth spaces, the **total space** E and the **base space** B, together with a **projection map**

$$E \xrightarrow{p} B\,.$$

To categorify the theory of bundles, we start by replacing smooth spaces by smooth 2-spaces:

Definition 28. *A* **2-bundle** *consists of*

- *a 2-space E (the* **total 2-space***),*
- *a 2-space B (the* **base 2-space***),*
- *a smooth map $p\colon E \to B$ (the* **projection***).*

In gauge theory we are interested in *locally trivial* 2-bundles. Ordinarily, a locally trivial bundle with fiber F is a bundle $E \xrightarrow{p} B$ together with an open cover U_i of B, such that the restriction of E to any of the U_i is equipped with an isomorphism to the trivial bundle $U_i \times F \to U_i$. To categorify this, we would need to define a '2-cover' of the base 2-space B. This is actually a rather tricky issue, since forming the 'union' of 2-spaces requires knowing how to compose a morphism in one 2-space with a morphism in another. While this issue can be addressed, we prefer to avoid it here by assuming that B is just an ordinary smooth space. In Example 11 we saw that any smooth space can be seen as a 'trivial' 2-space: one with only identity morphisms. So, in what follows we restrict our attention to this case.

Definition 29. *Given an open cover $\{U_i\}_{i\in I}$ of a smooth space B, we define the* **space of n-fold intersections** *to be the disjoint union:*

$$U^{[n]} = \coprod_{i_1,i_2,\ldots,i_n\in I} U_{i_1}\cap U_{i_2}\cap\ldots\cap U_{i_n}\,.$$

We write a point in $U^{[n]}$ as $(i_1,\ldots,i_n,x)$ if it lies in $U_{i_1}\cap\ldots\cap U_{i_n}$. The space $U^{[n]}$ comes with maps

$$\begin{aligned} j_{01\cdots(k-1)(k+1)\cdots n}\colon U^{[n]} &\to U^{[n-1]} \\ (i_1,\ldots,i_n,x) &\mapsto (i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n,x) \end{aligned} \tag{3.1}$$

that forget about the kth member of the n-fold intersection.

We can now state the definition of a locally trivial 2-bundle. First note that we can restrict a 2-bundle $E \xrightarrow{p} B$ to any subspace $U \subseteq B$ to obtain a 2-bundle which we denote by $E|_U \xrightarrow{p} U$. Then:

Definition 30. *Given a 2-space F, we define a* **locally trivial 2-bundle with fiber** *F to be a 2-bundle $E \xrightarrow{p} B$ and a cover U of the base space B equipped with equivalences*

$$P|_{U_i} \xrightarrow{t_i} U_i \times F$$

called **local trivializations** *such that these diagrams:*

$$\begin{tikzcd}
P|_{U_i} \arrow[rr, "t_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl] \\
& U_i &
\end{tikzcd}$$

commute for all $i \in I$.

Readers wise in the ways of categorification may ask why we did not merely require that these diagrams commute *up to natural isomorphism*. The reason is that U_i, as an ordinary space, has only identity morphisms when we regard it as a 2-space. Thus, for this diagram to commute up to natural isomorphism, it must commute 'on the nose'.

Readers less wise in the ways of categorification may find the above definition painfully abstract. In the next section, we translate its meaning into local gluing data — in other words, data that specify how to build a locally trivial 2-bundle from trivial ones over the patches U_i. In order to do this, we first need to extract *transition functions* from the local trivializations.

3.2 2-Bundles in Terms of Local Data

Suppose $E \xrightarrow{p} B$ is a locally trivial 2-bundle with fiber F. This means that B is equipped with an open cover U and for each open set U_i in the cover we have a local trivialization

$$t_i \colon P|_{U_i} \to U_i \times F$$

which is an equivalence. By Def. 15 this means that t_i is equipped with a specified weak inverse

$$\bar{t}_i \colon U_i \times F \to P|_{U_i}$$

together with invertible 2-maps

$$\tau_i \colon \bar{t}_i t_i \Rightarrow 1$$
$$\bar{\tau}_i \colon t_i \bar{t}_i \Rightarrow 1$$

In particular, this means that $\bar{t}_i$ is also an equivalence.

Now consider a double intersection $U_{ij} = U_i \cap U_j$. The composite of equivalences is again an equivalence, so we get an **autoequivalence**

$$t_j\bar{t}_i \colon U_{ij} \times F \to U_{ij} \times F$$

that is, an equivalence from this 2-space to itself. By the commutative diagram in Def. 30, this autoequivalence must act trivially on the U_{ij} factor, so

$$t_j\bar{t}_i(x, f) = (x, fg_{ij}(x))$$

for some smooth function g_{ij} from U_{ij} to the smooth space of autoequivalences of the fiber F. Note that we write these autoequivalences as acting on F from the right, as customary in the theory of bundles. We call the functions g_{ij} **transition functions**, since they are just categorified versions of the usual transition functions for locally trivial bundles.

In fact, for any smooth 2-space F there is a smooth 2-space $\mathcal{AUT}(F)$ whose objects are autoequivalences of F and whose morphisms are invertible 2-maps between these. The transition functions are maps

$$g_{ij} \colon U_{ij} \to \mathrm{Ob}(\mathcal{AUT}(F)).$$

The 2-space $\mathcal{AUT}(F)$ is a kind of 2-group, with composition of autoequivalences giving the product. However, is not the sort of 2-group we have been considering here, because it does not have 'strict inverses': the group laws involving inverses do not hold as equations, but only up to specified isomorphisms that satisfy coherence laws of their own. So, $\mathcal{AUT}(F)$ is a 'coherent' smooth 2-group in the sense of Baez and Lauda [3].

Next, consider a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$. In an ordinary locally trivial bundle the transition functions satisfy the equation $g_{ij}g_{jk} = g_{ik}$, but in a locally trivial 2-bundle this holds *only up to isomorphism.* In other words, there is a smooth map

$$h_{ijk} \colon U_{ijk} \to \mathrm{Mor}(\mathcal{AUT}(F))$$

such that for any $x \in U_{ijk}$,

$$h_{ijk}(x) \colon g_{ij}(x)g_{jk}(x) \xrightarrow{\sim} g_{ik}(x).$$

To see this, note that there is an invertible 2-map

$$t_k\tau_j\bar{t}_i : t_k\bar{t}_jt_j\bar{t}_i \Rightarrow t_k\bar{t}_i$$

defined by horizontally composing τ_j with t_k on the left and $\bar{t}_i$ on the right. Since

$$t_k\bar{t}_jt_j\bar{t}_i(x, f) = (x, fg_{ij}(x)g_{jk}(x))$$

while

$$t_k\bar{t}_i(x, f) = (x, fg_{ik}(x))$$

we have

$$t_k\tau_j\bar{t}_i(x, f) \colon (x, fg_{ij}(x)g_{jk}(x)) \to (x, fg_{ik}(x)).$$

Since this morphism must be the identity on the first factor, we have

$$t_k \tau_j \bar{t}_i(x,f) = (1_x, f h_{ijk}(x))$$

where $h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$ depends smoothly on x.

Similarly, in a locally trivial bundle we have $g_{ii} = 1$, but in a locally trivial 2-bundle there is a smooth map

$$k_i \colon U_i \to \mathrm{Mor}(\mathcal{AUT}(F))$$

such that for any $x \in U_i$,

$$k_i(x) \colon g_{ii}(x) \to 1.$$

To see this, recall that there is an invertible 2-map

$$\bar{\tau}_i \colon t_i \bar{t}_i \Rightarrow 1.$$

Since

$$t_i \bar{t}_i(x,f) = (x, f g_{ii}(x))$$

we have

$$\bar{\tau}_i(x,f) \colon (x, f g_{ii}(x)) \to (x,f),$$

and since this morphism must be the identity on the first factor, we have

$$\bar{\tau}_i(x,f) = (1_x, f k_i(x))$$

where $k_i(x) \colon g_{ii}(x) \to 1$ depends smoothly on x.

In short, the transition functions g_{ij} for a locally trivial 2-bundle satisfy the usual cocycle conditions up to specified isomorphisms h_{ijk} and k_i, which we call **2-transition functions**. These in turn, satisfy some cocycle conditions of their own:

Theorem 31. *Suppose $E \xrightarrow{p} B$ is a locally trivial 2-bundle with fiber F, and define the functions*

$$g_{ij} \colon U_{ij} \to \mathrm{Ob}(\mathcal{AUT}(F))$$
$$h_{ijk} \colon U_{ijk} \to \mathrm{Mor}(\mathcal{AUT}(F))$$
$$k_i \colon U_i \to \mathrm{Mor}(\mathcal{AUT}(F))$$

as above. Then:

- *h makes this diagram, called the* **associative law**, *commute for any $x \in U_{ijkl}$:*

$$\begin{tikzcd}
 & g_{ij}(x)\, g_{kl}(x)\, g_{lm}(x) \arrow[dl, "h_{ijk}(x)\, g_{kl}(x)"'] \arrow[dr, "g_{ij}(x)\, h_{jkl}(x)"] & \\
g_{il}(x)\, g_{lm}(x) \arrow[dr, "h_{ikl}(x)"'] & & g_{ij}(x)\, g_{jm}(x) \arrow[dl, "h_{ijl}(x)"] \\
 & g_{im}(x) &
\end{tikzcd}$$

- *k makes these diagrams, called the* **left and right unit laws***, commute for any $x \in U_{ij}$:*

$$\begin{tikzcd}
g_{ii}(x)\,g_{ij}(x) \arrow[r, "k_i(x)\,g_{ij}(x)"] \arrow[dr, "h_{iij}(x)"'] & 1\,g_{ij}(x) \arrow[d, "="] \\
& g_{ij}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
g_{ij}(x)\,1 \arrow[d, "="'] & g_{ij}(x)\,g_{jj}(x) \arrow[l, "g_{ij}(x)\,k_j(x)"'] \arrow[dl, "h_{ijj}(x)"] \\
g_{ij}(x) &
\end{tikzcd}$$

Proof. These are straightforward computations using the definitions of g, h, and k in terms of $t, \bar{t}$, τ and $\bar{\tau}$. ☐

The associative law and unit laws are analogous to those which hold in a monoid. They also have simplicial interpretations. In a locally trivial bundle, the transition functions give a commuting triangle for any triple intersection:

$$\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & & \bullet
\end{tikzcd}$$

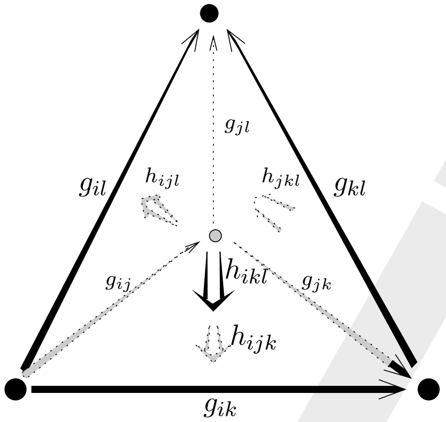
In a locally trivial 2-bundle, such triangles commute only up to isomorphism:

$$\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}"] \arrow[d, phantom, "h_{ijk}\Downarrow"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & {} & \bullet
\end{tikzcd}$$

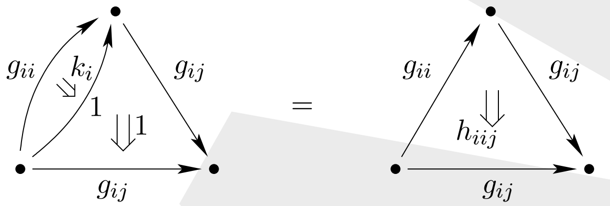
However, the associative law gives the following equation:

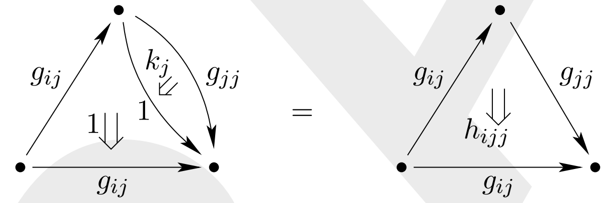
$$\begin{tikzcd}
\bullet \arrow[r, "g_{jk}"] \arrow[dr, phantom, "h_{jkl}\Searrow" near start, "h_{ikl}\Downarrow"' near end] & \bullet \arrow[d, "g_{kl}"] \arrow[dr, phantom, "="] & & \bullet \arrow[r, "g_{jk}"] \arrow[dr, "g_{il}"] & \bullet \arrow[d, "g_{kl}"] \\
\bullet \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{ik}"'] & \bullet & & \bullet \arrow[u, "g_{ij}"] \arrow[r, "g_{ik}"'] \arrow[ur, phantom, "\Downarrow h_{ijl}" near start, "h_{ijk}\Swarrow"' near end] & \bullet
\end{tikzcd}$$

We can think of the two sides of this equation as the front and back of a tetrahedron. So, in terms of the 3-dimensional diagrams familiar in 2-category theory, we obtain a commuting tetrahedron for each quadruple intersection:



We can also visualize the left and right unit laws simplicially, but they involve degenerate tetrahedra:





The associative law is familiar in the theory of gerbes, but the freedom of having $k_i \neq 1$ is not usually considered there, so the left and right unit laws usually go unmentioned. Gerbe cocycles are usually defined in terms of Čech cohomology and hence antisymmetric in the indices $i, j, k, \ldots$, in the sense that group-valued functions go their inverses upon an odd permutation of these indices. Whenever we derive nonabelian gerbe cocycles from 2-bundles with 2-connection we will thus have to restrict to case where $k_i = 1$ for all i. We suspect that in some sense every locally trivial 2-bundle is equivalent to one of this type. Proving this would require that we construct a 2-category of locally trivial 2-bundles.

We are now almost in a position to define $\mathcal{G}$-2-bundles for any smooth 2-group $\mathcal{G}$, and principal $\mathcal{G}$-2-bundles. We only need to understand how a 2-group can 'act' on a 2-space:

Definition 32. *A* **(strict) action** *of a smooth 2-group $\mathcal{G}$ on a smooth 2-space F is a smooth homomorphism*

$$\alpha \colon \mathcal{G} \to \mathcal{AUT}(F),$$

that is, a smooth map that preserves products and inverses.

Note in particular that every smooth 2-group has an action on itself via right multiplication.

Definition 33. *For any smooth 2-group $\mathcal{G}$, we say a 2-bundle $E \to M$ has $\mathcal{G}$ as its* **structure 2-group** *when g_{ij}, h_{ijk}, and k_i factor through an action $\mathcal{G} \to \mathcal{AUT}(F)$. In this case we also say P is a* **$\mathcal{G}$-2-bundle***. If furthermore $F = \mathcal{G}$ and $\mathcal{G}$ acts on F by right multiplication, we say P is a* **principal $\mathcal{G}$-2-bundle***.*

In other words, for a principal $\mathcal{G}$-2-bundle we can think of the functions g_{ij}, h_{ijk} and k_i as taking values in $\mathcal{G}$.

Summary. We may categorify the ordinary concept of a bundle and obtain the notion of '2-bundle' by replacing spaces everywhere by 2-spaces. A 2-bundle differs from an ordinary bundle essentially in that the fibers are categories instead of sets. Equations such as the cocycle condition for transition functions are thus naturally replaced by isomorphisms which satisfy new cocycle conditions of their own.

So far all of this pertained to 2-bundles (and nonabelian gerbes) not equipped with any sort of connection. In what follows, we categorify the notion of a connection for a principal bundle to obtain the notion of '2-connection'. We determine when these 2-connections allow for parallel transport along surfaces in a parametrization-independent manner, just as ordinary parallel transport along a curve does not depend on the parametrization of the curve. We also relate our concept of 2-connection to the concept of connection on a nonabelian gerbe.

4. 2-Connections

In this section we define the concept of '2-connection' and state our main result, Theorem 39, which describes 2-connections in terms of Lie-algebra-valued differential forms. The proof of this result occupies the rest of the paper.

4.1 Path Groupoids and 2-Groupoids

For a trivial bundle, the holonomy of a connection assigns elements of the structure group to paths in space. Similarly, 2-holonomy of a 2-connection assigns objects and morphisms of the structure 2-group to paths and surfaces in space. To make this precise we need the notion of a 'path 2-groupoid'.

We described the path groupoid of a smooth space U in Example 12. This has points of U as objects:

$$\bullet\, x$$

and thin homotopy classes of paths in U as morphisms:

$$\begin{tikzcd} x \bullet \arrow[r, bend left, "\gamma"] & \bullet\, y \end{tikzcd}$$

The path 2-groupoid also has 2-morphisms, which are thin homotopy classes of surfaces like this:

$$\begin{tikzcd} x \bullet \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_2"'{name=D}] & \bullet\, y \arrow[Rightarrow, from=U, to=D, "\Sigma"] \end{tikzcd}$$

We call these 'bigons':

Definition 34. *Given a smooth space M, a* **parametrized bigon** *in M is a smooth map*

$$\Sigma\colon [0,1]^2 \to M$$

which is constant near $s = 0$, constant near $s = 1$, independent of t near $t = 0$, and independent of t near $t = 1$. We call $\Sigma(\cdot, 0)$ the **source** *of the parametrized bigon Σ, and $\Sigma(\cdot, 1)$ the* **target**. *If Σ is a parametrized bigon with source γ_1 and target γ_2, we write $\Sigma\colon \gamma_1 \to \gamma_2$.*

Definition 35. *Suppose $\Sigma\colon \gamma_1 \to \gamma_2$ and $\Sigma'\colon \gamma_1' \to \gamma_2'$ are parametrized bigons in a smooth space M. A* **thin homotopy** *between Σ and Σ' is a smooth map*

$$H\colon [0,1]^3 \to M$$

with the following properties:

- *$H(s,t,0) = \Sigma(s,t)$ and $H(s,t,1) = \Sigma'(s,t)$,*
- *$H(s,t,u)$ is independent of u near $u = 0$ and near $u = 1$,*

- *For some thin homotopy F_1 from γ_1 to γ_1', $H(s,t,u) = F_1(s,u)$ for t near 0, and for some thin homotopy F_2 from γ_2 to γ_2', $H(s,t,u) = F_2(s,u)$ for t near 1,*
- *$H(s,t,u)$ is constant for $s = 0$ and near $s = 1$,*
- *H does not sweep out any volume: the rank of the differential $dH(s,t,u)$ is < 3 for all $s,t,u \in [0,1]$.*

A **bigon** *is a thin homotopy class $[\Sigma]$ of parametrized bigons.*

Definition 36. *The* **path 2-groupoid** *$\mathcal{P}_2(M)$ of a smooth space M is the 2-category in which:*

- *objects are points $x \in M$:*

$$\bullet\, x$$

- *morphisms are thin homotopy classes of paths γ in M that are constant near $s = 0$ and $s = 1$:*

$$\begin{tikzcd} x \arrow[r, bend left, "{[\gamma]}"] & y \end{tikzcd}$$

- *2-morphisms are bigons in M*

$$\begin{tikzcd} x \arrow[r, bend left=40, "{[\gamma_1]}", ""{name=U, below}] \arrow[r, bend right=40, "{[\gamma_2]}"', ""{name=D}] & y \arrow[Rightarrow, from=U, to=D, "{[\Sigma]}"] \end{tikzcd}$$

and whose composition operations are defined as:

- $$\begin{tikzcd} x \arrow[r, bend left, "{[\gamma_1]}"] & y \arrow[r, bend left, "{[\gamma_2]}"] & z \arrow[r, phantom, "="] & x \arrow[r, bend left, "{[\gamma_1 \circ \gamma_2]}"] & z \end{tikzcd}$$

  where

$$(\gamma_1 \circ \gamma_2)(s) := \begin{cases} \gamma_1(2s) & \textit{for } 0 \le s \le 1/2 \\ \gamma_2(2s-1) & \textit{for } 1/2 \le s \le 1 \end{cases}$$

- $$\begin{tikzcd} x \arrow[r, bend left=60, "{[\gamma_1]}", ""{name=A, below}] \arrow[r, "{[\gamma_2]}" description, ""{name=B, above}, ""{name=C, below}] \arrow[r, bend right=60, "{[\gamma_3]}"', ""{name=D}] & y \arrow[Rightarrow, from=A, to=B, "{[\Sigma_1]}"] \arrow[Rightarrow, from=C, to=D, "{[\Sigma_2]}"] \arrow[r, phantom, "="] & x \arrow[r, bend left=40, "{[\gamma_1]}", ""{name=E, below}] \arrow[r, bend right=40, "{[\gamma_3]}"', ""{name=F}] & y \arrow[Rightarrow, from=E, to=F, "{[\Sigma_1 \Sigma_2]}"] \end{tikzcd}$$

  where

$$(\Sigma_1 \Sigma_2)(s,t) := \begin{cases} \Sigma_1(s,2t) & \textit{for } 0 \le t \le 1/2 \\ \Sigma_2(s,2t-1) & \textit{for } 1/2 \le t \le 1 \end{cases}$$

- $$\begin{tikzcd} x \arrow[r, bend left, "{[\gamma_1]}"] \arrow[r, bend right, "\gamma'_1"'] \arrow[r, phantom, "\Downarrow {[\Sigma_1]}"] & y \arrow[r, bend left, "{[\gamma_2]}"] \arrow[r, bend right, "{[\gamma'_2]}"'] \arrow[r, phantom, "\Downarrow {[\Sigma_2]}"] & z \arrow[r, phantom, "="] & x \arrow[r, bend left, "{[\gamma_1 \circ \gamma_2]}"] \arrow[r, bend right, "{[\gamma'_1 \circ \gamma'_2]}"'] \arrow[r, phantom, "\Downarrow {[\Sigma_1 \circ \Sigma_2]}"] & z \end{tikzcd}$$

  where

$$(\Sigma_1 \circ \Sigma_2)(s,t) := \begin{cases} \Sigma_1(2s,t) & \textit{for } 0 \le s \le 1/2 \\ \Sigma_2(2s-1,t) & \textit{for } 1/2 \le s \le 1 \end{cases}$$

One can check that these operations are well-defined, where for vertical composition we must choose suitable representatives of the bigons being composed. One can also check that $\mathcal{P}_2(M)$ is indeed a 2-category. Furthermore, the objects, morphisms and 2-morphisms in $\mathcal{P}_2(M)$ all form smooth spaces, by an elaboration of the ideas in Example 12, and all the 2-category operations are then smooth maps. We thus say $\mathcal{P}_2(M)$ is a **smooth 2-category**: that is, a 2-category in C^∞. Indeed, the usual definitions [48] of 2-category, 2-functor, pseudonatural transformation, and modification can all be internalized in C^∞, and we use these 'smooth' notions in what follows. Furthermore, both morphisms and 2-morphisms in $\mathcal{P}_2(M)$ have strict inverses, and the operations of taking inverses are smooth, so we say $\mathcal{P}_2(M)$ is a **smooth 2-groupoid**.

4.2 Holonomy Functors and 2-Functors

We obtain the notion of '2-connection' by categorifying the usual notion of connection. To do this, it is useful to describe connections in terms of parallel transport. By trivializing a principal G-bundle over open sets U_i covering the base space, we can think of parallel transport as giving a functor from the path groupoid of each of these open sets to G. These functors must agree on the intersections $U_{ij} = U_i \cap U_j$ in the following sense:

Theorem 37. *For any smooth group G and smooth space B, suppose $E \to B$ is a principal G-bundle equipped with local trivializations over open sets $\{U_i\}_{i \in I}$ covering B. Let g_{ij} be the transition functions. Then there is a one-to-one correspondence between connections on E and data of the following sort:*

- *for each $i \in I$ a smooth map between smooth 2-spaces:*

$$\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$$

  called the **local holonomy functor**, *from the path groupoid of U_i to the group G regarded as a smooth 2-space with a single object $\bullet$,*

such that:

- *for each $i, j \in I$, the transition function g_{ij} defines a smooth natural isomorphism:*

$$\mathrm{hol}_i|_{U_{ij}} \xrightarrow{g_{ij}} \mathrm{hol}_j|_{U_{ij}}$$

  called the **transition natural isomorphism**. *In other words, this diagram commutes:*

$$\begin{tikzcd}
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "\mathrm{hol}_i(\gamma)"'] & \bullet \arrow[d, "\mathrm{hol}_j(\gamma)"] \\
\bullet \arrow[r, "g_{ij}(y)"'] & \bullet
\end{tikzcd}$$

for any path $x \xrightarrow{\gamma} y$ *in* U_{ij}.

Proof. To avoid distracting the reader with technicalities we postpone the proof to Proposition 42. ☐

In addition, it is worth noting that whenever we have a connection, for each $i, j, k \in I$ this triangle commutes:

$$\begin{tikzcd}
 & \mathrm{hol}_j \arrow[dr, "g_{jk}"] & \\
\mathrm{hol}_i \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & & \mathrm{hol}_k
\end{tikzcd}$$

Starting with the above definition, by differentiating one obtains this familiar result:

- The local holonomy functors hol_i are specified by 1-forms

  $$A_i \in \Omega^1(U_i, \mathfrak{g}) \ .$$

- The transition natural isomorphisms g_{ij} are specified by smooth functions

  $$g_{ij} \colon U_{ij} \to G \,,$$

  satisfying the equation

  $$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1}$$

  on U_{ij}.

- The commuting triangle for the triple intersection U_{ijk} is equivalent to the equation

  $$g_{ij} g_{jk} = g_{ik}$$

  on U_{ijk}.

Categorifying all this, we make the following definition:

Definition 38. *For any smooth 2-group $\mathcal{G}$ and any smooth space B, suppose that $E \to B$ is a principal $\mathcal{G}$-2-bundle equipped with local trivializations over open sets $\{U_i\}_{i\in I}$ covering B. Let the transition functions g_{ij}, h_{ijk} and k_i be given as in Theorem 31. Suppose for simplicity that $k_i = 1$. Then a* **2-connection** *on E consists of the following data:*

- *for each $i \in I$ a smooth 2-functor*

$$\begin{array}{cccc} \mathrm{hol}_i : & \mathcal{P}_2(U_i) & \to & \mathcal{G} \end{array}$$

$$\begin{tikzcd} x \arrow[r, bend left=40, "\gamma"{name=U}] \arrow[r, bend right=40, "\eta"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\Sigma"] & y \end{tikzcd} \mapsto \begin{tikzcd} \bullet \arrow[r, bend left=40, "\mathrm{hol}_i(\gamma)"{name=U}] \arrow[r, bend right=40, "\mathrm{hol}_i(\eta)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\mathrm{hol}_i(\Sigma)"] & \bullet \end{tikzcd}$$

  called the **local holonomy 2-functor***, from the path 2-groupoid $\mathcal{P}_2(U_i)$ to the 2-group $\mathcal{G}$ regarded as a smooth 2-category with a single object $\bullet$,*

such that:

- *For each i, j a pseudonatural isomorphism:*

$$g_{ij} \colon \mathrm{hol}_i|_{\mathcal{P}(U_i \cap U_j)} \to \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

  extending the transition function g_{ij}. In other words, for each path $\gamma \colon x \to y$ in $U_i \cap U_j$ a morphism in G:

$$\begin{tikzcd}[row sep=huge, column sep=huge] \bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "\mathrm{hol}_i(\gamma)"'] & \bullet \arrow[d, "\mathrm{hol}_j(\gamma)"] \\ \bullet \arrow[r, "g_{ij}(y)"'] \arrow[ur, phantom, "\overset{g_{ij}(\gamma)}{\Nearrow}"] & \bullet \end{tikzcd}$$

  depending smoothly on γ, such that this diagram commutes:

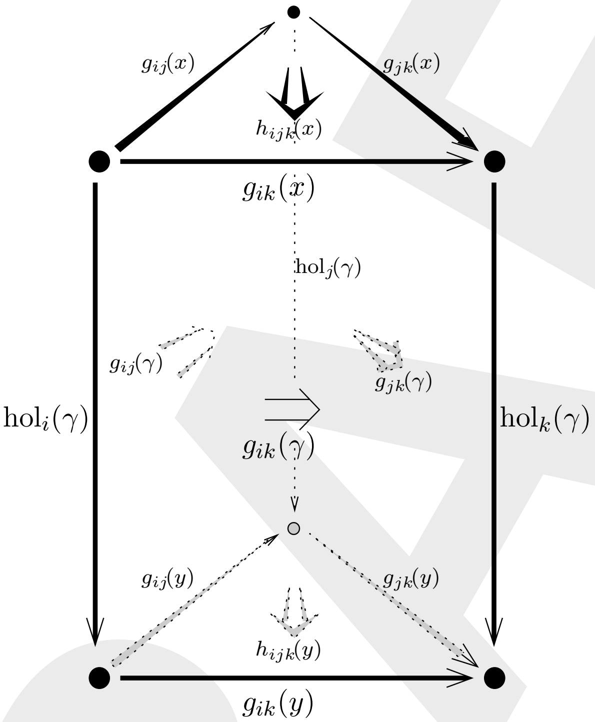
$$\begin{tikzcd}[row sep=7em, column sep=10em] \bullet \arrow[r, "g_{ij}(x)"] \arrow[d, bend right=50, "\mathrm{hol}_i(\gamma)"'{name=A}] \arrow[d, dotted, bend left=30, "\mathrm{hol}_i(\eta)"{name=B}] \arrow[Rightarrow, dotted, from=A, to=B, "\mathrm{hol}_i(\Sigma)"] & \bullet \arrow[d, bend right=30, "\mathrm{hol}_j(\gamma)"'{name=C}] \arrow[d, bend left=50, "\mathrm{hol}_j(\eta)"{name=E}] \arrow[Rightarrow, from=C, to=E, "\mathrm{hol}_j(\Sigma)"] \\ \bullet \arrow[r, "g_{ij}(y)"'] \arrow[ur, phantom, "\overset{g_{ij}(\gamma)}{\Nearrow}\quad \underset{g_{ij}(\eta)}{\Swarrow}"] & \bullet \end{tikzcd}$$

  for any bigon $\Sigma \colon \gamma \Rightarrow \eta$ in U_{ij},

- *for each $i, j, k \in I$ the transition function h_{ijk} defines a modification:*

$$\begin{tikzcd}
& \mathrm{hol}_j \arrow[dr, "g_{jk}"] \arrow[d, phantom, "h_{ijk} \Downarrow"] & \\
\mathrm{hol}_i \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & & \mathrm{hol}_k
\end{tikzcd}$$

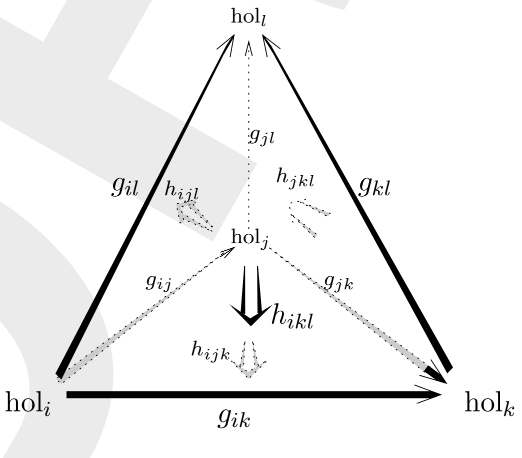
In other words, this diagram commutes:



(4.1)

for any bigon $\Sigma\colon \gamma \Rightarrow \eta$ in U_{ijk}.

In addition, it is worth noting that whenever we have a 2-connection, for each $i, j, k, l \in I$ this tetrahedron commutes:



In analogy to the situation for ordinary connections on bundles, one would like to obtain 2-connections from Lie-algebra-valued differential forms. This is our next result. In what follows, (G, H, t, α) will be the smooth crossed module corresponding to the smooth 2-group $\mathcal{G}$. We think of the transition function g_{ij} as taking values in $\mathrm{Ob}(\mathcal{G}) = G$, and think of h_{ijk} as taking values in H. Actually h_{ijk} takes values in $\mathrm{Mor}(\mathcal{G}) \cong G \rtimes H$, but its G component is determined by its source, so only its H component is interesting. In these terms, the fact that

$$h_{ijk}(x)\colon g_{ij}(x)g_{jk}(x) \xrightarrow{\sim} g_{ik}(x)$$

translates into the equation

$$g_{ij}(x)\, g_{jk}(x)\, t(h_{ijk}(x)) = g_{ik}(x),$$

and the associative law of Theorem 31 (i.e. the above tetrahedron) becomes a cocycle condition familiar from the theory of nonabelian gerbes:

$$h_{ijk}\, h_{ikl} = \alpha(g_{ij})(h_{jkl})\, h_{ijl}\,. \tag{4.2}$$

Theorem 39. *For any smooth 2-group $\mathcal{G}$, suppose that $E \to B$ is a principal $\mathcal{G}$-2-bundle equipped with local trivializations over open sets $\{U_i\}_{i\in I}$ covering B, with the transition functions g_{ij}, h_{ijk} and k_i given as in Theorem 31. Suppose for simplicity that $k_i = 1$. Let (G, H, t, α) be the smooth crossed module corresponding to $\mathcal{G}$, and let $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ be the corresponding differential crossed module. Then there is a one-to-one correspondence between 2-connections on E and Lie-algebra-valued differential forms (A_i, B_i, a_{ij}) satisfying certain equations, as follows:*

1. *The local holonomy 2-functor hol_i is specified by differential forms*

   $$A_i \in \Omega^1(U_i, \mathfrak{g})$$
   $$B_i \in \Omega^2(U_i, \mathfrak{h})$$

   satisfying

   $$F_{A_i} + dt(B_i) = 0\,,$$

   where $F_{A_i} = \mathbf{d}A_i + A_i \wedge A_i$ is the curvature 2-form of A_i.

2. *The transition pseudonatural isomorphism $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ is specified by the transition functions g_{ij} together with differential forms*

   $$a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$$

   satisfying the equations:

   $$\begin{aligned} A_i &= g_{ij} A_j g_{ij}^{-1} + g_{ij}\mathbf{d}g_{ij}^{-1} \\ B_i &= \alpha(g_{ij})(B_i) + k_{ij} \end{aligned}$$

   on U_{ij}, where

   $$k_{ij} = \mathbf{d}a_{ij} + a_{ij} \wedge a_{ij} + d\alpha(A_i) \wedge a_{ij}\,.$$

3. *The modification* $g_{ij} \circ g_{jk} \xrightarrow{h_{ijk}} g_{ik}$ *is specified by the transition functions* h_{ijk}. *For this, the differential forms* a_{ij} *are required to satisfy the equation:*

$$a_{ij} + \alpha(g_{ij}) a_{jk} = h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + \mathbf{d}h_{ijk}\, h_{ijk}^{-1} + d\alpha(A_i)(h_{ijk})\, h_{ijk}^{-1}.$$

on U_{ijk}.

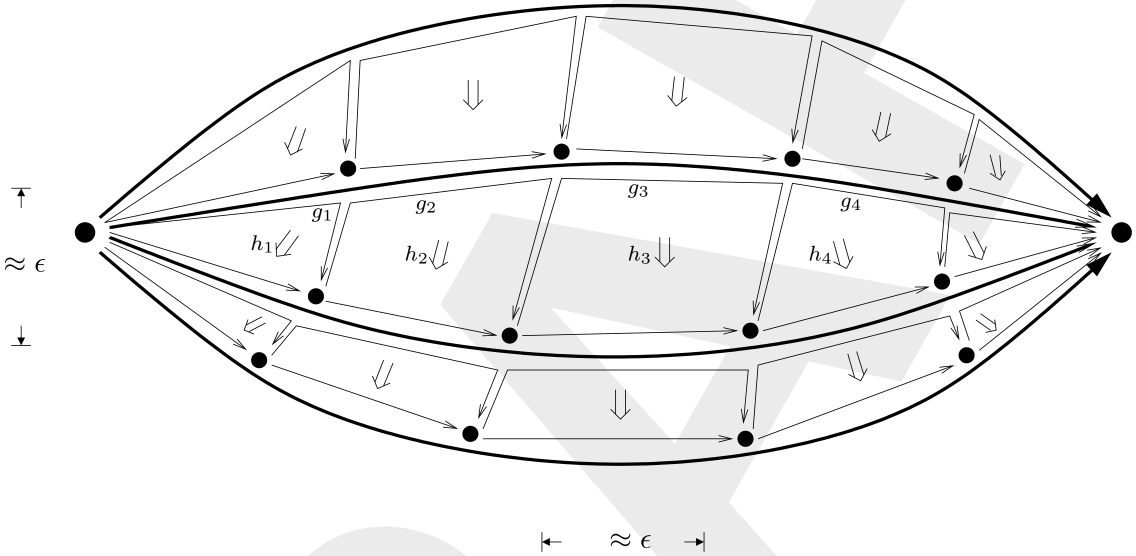
Proof. The idea behind the Proof of part 1. is sketched in §5.1. The full proof is the content of §6.5. The proof for part 2. is given in §5.2. Again, the idea is quite simple, but the proof needs some facts only developed in §6. The proof of part 3. appears in §5.3. Part 4. was discussed before in §3.2. □

5. 2-Connections from Local Data

In this subsection we sketch the proof of Theorem 39 in a way that points out the underlying mechanisms. Several technicalities that these proofs rely on are then discussed in detail in §6.

5.1 Definition on Single Overlaps

Consider any bigon Σ in a patch U_i, i.e. a 2-morphism in $\mathcal{P}_2(U_i)$ (Def. 36), and consider a local 2-holonomy functor $\mathrm{hol}_i\colon \mathcal{P}_2(U_i) \to \mathcal{G}$ (Def. 38). Since hol_i is a functor, the 2-group 2-morphism which it associates to Σ can be computed by dividing Σ into many small sub-bigons, evaluating hol_i on each of these and composing the result in $\mathcal{G}$. This is illustrated in the following sketchy figure.



Here the j-th 2-morphism is supposed to be given by

$$\mathrm{hol}(\Sigma_j) = (g_j, h_j) \in \mathcal{G}$$

with $g \in G$ and $h \in H$. By the rules of 2-group multiplication (Prop. 18) the total horizontal product

$$(g^{\mathrm{tot}}, h^{\mathrm{tot}}) \equiv (g_1, h_1) \cdot (g_2, h_2) \cdot (g_3, h_3) \cdots$$

of all these 2-morphisms is given by

$$\begin{aligned} g^{\mathrm{tot}} &= g_1\, g_2\, g_3 \cdots g_N \\ h^{\mathrm{tot}} &= h_1\, \alpha(g_1)(h_2)\ \alpha(g_1 g_2)(h_3) \cdots \alpha(g_1 g_2 g_3 \cdots g_{N-1})(h_N)\ . \end{aligned}$$

The products of the g_j can be addressed as a *path holonomy* along the upper edges, which, for reasons to become clear shortly, we shall write as

$$g_1\, g_2 \cdots g_j \equiv (W_{j+1})^{-1}\, .$$

Now suppose the group elements come from algebra elements $A_j \in \mathfrak{g}$ and $B_j \in \mathfrak{h}$ as

$$\begin{aligned} g_j &\equiv \exp(\epsilon A_j) \\ h_j &\equiv \exp\left(\epsilon^2 B_j\right) \end{aligned} \tag{5.1}$$

where

$$\epsilon \equiv 1/N \,,$$

then

$$h^{\mathrm{tot}} = 1 + \epsilon^2 \sum_{j=1}^{N} \alpha\left(W_j^{-1}\right)(B_j) + \mathcal{O}\left(\epsilon^4\right) \,.$$

Using the notation

$$\begin{aligned} W_j &\equiv W(1 - j\epsilon, 1) \\ B_j &\equiv B(1 - \epsilon j) \end{aligned}$$

we have

$$h^{\mathrm{tot}} = 1 + \epsilon \int_0^1 d\sigma \; \alpha\left(W^{-1}(\sigma,1)\right)(B(\sigma)) + \mathcal{O}\left(\epsilon^3\right) \,.$$

Finally, imagine that the $\mathcal{G}$-labels h_k^{tot} of many such thin horizontal rows of 'surface elements' are composed *vertically*. Each of them comes from algebra elements

$$B_k(\sigma) \equiv B(\sigma, k\epsilon)$$

and holonomies

$$W_k(\sigma, 1) \equiv W_{k\epsilon}(\sigma, 1)$$

as

$$h_k^{\mathrm{tot}} \equiv 1 + \epsilon \int_0^1 d\sigma \; \alpha\left(W_{k\epsilon}^{-1}(\sigma,1)\right)(B(\sigma,k\epsilon)) + \mathcal{O}\left(\epsilon^3\right) \,.$$

In the limit of vanishing ϵ their total vertical product is

$$\lim_{\epsilon = 1/N \to 0} h_0^{\mathrm{tot}} h_\epsilon^{\mathrm{tot}} h_{2\epsilon}^{\mathrm{tot}} \cdots h_1^{\mathrm{tot}} = \mathrm{P}\exp\left(\int_0^1 d\tau \; \mathcal{A}(\tau)\right)$$

for

$$\mathcal{A}(\tau) = \int_0^1 d\sigma \; \alpha\left(W_\tau^{-1}(\sigma,1)\right)(B(\sigma,\tau)) \,, \tag{5.2}$$

P where denotes path ordering with respect to τ.

Thinking of each of these vertical rows of surface elements as paths (in the limit $\epsilon \to 0$), this shows roughly how the computation of total 2-group elements from vertical and horizontal products of many 'small' 2-group elements can be reformulated as the holonomy of a connection on path space of the form 5.2. This way the local 2-holonomy functor hol_i comes from a 1-form $A_i \in \Omega^1(U_i, \mathfrak{g})$ and a 2-form $B_i \in \Omega^2(U_i, \mathfrak{h})$ that arise as the continuum limit of the construction in 5.1. This is made precise in §6.5.

There it is discussed that given a bigon $\gamma \xrightarrow{\Sigma} \tilde{\gamma}$ the 2-group morphism

$$\mathrm{hol}_i(\Sigma) = (W_i[\gamma_1] \in G,\ \mathcal{W}_i^{-1}[\Sigma] \in H) \tag{5.3}$$

is obtained from the holonomy $W_i[\gamma]$ of A_i along γ and the inverse of the path space holonomy $\mathcal{W}_i^{-1}[\Sigma]$ of $\mathcal{A}_{(A_i,B_i)}$ along a path in path space that maps to Σ.

But not every pair (A, B) corresponds to a local holonomy-functor. As first noticed in [9] there is a consistency condition which can be understood as follows:

Let $g_j \xrightarrow{h_j} g'_j$ be the jth 2-group 2-morphism in the above figure. The nature of 2-groups (Prop. 18) requires that

$$t(h_j) = g'_j g_j^{-1}\,.$$

But, in the above sense, the left hand side is given by $\exp\left(\epsilon^2 dt(B)_j\right)$, while the right hand side is $\approx \exp\left(-\epsilon F_{A_j}\right)$, where F_{A_j} denotes the curvature 2-form of A evaluated on a 2-vector tangent to Σ_j. Hence we get the condition

$$dt(B) + F_A = 0\,.$$

This is the content of Prop. 66 (p. 65). See also Prop. 60 (p. 62).

5.2 Transition Law on Double Overlaps

Proposition 40.

The 2-commutativity of the diagram 4.1 (p. 41) is equivalent to the equations

$$\begin{aligned} A_i &= g_{ij} A_j g_{ij}^{-1} + g_{ij}\mathbf{d}g_{ij}^{-1} - dt(a_{ij}) \\ B_i &= \alpha(g_{ij})(B_i) + k_{ij}\,. \end{aligned}$$

Proof.

The 2-commutativity of the diagram is equivalent to the equality of the 2-morphism on its left face with the composition of the 2-morphisms on the front, back and right faces:

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=50, "\mathrm{hol}_i(\gamma)"] \arrow[r, bend right=50, "\mathrm{hol}_i(\tilde{\gamma})"'] \arrow[r, phantom, "\Downarrow \mathrm{hol}_i(\Sigma)"] & \bullet & = & \bullet \arrow[r, "g_{ij}(x)"] \arrow[rrr, bend left=70, "\mathrm{hol}_i(\gamma)"] \arrow[rrr, bend right=70, "\mathrm{hol}_i(\tilde{\gamma})"'] & \bullet \arrow[r, bend left=50, "\mathrm{hol}_j(\gamma)"] \arrow[r, bend right=50, "\mathrm{hol}_j(\tilde{\gamma})"'] \arrow[r, phantom, "\Downarrow \mathrm{hol}_j(\Sigma)"] \arrow[r, phantom, yshift=45pt, "\Downarrow a_{ij}(\gamma)"] \arrow[r, phantom, yshift=-45pt, "\Downarrow \bar{a}_{ij}(\tilde{\gamma})"] & \bullet \arrow[r, "g_{ij}^{-1}(y)"] & \bullet \end{tikzcd} \tag{5.4}$$

Recall from 5.3 that $\mathrm{hol}_i(\Sigma)$ has source $\mathrm{hol}_i(\gamma) = W_i[\gamma]$. So we write

$$a_{ij}(\gamma) \equiv (W_i[\gamma] \in G,\ E(a_{ij})\,[\gamma] \in H)\ ,$$

where

$$a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$$

is a 1-form (which we find convenient to denote by the same symbol as the 2-morphism $a_{ij}(\gamma)$ that it is associated with) and where E is a function whose nature is to be determined by the source/target matching condition. This says that

$$t(E(a_{ij}[\gamma]))\,W_i[\gamma] = g_{ij}(x)\,W_j[\gamma]g_{ij}^{-1}(y)\ . \tag{5.5}$$

Expressions like this are handled by Prop. 52 (p. 59). In order to apply it conveniently we take the inverse on both sides to get

$$W_i[\gamma^{-1}]t(E(a_{ij})\,[\gamma])^{-1} = g_{ij}W_j[\gamma^{-1}]g_{ij}^{-1} \tag{5.6}$$

(using $W[\gamma^{-1}] = W^{-1}[\gamma]$). Then the proposition tells us that $t(E(a_{ij})\,[\gamma])^{-1}$ is of the form

$$t(E(a_{ij})\,[\gamma])^{-1} = \lim_{\epsilon=1/N\to 0}\left(1+\epsilon\oint_{A_i}(\alpha)\right)\left(1+\epsilon\oint_{A_i+\epsilon\alpha}(\alpha)\right)\cdots\left(1+\epsilon\oint_{A_i+(1-\epsilon)a^1_{ij}}(\alpha)\right)_{|\gamma^{-1}}\ , \tag{5.7}$$

where the right hand side is evaluated at γ^{-1}, and where $\alpha \in \Omega^1(U_{ij}, \mathfrak{g})$ is given by

$$\alpha \equiv g^1_{ij}(\mathbf{d} + A_j)(g^1_{ij})^{-1} - A_i\,.$$

The 1-form α must take values in the image of dt, and it is the corresponding pre-image which we denote by a_{ij}, so that $dt(a_{ij}) = \alpha$:

$$dt(a_{ij}) = g^1_{ij}(d + A_j)(g^1_{ij})^{-1} - A_i\,. \tag{5.8}$$

This is the first of the two equations to be derived.

It follows that $E(a_{ij})\,[\gamma]$ itself is given by

$$(E(a_{ij})\,[\gamma])^{-1} = \lim_{\epsilon=1/N\to 0}\left(1+\epsilon\oint_{A_i}(a_{ij})\right)\left(1+\epsilon\oint_{A_i+\epsilon dt(a_{ij})}(a_{ij})\right)\cdots\left(1+\epsilon\oint_{A_i+(1-\epsilon)dt(a_{ij})}(a_{ij})\right)_{|\gamma^{-1}}\ .$$

Now that we have determined the 2-morphism $E(a_{ij})\,[\gamma]$, we can evaluate the diagrams in equation 5.4. Recalling again equation 5.3, one sees that the equality of the 2-morphism on the left hand with that on the right means that

$$\mathcal{W}_i^{-1}(\Sigma) = (E(a_{ij})\,[\tilde{\gamma}])^{-1}\mathcal{W}_j^{-1}(\Sigma)\,E(a_{ij})\,[\gamma]\,.$$

This is nothing but a gauge transformation of path space holonomy. Using Prop. 62 it implies the second of two equations to be proven. □

5.3 Transition Law on Triple Overlaps

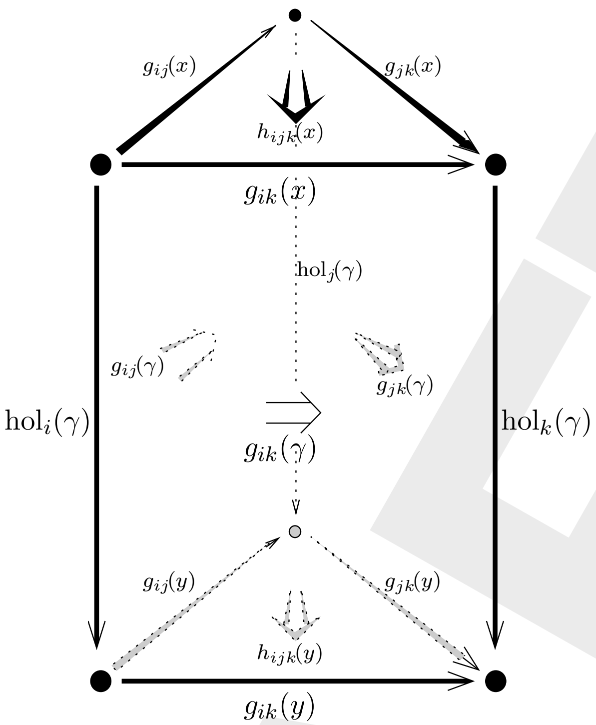
Proposition 41 *The 2-commutativity of the diagram 4.1 (p. 42) is equivalent to the equation 2.22 (p. 29)*

$$a_{ij} + \alpha(g_{ij})(a_{jk}) = h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + \mathbf{d}h_{ijk}\, h_{ijk}^{-1} + d\alpha(A_i)(h_{ijk})\, h_{ijk}^{-1}\,.$$

Proof. Since our target category $\mathcal{G}$ is a strict 2-group, so that (when regarded as a 2-category with a single object) all of its 1- and 2-morphisms are invertible, the diagram 4.1 expressing the modifications on U_{ijk} can be simplified. Using the transition diagram 4.1 we can equate the composition of the 2-morphisms $\mathrm{hol}_i(\Sigma)$ and $\mathrm{hol}_k(\Sigma)$ as well as the 2-morphisms $a_{ik}[\tilde{\gamma}]$ on the front of this diagram with the single 2-morphism $a_{ik}[\gamma]$ and hence get rid of the dependency on $\tilde{\gamma}$ and Σ:

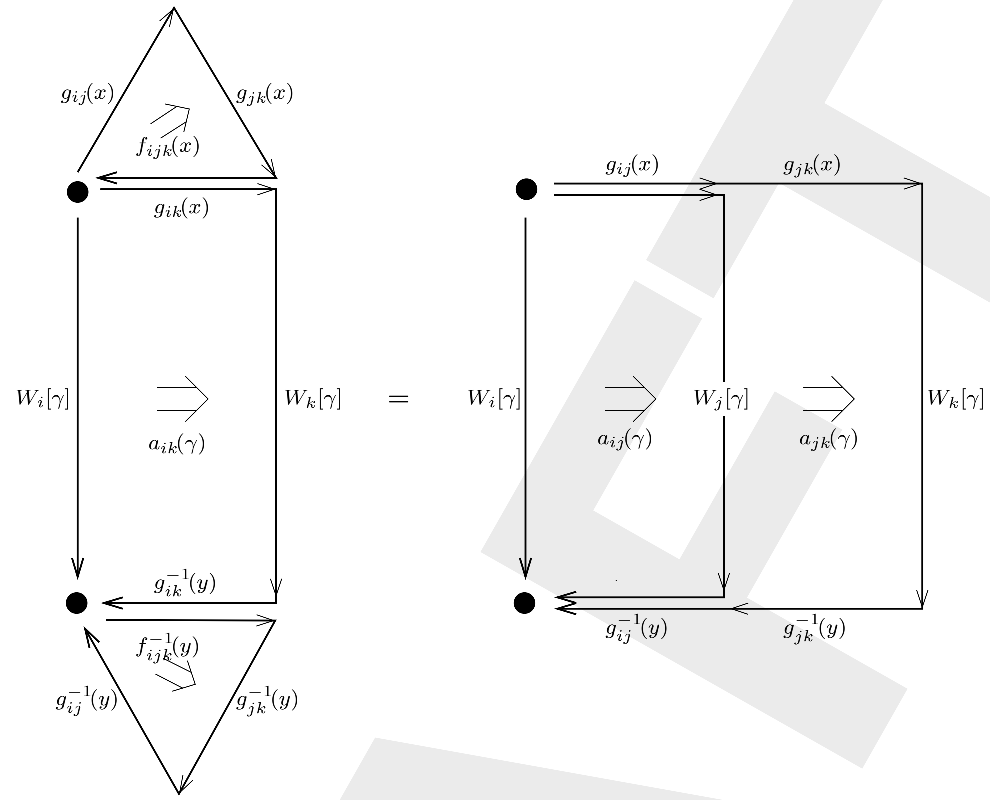
$$\begin{tikzcd}[row sep=large, column sep=huge]
\bullet \arrow[rr, bend left=25, "g_{ij}\circ g_{jk}(x)", ""{name=U1, below}] \arrow[rr, bend right=20, "g_{ik}(x)"{description}, ""{name=D1, above}] \arrow[ddd, "\mathrm{hol}_i(\gamma)"'] \arrow[Rightarrow, from=D1, to=U1, "f_{ijk}(x)"'] & & \bullet \arrow[ddd, "\mathrm{hol}_k(\gamma)"] \\
& {} \arrow[d, Rightarrow, "a_{ik}(\gamma)"'] & {} \arrow[dl, Rightarrow, dotted, "a_{ij}\cdot a_{jk}(\gamma)"] \\
& {} & \\
\bullet \arrow[rr, dotted, bend left=25, "g_{ij}\circ g_{jk}(y)", ""{name=U2, below}] \arrow[rr, bend right=20, "g_{ik}(y)"{description}, ""{name=D2, above}] \arrow[Rightarrow, dotted, from=D2, to=U2, "f_{ijk}(y)"'] & & \bullet
\end{tikzcd}$$

In order to emphasize the structure of this diagram it is useful to make the triangular shape of the top and bottom explicit:



(5.9)

The 2-commutativity of this diagram is equivalent to the following equality between the 2-morphism obtained from its top, bottom and front face and the 2-morphism obtained from the two faces on the back:



THIS DIAGRAM NEEDS TO BE FIXED!!!

In terms of group elements this means that

$$\alpha(\mathrm{hol}_i(\gamma))(h_{ijk}(y))\; g_{ik}(\gamma)\; h_{ijk}(x)^{-1} = g_{ij}(\gamma)\; \alpha(g_{ij}(x))(g_{jk}(\gamma))\,.$$

Now expand around the point x to get the differential version of this statement:

$$\begin{aligned} \mathrm{hol}_i(\gamma) &\approx 1 + \epsilon A_i\big(\gamma'\big) \\ g_{ik}(\gamma) &\approx 1 + \epsilon a_{ij}\big(\gamma'\big) \\ h_{ijk}(y) &\approx h_{ijk}(x) + \epsilon(\mathbf{d}(h_{ijk}))\big(\gamma'\big)\ . \end{aligned}$$

Here $\gamma' = \frac{d}{d\sigma}\gamma(\sigma)\,|_{\sigma=0}$ is the tangent vector to γ at $x = \gamma(0)$. Substituting these into the above equation and collecting terms of first order in ϵ yields the promised equation. $\Box$

6. 2-Connections in Terms of Connections on Path Space

6.1 Connections on Smooth Spaces

In this section we prove Proposition 37 and in the process develop some useful machinery concerning path spaces. To prove this proposition, it suffices to show it in the special case of a trivial bundle (EXPLAIN WHY):

Proposition 42. *Let G be a smooth group and X a smooth space. Given any $\mathfrak{g}$-valued 1-form A on X, there is a smooth map between smooth 2-spaces*

$$\mathrm{hol}\colon \mathcal{P}_1(X) \to G$$

given by

$$\mathrm{hol}([\gamma]) = \mathrm{P}\exp \int_\gamma A$$

where γ is any representative of the thin homotopy class $[\gamma]$. This gives a one-to-one correspondence between elements $A \in \Omega^1(X, \mathfrak{g})$ and smooth maps $\mathrm{hol}\colon \mathcal{P}_1(X) \to G$.

Proof. Any $\mathfrak{g}$-valued 1-form A on X defines a smooth map of 2-spaces $\mathrm{hol}\colon \mathcal{P}_1(X) \to G$ via the above formula, since we are assuming all our smooth groups are exponentiable (see Example 21). We must show that any smooth functor $\mathrm{hol}\colon \mathcal{P}_1(X) \to G$ comes from a unique $\mathfrak{g}$-valued 1-form A in this way. We construct A by differentiating hol.

To do this, we must clarify the relation between morphisms in $\mathcal{P}_1(X)$ and paths in X. Recall that morphisms in $\mathcal{P}_1(X)$ are equivalence classes of smooth maps $\gamma\colon [0,1] \to X$ for which $\gamma(t)$ is constant for t near 0 and t near 1. Let PX stand for the space of *all* smooth maps $\gamma\colon [0,1] \to X$.

Lemma 43. *Suppose $f\colon [0,1] \to [0,1]$ is any smooth function with $f(t) = 0$ for t near 0 and $f(t) = 1$ for t near 1. Then there is a smooth map*

$$\begin{aligned} p\colon PX &\to \mathrm{Mor}(\mathcal{P}_1(X)) \\ \gamma &\mapsto [\gamma \circ f] \end{aligned}$$

where $[\gamma \circ f]$ is the thin homotopy class of $\gamma \circ f$. Moreover, p does not depend on the choice of f.

Proof. The map p is clearly well-defined for any fixed f of the above sort; to show p is smooth it suffices to show that the map

$$\begin{aligned} PX &\to PX \\ \gamma &\mapsto \gamma \circ f \end{aligned}$$

is smooth, which is straightforward. To show the map p is independent of f, suppose $f_0, f_1\colon [0,1] \to [0,1]$ are both smooth functions that equal 0 near 0 and 1 near 1. Then there exists a smooth function $F\colon [0,1]^2 \to [0,1]$ such that:

- $F(s,t) = f_0(t)$ for s near 0
- $F(s,t) = f_1(t)$ for s near 1
- $F(s,t) = 0$ for t near 0
- $F(s,t) = 1$ for t near 1

It follows that

$$H(s,t) = \gamma(F(s,t))$$

is a thin homotopy from $\gamma \circ f_0$ to $\gamma \circ f_1$, so $[\gamma \circ f_0] = [\gamma \circ f_1]$. ☐

We also need to consider **Moore paths**, which are smooth maps $\gamma\colon [0,T] \to X$ for arbitrary $T \geq 0$. Note that a Moore path for $T = 0$ is just a point of X. We denote the set of all Moore paths by MX. This becomes a smooth space where a plot (see the Appendix) is a function

$$\begin{array}{rcl} \phi\colon \mathbb{R}^n & \to & MX \\ z & \mapsto & \phi_z \end{array}$$

such that $\phi_z\colon [0,T_z] \to X$, where $T_z \in [0,\infty]$ is a smooth function of z and $\phi_z(t) \in X$ is a smooth function of z and t.

Here is the relation between Moore paths and paths parametrized by the unit interval:

Lemma 44 *There is a smooth map*

$$q\colon MX \to PX$$

sending any Moore path $\gamma\colon [0,T] \to X$ to the path

$$(q\gamma)(t) = \gamma(Tt).$$

Proof. Straightforward. ☐

Using the smooth maps

$$MX \xrightarrow{q} PX \xrightarrow{p} \mathrm{Mor}(\mathcal{P}_1(X))$$

we can define holonomies for Moore paths. Namely, given any smooth map $\mathrm{hol}\colon \mathcal{P}_1(X) \to G$, we get a smooth map

$$\mathrm{Hol} := \mathrm{hol} \circ q \circ p$$

assigning an element of G to any Moore path.

To prove the lemma, it suffices to show there exists a unique $A \in \Omega^1(X, \mathfrak{g})$ with

$$\mathrm{Hol}(\gamma) = \mathrm{P}\exp \int_\gamma A$$

for any Moore path $\gamma\colon [0,T] \to X$ with $T > 0$. For this, it suffices to show there exists a unique A such that for any Moore path $\gamma\colon [0,T] \to X$ with $T > 0$ we have

$$\frac{d}{ds}\mathrm{Hol}(\gamma_s) = A(\gamma'(s))\mathrm{Hol}(\gamma_s) \tag{6.1}$$

for all $s \in [0,T]$, where $\gamma_s\colon [0,s] \to X$ is the Moore path given by restricting γ to $[0,s]$.

In fact, it suffices to show there exists a unique A such that equation (6.1) holds at $s = 0$. The reason is that for any $s_0 \in [0,T]$ we can write the Moore path γ as a 'concatenation' of the Moore paths γ_{s_0} and $\eta\colon [0, T - s_0] \to X$, where

$$\eta(t) = \gamma(t - s_0).$$

So, since hol is a functor, we have

$$\mathrm{Hol}(\gamma_s) = \mathrm{Hol}(\eta_{s-s_0})\mathrm{Hol}(\gamma_{s_0})$$

for $s_0 \le s \le T$. It follows that

$$\begin{aligned} \tfrac{d}{ds}\mathrm{Hol}(\gamma_s)|_{s=s_0} &= \tfrac{d}{ds}\mathrm{Hol}(\eta_{s-s_0})\mathrm{Hol}(\gamma_{s_0})|_{s=s_0} \\ &= \tfrac{d}{ds}\mathrm{Hol}(\eta_s)|_{s=0}\ \mathrm{Hol}(\gamma_{s_0}). \end{aligned}$$

So, if equation (6.1) holds at $s = 0$ for all Moore paths (in particular for η), we have

$$\begin{aligned} \tfrac{d}{ds}\mathrm{Hol}(\gamma_s)|_{s=s_0} &= A(\eta'(0))\mathrm{Hol}(\eta_0)\mathrm{Hol}(\gamma_{s_0}) \\ &= A(\gamma'(s_0))\mathrm{Hol}(\gamma_{s_0}) \end{aligned}$$

since $\mathrm{Hol}(\eta_0) = 1$ and $\eta'(0) = \gamma'(s_0)$. Thus, we have shown that equation (6.1) holds for γ at $s = s_0$, given that it holds for η at $s = 0$.

In short, we must show there exists a unique $\mathfrak{g}$-valued 1-form A such that for any Moore path $\gamma\colon [0,T] \to X$ with $T > 0$,

$$\frac{d}{ds}\mathrm{Hol}(\gamma_s)|_{s=0} = A(\gamma'(0)).$$

Since any 1-form on X is determined by its values on tangent vectors of paths (see the Appendix), this formula uniquely determines A. However, we need to check that A exists: in other words, that it is well-defined and smooth. For the first, suppose we have two Moore paths γ and η starting at $x \in X$ with $\gamma'(0) = \eta'(0)$; we need to check that

$$\frac{d}{ds}\mathrm{Hol}(\gamma_s)|_{s=0} = \frac{d}{ds}\mathrm{Hol}(\eta_s)|_{s=0}.$$

For this, it suffices by the chain rule to check that

$$\frac{d}{ds}\gamma_s|_{s=0} = \frac{d}{ds}\eta_s|_{s=0}$$

as tangent vectors in MX.

To check this, note that there is a smooth map

$$r: MX \to MMX$$

sending any Moore path $\gamma: [0, T] \to X$ to the Moore path $\Gamma: [0, T] \to MX$ given by $\Gamma(s)(t) = \gamma_s(t)$. Quite generally, the tangent vector of a Moore path in a smooth space can be thought of as an equivalence class of Moore paths (see the Appendix). In particular, the tangent vector $\gamma'(0) \in T_x X$ is an equivalence class of Moore paths in X starting at x. The smooth map r sends any Moore path in this equivalence class to a Moore path in MX representing the tangent vector $\frac{d}{ds}\gamma_s|_{s=0}$. Since $\gamma'(0) = \eta'(0)$, γ and η lie in the same equivalence class, so $r\gamma$ and $r\eta$ lie in the same equivalence class, which means that

$$\frac{d}{ds}\gamma_s|_{s=0} = \frac{d}{ds}\eta_s|_{s=0}$$

as desired.

NEED TO CHECK SMOOTHNESS.... ☐

6.2 More stuff...

As we have seen, the space of all paths in a manifold or more general smooth space is itself a smooth space. This allows us to study the notion of holonomy for paths in path space. A path in the path space of U gives rise to a (possibly degenerate) surface in U and hence its path space holonomy gives rise to a notion of surface holonomy in U.

In this section we first discuss basic concepts of differential geometry on path spaces and then apply them to define path space holonomy. Using that, a 2-functor hol_i from the 2-groupoid of bigons in U_i (to be defined below) to the structure 2-group is defined and shown to be consistent.

Throughout the following, various p-forms taking values in Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are used, where $\mathfrak{g}$ and $\mathfrak{h}$ are part of a differential crossed module $\mathcal{C}$ (Def. 23). Elements of a basis of $\mathfrak{g}$ will be denoted by T_a with $a \in (1, \ldots, \dim(\mathfrak{g}))$ and those of a basis of $\mathfrak{h}$ by S_a with $a \in (1, \ldots, \dim(\mathfrak{h}))$. Arbitrary elements will be expanded as $A = A^a T_a$.

Given a $\mathfrak{g}$-valued 1-form A we define the **gauge covariant exterior derivative** by

$$\begin{aligned} \mathbf{d}_A\omega &\equiv [\mathbf{d} + A, \omega] \\ &\equiv \mathbf{d}\omega + A^a \wedge d\alpha(T_a)(\omega) \end{aligned}$$

and the **curvature** by

$$\begin{aligned} F_A &\equiv (\mathbf{d} + A)^2 \\ &\equiv \mathbf{d}A + \frac{1}{2}A^a \wedge A^b \,[T_a, T_b] \,. \end{aligned}$$

Differential calculus on spaces of *parametrized* paths can be handled rather easily. We start by establishing some basic facts on parametrized paths and then define the *path groupoid* by considering thin homotopy equivalence classes of parametrized paths.

Definition 45. *Given a manifold U, the* **based parametrized path space** $P_x^y(U)$ *of U with source $x \in U$ and target $y \in U$ is the space of smooth maps*

$$\begin{aligned} X\colon [0,1] &\to U \\ \sigma &\mapsto X(\sigma) \end{aligned} \qquad (6.2)$$

with $X(\sigma) = x$ for σ in a neighborhood of 0 and $X(\sigma) = y$ for σ in a neighborhood of 1. When source and target coincide:

$$\Omega_x(U) := P_x^x(U)$$

is called the **based loop space** *of U based at x.*

The constancy condition at the boundary is known as the property of having **sitting instant**; compare for instance [51]. It serves to ensure that the composition of two smooth parametrized paths is again a smooth parametrized path.

In the study of differential forms on parametrized path space the following notions play an important role (*cf.* [50], section 2):

Definition 46.

1. *Given any path space $P_s^t(U)$ (Def. 45), the 1-parameter family of maps*

$$\begin{aligned} e_\sigma : P_s^t(U) &\to U \qquad (\sigma \in (0,1)) \\ \gamma &\mapsto \gamma(\sigma) \end{aligned}$$

maps each path to its position in U at parameter value σ.

2. *Given any differential p-form $\omega \in \Omega^p(U)$ the pullback to $P_s^t(U)$ by e_σ shall be denoted simply by*

$$\omega(\sigma) \equiv e_\sigma^*(\omega) \ .$$

3. *The contraction of $\omega(\sigma)$ with the vector*

$$\gamma' \equiv \frac{d}{d\sigma}\gamma$$

is denoted by $\iota_{\gamma'}\omega(\sigma)$.

A special class of differential forms on path space play a major role:

Definition 47. *Given a familiy $\{\omega_i\}_{i=1}^N$ of differential forms on a manifold U with degree*

$$\deg(\omega_i) \equiv p_i + 1$$

one gets a differential form (see Def. 46)

$$\Omega_{\{\omega_i\},(\alpha,\beta)}(\gamma) \equiv \oint\limits_{X|_\alpha^\beta} (\omega_1, \dots, \omega_n) \ \equiv \int\limits_{\alpha<\sigma_i<\sigma_{i+1}<\beta} \iota_{\gamma'}\omega_1(\sigma^1) \wedge \cdots \wedge \iota_{\gamma'}\omega_N(\sigma^N)$$

of degree

$$\deg\big(\Omega_{\{\omega_i\}}\big) = \sum_{i=1}^{N} p_i \,,$$

on any based parametrized path space $P_s^t(U)$ (Def. 45).

For $\alpha = 0$ and $\beta = 1$ we write

$$\Omega_{\{\omega_i\}} \equiv \Omega_{\{\omega_i\},(0,1)} \,.$$

These path space forms are known as **multi integrals** *or* **iterated integrals** *or* **Chen forms** *(cf. [50, 52]).*

It turns out that the exterior derivative on path space maps Chen forms to Chen forms in a nice way:

Proposition 48. *The action of the path space exterior derivative on Chen forms is given by:*

$$\mathbf{d} \oint (\omega_1, \cdots, \omega_n) = (\tilde{\mathbf{d}} + \tilde{M}) \oint (\omega_1, \cdots, \omega_n) \,, \tag{6.3}$$

where:

$$\begin{aligned} \tilde{\mathbf{d}} \oint (\omega_1, \cdots, \omega_n) &\equiv -\sum_k (-1)^{\sum_{i<k} p_i} \oint (\omega_1, \cdots, \mathbf{d}\omega_k, \cdots, \omega_n) \\ \tilde{M} \oint (\omega_1, \cdots, \omega_n) &\equiv -\sum_k (-1)^{\sum_{i<k} p_i} \oint (\omega_1, \cdots, \omega_{k-1} \wedge \omega_k, \cdots, \omega_n) \,, \end{aligned}$$

and we have:

$$\begin{aligned} \tilde{\mathbf{d}}^2 &= 0 \\ \tilde{M}^2 &= 0 \\ \tilde{\mathbf{d}}\tilde{M} + \tilde{M}\tilde{\mathbf{d}} &= 0 \,. \end{aligned} \tag{6.4}$$

Proof. See [50, 52]. □

6.3 The Standard Connection 1-Form on Path Space

There are many 1-forms on path space that one could consider as local connection 1-forms in order to define a local holonomy on path space. Here we restrict attention to a special class, to be called the *standard connection 1-forms* (Def. 54), because, as is shown in §6.5, these turn out to be the ones which compute local 2-group holonomy. (This same 'standard connection 1-form' can however also be motivated from other points of view, as done in [10, 11].)

Holonomy and parallel transport. In order to set up some notation and conventions and for later references, the following gives a list of well-known definitions and facts that are crucial for the further developments:

Definition 49. *Given a path space $P_x^y(U)$ (Def. 45), a $\mathfrak{g}$-valued 1-form A on U, and an $\mathfrak{h}$-valued 2-form B on U, we make the following definitions:*

1. *The* **line holonomy** *of A along a given path γ is denoted by*

$$\begin{aligned} W_A[\gamma]\big(\sigma^1,\sigma^2\big) &\equiv \mathrm{P}\exp\left(\int\limits_{\gamma|_{\sigma^1}^{\sigma^2}} A\right) \\ &\equiv \sum_{n=0}^{\infty} \oint\limits_{\gamma|_{\sigma^1}^{\sigma^2}} (A^{a_1},\dots,A^{a_n}) T_{a_1}\cdots T_{a_n}\,. \end{aligned} \tag{6.5}$$

2. *The* **parallel transport** *of elements in $T \in \mathfrak{g}$ and $S \in \mathfrak{h}$ is written*

$$\begin{aligned} T^{W_A[\gamma]}(\sigma) &\equiv W_A^{-1}[\gamma|_\sigma^1] T(\sigma)\, W_A[\gamma](\sigma,1) \\ &= \sum_{n=0}^{\infty} \oint_{\gamma|_\sigma^1} (-A^{a_1},\cdots,-A^{a_n})\ [T_{a_n},\cdots[T_{a_1},T(\sigma)]\cdots]\ , \\ S^{W_A[\gamma]}(\sigma) &\equiv \alpha\big(W_A[\gamma|_\sigma^1]\big)(S(\sigma)) \\ &\equiv \sum_{n=0}^{\infty} \oint_{\gamma|_\sigma^1} (-A^{a_1},\cdots,-A^{a_n})\, d\alpha(T_{a_n})\circ\cdots\circ d\alpha(T_{a_1})(S(\sigma))\ . \end{aligned} \tag{6.6}$$

For convenience the dependency $[\gamma]$ on the path γ will often be omitted.

Proposition 50. *Parallel transport (Def. 49) has the following properties:*

1. *Let $\sigma_1 \leq \sigma_2 \leq \sigma_3$ then*

$$W_A[\gamma](\sigma_1,\sigma_2)\circ W_A[\gamma](\sigma_2,\sigma_3) = W_A[\gamma](\sigma_1,\sigma_3)\ .$$

2. *Conjugation of elements in $\mathfrak{g}$ with parallel tranport of elements in $\mathfrak{h}$ yields*

$$W_A(\sigma,1)\big(d\alpha(T)(\sigma)\big(W_A^{-1}(\sigma,1)(S)\big)\big) = d\alpha\big(T^{W_A}(\sigma)\big)(S)\ . \tag{6.7}$$

3. *Given a G-valued 0-form $g \in \Omega^0(U,G)$ and a path $\gamma \in P_x^y(U)$ we have*

$$g(x)\, W_A[\gamma](g(y))^{-1} = W_{(gAg^{-1}+g^{-1}\mathbf{d}g)}[\gamma]\,. \tag{6.8}$$

4. *Given a G-valued 0-form $g \in \Omega^0(U, G)$ and a based loop $\gamma \in P_x^x(U)$ we have*

$$\alpha(\phi(x))(W_A[\gamma](\sigma,1)(S(\sigma))) = W_{A'}[\gamma](\sigma,1)(\alpha(\phi(\gamma(\sigma)))(S(\sigma))) \tag{6.9}$$

with

$$A' \equiv \phi A \phi^{-1} + \phi(d\phi^{-1})\,.$$

Integrals of p-forms pulled back to a path and parallel tranported to some base point play an important role for path space holonomy. Following [52, 11] we introduce special notation to take care of that automatically:

Definition 51. *A natural addition to the notation 47 for iterated integrals in the presence of a $\mathfrak{g}$-valued 1-form A is the abbreviation*

$$\oint_A (\omega_1, \dots \omega_N) \equiv \oint \left(\omega_1^{W_A}, \dots \omega_N^{W_A}\right)\,,$$

where $(\cdot)^{W_A}$ is defined in def 49. When Lie algebra indices are displayed on the left they are defined to pertain to the parallel tranported object:

$$\oint_A (\dots, \omega^a, \dots) \equiv \oint (\dots, (\omega^{W_A})^a, \dots)\,. \tag{6.10}$$

Using this notation first of all the following fact can be conveniently stated, which plays a central role in the analysis of the transition law for the 2-holonomy in §5.2:

Proposition 52. *The difference in line holonomy (Def. 49) along a given loop with respect to two different 1-forms A and A' can be expressed as*

$$(W_A[\gamma])^{-1} W_{A'}[\gamma] = \lim_{\epsilon = 1/N \to 0} \left(1 + \epsilon \oint\limits_A (\alpha)\right) \left(1 + \epsilon \oint\limits_{A+\epsilon(\alpha)} (\alpha)\right) \cdots \left(1 + \epsilon \oint\limits_{A'-\epsilon(\alpha)} (\alpha)\right)_\gamma\,,$$

with $\alpha \equiv A' - A$.

Proof.

First note that from def. 49 it follows that

$$\oint\limits_A (\alpha) = \int\limits_0^1 d\sigma (W_A[\gamma](\sigma,1))^{-1} \iota_{\gamma'} \alpha(\sigma)\, W_A[\gamma](\sigma,1)\,.$$

This implies that

$$W_A[\gamma] \left(1 + \epsilon \oint_A (\alpha)\right)_\gamma = W_{A+\epsilon(\alpha)}[\gamma] + \mathcal{O}(\epsilon^2)\,.$$

The proposition follows by iterating this. □

Exterior derivative and curvature for Chen forms. The exterior derivative on path space maps Chen forms to Chen forms (Prop. 48). Since we shall be interested in Chen forms involving parallel transport (Def. 51), it is important to know also the particular action of the exterior derivative on these:

Proposition 53. *The action of the path space exterior derivative on* $\oint_A(\omega)$ *is*

$$\mathbf{d}\oint_A(\omega) = -\oint_A(\mathbf{d}_A\omega) - (-1)^{\deg(\omega)}\oint_A\left(d\alpha(T_a)(\omega)\,,F_A^a\right)\,. \tag{6.11}$$

(Recall the convention 6.10).

Proof.

This is a straightforward, though somewhat tedious, computation using prop 48. □

We have restricted attention here to just a single insertion, i.e. $\oint_A(\omega)$ instead of $\oint_A(\omega_1,\dots,\omega_n)$, because this is the form that the *standard connection 1-form* has:

Definition 54. *Given a* $\mathfrak{g}$*-valued 1-form* A *and an* $\mathfrak{h}$*-valued 2-form* B *on* U*, the* $\mathfrak{h}$*-valued 1-form on* $P_s^t(U)$

$$\mathcal{A}_{(A,B)} \equiv \oint_A(B)\,.$$

is called the **standard local connection 1-form on path space**.

(See for example [10, 53, 11])

Given a connection, one wants to know its curvature:

Corollary 55 *The curvature of the standard path space 1-form* $\mathcal{A}_{(A,B)}$ *(Def. 54) is*

$$\mathcal{F}_{\mathcal{A}} = -\oint_A(\mathbf{d}_A B) - \oint_A\left(d\alpha(T_a)(B)\,,(F_A + dt(B))^a\right)\,. \tag{6.12}$$

Proof. Use Prop. 53. □

Definition 56. *Given a standard path space connection 1-form* $\mathcal{A}_{(A,B)}$ *(Def. 54) coming from a* $\mathfrak{g}$*-valued 1-form* A *and an* $\mathfrak{h}$*-valued 2-form* B

- *the 3-form*

  $$H \equiv \mathbf{d}_A B \tag{6.13}$$

  is called the **curvature 3-form**,

- *the 2-form*

$$\tilde{F} \equiv F_A + dt(B) \tag{6.14}$$

is called the **fake curvature 2-form***.*

The term 'fake curvature' has been introduced in [13]. The notation $\tilde{F}$ follows [9]. The curvature 3-form was used in [2].

Using this notation the local path space curvature reads

$$\mathcal{F}_{\mathcal{A}} = -\oint_A (H) - \oint_A \Big(d\alpha(T_a)(B)\,,\tilde{F}^a\Big)\;. \tag{6.15}$$

6.4 Path Space Line Holonomy and Gauge Transformations

With the usual tools of differential geometry available for path space, the holonomy on path space is defined as usual:

Definition 57. *Given a path space 1-form $\mathcal{A}$ and a path Σ in path space the* **path space line holonomy** *of $\mathcal{A}$ along Σ is*

$$\mathcal{W}_{\mathcal{A}}(\Sigma) \equiv \mathrm{P}\exp\left(\int_\Sigma \mathcal{A}\right)\;.$$

Note that by definition P here indicates path ordering with objects at higher parameter value to the *right* of those with lower parameter value, just as in the definition of ordinary line holonomy in (Def. 49).

Path space line holonomy has a richer set of gauge transformations than holonomy on base space. In fact, ordinary gauge transformations on base space correspond to *constant* ('global') gauge transformations on path space in the following sense:

Proposition 58. *Given a path space line holonomy (Def. 57) coming from a standard path space connection 1-form (Def. 54) $\mathcal{A}_{(A,B)}$ in a based loop space $P_x^x(U)$ as well as a G-valued 0-form $\phi \in \Omega^0(U,G)$ we have*

$$\alpha(\phi(x))\Big(\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma)\Big) = \mathcal{W}_{\mathcal{A}_{(A',B')}}(\Sigma)$$

with

$$\begin{aligned} A' &= \phi A \phi^{-1} + \phi(d\phi^{-1}) \\ B' &= \alpha(\phi)(B)\;. \end{aligned}$$

Proof. Write out the path space holonomy in infinitesimal steps and apply 6.9 on each of them. □

The usual notion of gauge transformation is obtained by conjugation:

Definition 59. *Given the path space holonomy* $\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0})$ *(Def. 57) of a standard local path space connection 1-form* $\mathcal{A}_{(A,B)}$ *(Def. 54) along a path* Σ *in* $P^t_s(U)$ *with endpaths* γ_0 *and* γ_1*, an* **infinitesimal path space holonomy gauge transformation** *is a 1-parameter familiy maps*

$$\begin{aligned} \mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0}) &\mapsto \left(1 - \epsilon \oint_A (a)\right)_{\gamma_0} \mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0}) \left(1 + \epsilon \oint_A (a)\right)_{\gamma_1} \\ &\equiv \mathrm{Ad}^{\gamma_1}_{\gamma_0} \left(1 - \epsilon \oint_A (a)\right) \left(\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0})\right) , \end{aligned}$$

for $\epsilon \in \mathbb{R}$ *and for a any 1-form*

$$a \in \Omega^1(U, \mathfrak{h}) .$$

This yields a new sort of gauge transformation in terms of the 1-form A and the 2-form B:

Proposition 60. *Infinitesimal path space holonomy gauge transformations (Def. 59) for the holonomy of a standard path space connection 1-form* $\mathcal{A}_{(A,B)}$ *and arbitrary transformation parameter* a *yields to first order in the parameter* ϵ *the path space holonomy of a transformed standard path space connection 1-form* $\mathcal{A}_{(A',B')}$ *with*

$$\begin{aligned} A' &= A + dt(a) \\ B' &= B - \mathbf{d}_A a \end{aligned} \tag{6.16}$$

if and only if the fake curvature (Def. 56) vanishes.

(This was originally considered in [11] for the special case $G = H$, $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$.)
Proof.

As for any holonomy, the gauge transformation induces a transformation of the connection 1-form $\mathcal{A} \to \mathcal{A}'$ given by

$$\begin{aligned} \mathcal{A}' &= \left(1 - \epsilon \oint_A (a)\right) (\mathbf{d} + \mathcal{A}) \left(1 + \epsilon \oint_A (a)\right) \\ &= \mathcal{A} + \epsilon \, \mathbf{d}_{\mathcal{A}} \oint_A (a) + \mathcal{O}(\epsilon^2) . \end{aligned} \tag{6.17}$$

Using 6.11 one finds (using the notation 6.10)

$$\mathcal{A} + \epsilon \, \mathbf{d}_{\mathcal{A}} \oint_A (a) = \oint_{A'} (B') + \epsilon \oint_A (d\alpha(T_a)(a) , (dt(B) + F)^a) + \mathcal{O}(\epsilon^2) .$$

Since a is by assumption arbitrary, the last line is equal to a standard connection 1-form to order ϵ if and only if $dt(B) + F = 0$. $\square$

The above infinitesimal gauge transformation is easily integrated to a finite gauge transformation:

Definition 61. *A* **finite path space holonomy gauge transformation** *is the integration of infinitesimal path space holonomy gauge transformations (Def. 59), i.e. it is a map for any $a \in \Omega^1(U, \mathfrak{h})$ given by*

$$\mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{\gamma_0}^{\gamma_1}\right) \mapsto \lim_{\epsilon = 1/N \to 0} \underbrace{\mathrm{Ad}_{\gamma_0}^{\gamma_1}\left(1 - \epsilon \oint_{A+dt(a)} (a)\right) \cdots \mathrm{Ad}_{\gamma_0}^{\gamma_1}\left(1 - \epsilon \oint_{A} (a)\right)}_{N \text{ factors}} \left(\mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{\gamma_0}^{\gamma_1}\right)\right) .$$

Proposition 62. *A finite path space holonomy gauge transformation (Def. 61) of the holonomy of a standard path space connection 1-form $\mathcal{A}_{(A,B)}$ is equivalent to a transformation*

$$\mathcal{A}_{(A,B)} \mapsto \mathcal{A}_{(A',B')}$$

where

$$\begin{aligned} A' &= A + dt(a) \\ B' &= B - \underbrace{(d_A a + a \wedge a)}_{\equiv k_a} \end{aligned} \tag{6.18}$$

Proof. This is a standard computation. □

In summary the above yields two different notions of gauge transformations on path space:

1. If the path space in question is a based loop space then according to Prop. 58 a gauge transformation on target space yields an ordinary gauge transformation of (A, B):

   $$\begin{aligned} A &\mapsto \phi A \phi^{-1} + \phi(d\phi^{-1}) \\ B &\mapsto \alpha(\phi)(B) \ . \end{aligned}$$

   We shall call this a **2-gauge transformation of the first kind**.

2. A gauge transformation in path space itself yields, according to Prop. 62, a transformation

   $$\begin{aligned} A &\mapsto A + dt(a) \\ B &\mapsto B - (d_A a + a \wedge a) \, . \end{aligned}$$

   We shall call this a **2-gauge transformation of the second kind.**

Recall that according to Prop. 60 this works precisely when (A, B) defines a standard connection 1-form (Def. 54) on path space for which the 'fake curvature' vanishes: $\tilde{F} = dt(B) + F_A = 0$.

In the context of loop space these two transformations and the conditions on them were discussed for the special case $G = H$ and $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$ in [11]. In the context of

2-groups and higher lattice gauge theory they were found in section 3.4 of [9]. They also appear in the transition laws for nonabelian gerbes [13, 14, 15], as is discussed in detail in §2.5. The same transformation for the special case where all groups are abelian is well known from abelian gerbe theory [54] and also from string theory (e.g. section 8.7 of [55]).

With holonomy on path space understood, it is now possible to use the fact that every path in path space maps to a (possibly degenerate) surface in target space in order to get a notion of (local) surface holonomy. That is the content of the next subsection.

6.5 The Local 2-Holonomy Functor

Definition 63. *Given a smooth space U and a smooth 2-group $\mathcal{G}$ a* **local 2-holonomy** *is a smooth 2-functor*

$$\mathrm{hol}\colon \mathcal{P}_2(U) \to \mathcal{G}$$

from the path 2-groupoid of U (Def. 36) to $\mathcal{G}$.

We want to construct a local 2-holonomy from a standard path space connection 1-form (Def. 54). In order to do so we first construct a 'pre-2-holonomy' for any standard path space connection 1-form and then determine under which conditions this actually gives a true 2-holonomy. It turns out that the necessary and sufficient conditions for this is the vanishing of the fake curvature (Def. 56).

Definition 64. *Given a standard path space connection 1-form (Def. 54) and given any parametrized bigon (Def. 34) $\Sigma : [0,1]^2 \to U$ with source edge $\gamma_1 \equiv \Sigma(\cdot, 0)$ and target edge $\gamma_2 \equiv \Sigma(\cdot, 1)$, the triple $(g_1, h, g_2) \in G \times H \times G$ with*

$$\begin{aligned} g_i &\equiv W_A(\gamma_i) \\ h &\equiv \mathcal{W}_{\mathcal{A}}^{-1}(\Sigma(1-\cdot,\cdot)) \end{aligned} \tag{6.19}$$

is called the **local pre-2-holonomy** *of Σ associated with $\mathcal{A}$.*

(The unexpected inverse and parameter inversion here is just due to the interplay of our conventions on signs and orientations, as will become clear shortly.)

In order for a pre-2-holonomy to give rise to a true 2-holonomy two conditions have to be satisfied:

1. The triple (g_1, h, g_2) has to specify a 2-group element. By Prop. 18 this is the case if and only if $g_2 = t(h)\, g_1$.
2. The pre-2-holonomy has to be invariant under thin homotopy in order to be well defined on bigons.

The solution of these conditions is the content of Prop. 68 below. In order to get there the following considerations are necessary:

In order to analyze the first of the above two points consider the behaviour of the pre-2-holonomy under changes of the target edge.

Given a path space $P_s^t(U)$ and a $\mathfrak{g}$-valued 1-form with line holonomy holonomy $W_A[\gamma]$ on $\gamma \in P_s^t$ (Def. 49) the **change in holonomy** of W_A as one changes γ is well known to be given by the following:

Proposition 65. *Let $\rho : \tau \mapsto \gamma(\tau)$ be the flow generated by the vector field D on P_s^t, then*

$$\frac{d}{d\tau} W_A^{-1}[\gamma(0)] W_A[\gamma(\tau)]\bigg|_{\tau=0} = -\left(\oint_A (F_A)\right)(D) \,. \tag{6.20}$$

(Note that the right hand side denotes evaluation of the path space 1-form $\oint_A (F_A)$ on the path space vector field D.)

Proof. The proof is standard. The only subtlety is to take care of the various conventions for signs and orientations which give rise to the minus sign in 6.20. □

Proposition 66. *For the pre-2-holonomy (Def. 64) of parametrized bigons Σ associated with the standard connection 1-form $\mathcal{A}_{(A,B)}$ to specify 2-group elements, i.e. for the triples (g_1, h, g_2) to satisfy $g_2 = t(h)\, g_1$, we must have*

$$dt(B) + F_A = 0\,.$$

Proof. According to def. 64 the condition $g_2 = t(h)\, g_1$ translates into

$$\begin{aligned} t(h) &= W_A(\gamma_2)\, W_A^{-1}(\gamma_1) \\ &= W_A^{-1}(\gamma_2^{-1})\, W_A(\gamma_1^{-1}) \,. \end{aligned}$$

Now let there be a flow $\tau \mapsto \gamma_\tau$ on $P_s^t(U)$ generated by a vector field D and choose $\gamma_2^{-1} = \gamma_\tau$ and $\gamma_1^{-1} = \gamma_0$. Then according to Prop. 65 we have

$$\frac{d}{d\tau} W_A^{-1}(\gamma_2^{-1})\, W_A(\gamma_1^{-1}) = +\left(\oint_A (F_A)\right)_{\gamma_0} (D) \,,$$

where the plus sign is due to the fact that D here points opposite to the D in Prop. 65.

Applying the same τ-derivative on the left hand side of 6.21 yields

$$-\left(\oint_A (dt(B))\right)(D) = \left(\oint_A (F_A)\right)(D) \,.$$

(Here the minus sign on the left hand side comes from the fact that we have identified $t(h)$ with the *inverse* path space holonomy $\mathcal{W}_A^{-1}$. This is necessary because the ordinary path space holonomy is path-ordered to the right, while we need $t(h)$ to be path ordered to the left.)

This can be true for all D only if $-dt(B) = F_A$. □

This is nothing but the **nonabelian Stokes theorem**. (Compare for instance [56] and references given there.)

Next it needs to be shown that a pre-2-holonomy with $dt(B) + F_A = 0$ is invariant under thin homotopy:

Proposition 67. *The standard path space connection 1-form $\mathcal{A}_{(A,B)}$ (Def. 54) is inavriant under thin homotopy precisely when the path space 2-form*

$$\oint_A (d\alpha(T_a)(B)\,, (F_A + dt(B))^a) \tag{6.21}$$

vanishes on all pairs path space vector fields that generate thin homotopy flows.

Proof. For the special case $G = H$ and $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$ this was proven by [12]. The full proof is a straightforward generalization of this special case:

Consider a path Σ in path space with tangent vector T and let D be any vector field on $P_s^t(U)$. By a standard result the path space holonomy $\mathcal{W}(\Sigma)$ is invariant under the flow generated by D iff the curvature of $\mathcal{A}$ vanishes on T and D, $\mathcal{F}(T, D) = 0$.

But from corollary 55 we know that $\mathcal{F} = -\oint(\mathbf{d}_A B) - \oint(d\alpha(T_a)(B)\,, (F_A + dt(B))^a)$. It is easy to see that $\oint(\mathbf{d}_A B)$ vanishes on all pairs of tangent vectors that generate thin homotopy transformations of Σ and that the remaining term vanishes on (T, D) for all D if it vanishes on all pairs of tangent vector that generate thin homotopy transformations. □

Now we can finally prove the following:

Proposition 68. *The pre-2-holonomy (Def. 64) induces a true local 2-holonomy (Def. 63)*

$$\begin{tikzcd}
\mathrm{hol}_i : & & \mathcal{P}_2(U_i) & & \rightarrow & & \mathcal{G} & \\
& x \arrow[rr, bend left, "\gamma"{name=U}] \arrow[rr, bend right, "\gamma'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\Sigma"] & & y & \mapsto & \bullet \arrow[rr, bend left, "W_i[\gamma]"{name=U2}] \arrow[rr, bend right, "W_i[\gamma']"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\mathcal{W}_i^{-1}(\Sigma)"] & & \bullet
\end{tikzcd}$$

if and only if the fake curvature (Def. 56) vanishes.

Proof.

We have already shown that for $dt(B) + F_A = 0$ the pre-2-holonomy indeed maps into a 2-group (Prop. 66) and that its values are well defined on bigons (Prop. 67). What remains to be shown is functoriality, i.e. that the pre-2-holonomy respects the composition of bigons and 2-group elements.

First of all it is immediate that composition of paths is respected, due to the properties of ordinary holonomy. Vertical composition of 2-holonomy (being composition of ordinary holonomy in path space) is completely analogous. The fact that pre-2-holonomy involves the *inverse* path space holonomy takes care of the nature of the vertical product in the 2-group, which reverses the order of factors: In the diagram

$$\begin{tikzcd}
\mathcal{G} & \bullet \arrow[rr, bend left=60, "{W_A[\gamma_1]}"] \arrow[rr, "{W_A[\gamma_2]}"] \arrow[rr, bend right=60, "{W_A[\gamma_3]}"'] \arrow[rr, phantom, shift left=3, "{\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]}"] \arrow[rr, phantom, shift right=3, "{\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]}"] & & \bullet & = & \bullet \arrow[rr, bend left=40, "{W_A[\gamma_1]}"] \arrow[rr, bend right=40, "{W_A[\gamma_3]}"'] \arrow[rr, phantom, "{\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1\circ\Sigma_2]}"] & & \bullet \\
\Uparrow \mathrm{hol} & & \Uparrow & & & & \Uparrow & \\
\mathcal{P}_2(U) & x \arrow[rr, bend left=60, "\gamma_1"] \arrow[rr, "\gamma_2"] \arrow[rr, bend right=60, "\gamma_3"'] \arrow[rr, phantom, shift left=3, "{\Downarrow [\Sigma_1]}"] \arrow[rr, phantom, shift right=3, "{\Downarrow [\Sigma_2]}"] & & y & = & x \arrow[rr, bend left=40, "\gamma_1"] \arrow[rr, bend right=40, "\gamma_3"'] \arrow[rr, phantom, "{\Downarrow [\Sigma_1\circ\Sigma_2]}"] & & y
\end{tikzcd}$$

the top right bigon must be labeled (according to the properties of 2-groups described in Prop. 18) by

$$\begin{aligned}
(W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]) \circ (W_A[\gamma_2], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]) &= (W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]) \\
&= (W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1 \circ \Sigma_2])\,,
\end{aligned}$$

which indeed is the label associated by the hol-functor in the right column of the diagram.

So far we have suppressed in these formulas the reversal 6.19 in the first coordinate of Σ, since it plays no role for the above. This reversal however is essential in order for the hol-functor to respect horizontal composition.

In order to see this it is sufficient to consider *whiskering*, i.e. horizontal composition with identity 2-morphisms.

When whiskering from the left we have

$$\begin{tikzcd}
\mathcal{G} & \bullet \arrow[r, "{W_A[\gamma_1]}"] & \bullet \arrow[r, bend left=40, "{W_A[\gamma_2]}"] \arrow[r, bend right=40, "{W_A[\gamma_2']}"'] \arrow[r, phantom, "{\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma]}"] & \bullet & = & \bullet \arrow[r, bend left=40, "{W_A[\gamma_1\circ\gamma_2]}"] \arrow[r, bend right=40, "{W_A[\gamma_1\circ\gamma_2']}"'] \arrow[r, phantom, "{\Downarrow \alpha(W_A[\gamma_1])(\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma])}"] & \bullet \\
\Uparrow \mathrm{hol} & & \Uparrow & & & \Uparrow & \\
\mathcal{P}_2(U) & x \arrow[r, "\gamma_1"] & y \arrow[r, bend left=40, "\gamma_2"] \arrow[r, bend right=40, "\gamma_2'"'] \arrow[r, phantom, "{\Downarrow [\Sigma]}"] & z & = & x \arrow[r, bend left=40, "\gamma_1\circ\gamma_2"] \arrow[r, bend right=40, "\gamma_1\circ\gamma_2'"'] \arrow[r, phantom, "{\Downarrow [\Sigma]}"] & z
\end{tikzcd}$$

Evaluating the line holonomy in path space for this situation involves taking the path ordered exponential of (*cf.* 6.6)

$$\int\limits_{(\gamma_1\circ\gamma_2)^{-1}} d\sigma\; \alpha\big(W_A^{-1}[(\gamma_1 \circ \gamma_2)^{-1}|_\sigma^1]\big)(B(\sigma))$$

evaluated on the tangent vector to the whiskered Σ. Since this vanishes on γ_1 and using the reparameterization invariance of W_A the above equals

$$\cdots = \alpha(W_A[\gamma_1]) \left(\int\limits_{\gamma_2^{-1}} d\sigma\ \alpha\big(W_A^{-1}[\gamma_2^{-1}|_\sigma^1]\big)(B(\sigma)) \right) .$$

Hence the above diagram does commute. In this computation the path reversal is essential, which of course is related to our convention that parallel transport be to the point with parameter $\sigma = 1$. A simple plausibility argument for this was given at the beginning of §6.3.

Finally, whiskering to the right is trivial, since we can simply use reparametrization invariance to obtain

$$\int\limits_{(\gamma_1\circ\gamma_2)^{-1}} d\sigma\ \alpha\big(W_A[(\gamma_1 \circ \gamma_2)^{-1}|_\sigma^1]\big)(B(\sigma)) = \int\limits_{\gamma_1^{-1}} d\sigma\ \alpha\big(W_A[\gamma_1^{-1}|_{\sigma^1}]\big)(B(\sigma)) \ ,$$

because for right whiskers the integrand vanishes on γ_2.

Since general horizontal composition is obtained by first whiskering and then composing vertically, this also proves that the hol-functor respects general horizontal composition.

In summary, this shows that a pre-2-holonomy with vanishing fake curvature (Def. 56) $dt(B) + F_A = 0$ defines a 2-functor hol: $\mathcal{P}_2(U) \to \mathcal{G}$ and hence a local strict 2-holonomy. □

7. Appendix: Smooth Spaces

It is a sad fact of life that the category of finite-dimensional smooth manifolds Diff is not closed under many useful constructions. For example, a subspace or quotient space of a manifold is usually not a manifold, nor is the space of smooth maps from one manifold to another. Various approaches to remedying this problem have been proposed; here we use the last of several variants proposed by Chen [33, 34] in his work on path integrals.

In what follows, we use **convex set** to mean a convex subset of $\mathbb{R}^n$, where n is arbitrary (not fixed). Any convex set inherits a topology from its inclusion in $\mathbb{R}^n$. We say a map f between convex sets is **smooth** if arbitrarily high derivatives of f exist and are continuous, using the usual definition of derivative as a limit of a quotient.

Definition 69. *A* **smooth space** *is a set X equipped with, for each convex set C, a collection of functions $\phi\colon C \to X$ called* **plots in** *X, such that:*

1. *If $\phi\colon C \to X$ is a plot in X, and $f\colon C' \to C$ is a smooth map between convex sets, then $\phi \circ f$ is a plot in X,*
2. *If $i_\alpha\colon C_\alpha \to C$ is an open cover of a convex set C by convex subsets C_α, and $\phi\colon C \to X$ has the property that $\phi \circ i_\alpha$ is a plot in X for all α, then ϕ is a plot in X.*
3. *Every map from a point to X is a plot in X.*

Definition 70. *A* **smooth map** *from the smooth space X to the smooth space Y is a map $f\colon X \to Y$ such that for every plot ϕ in X, $\phi \circ f$ is a plot in Y.*

It is straightforward to check that there is a category C^∞ whose objects are smooth spaces and whose morphisms are smooth maps.

We make any smooth manifold X into a smooth space by decreeing that the plots $\phi\colon C \to X$ are precisely those maps that are smooth in the usual sense. Given this, one can check that a map between smooth manifolds is smooth in the sense defined above precisely when it is smooth in the usual sense, so no confusion arises. In other words, Diff is a full subcategory of C^∞.

The same procedure lets us make any smooth manifold with boundary into a smooth space. Here we take advantage of plots $\phi\colon C \to X$ where C is a half-space.

The one-point space $*$ is a smooth space such that all maps $\phi\colon C \to *$ are plots. By item 3 of Def. 69, all maps from $*$ into a smooth space are smooth. Since smooth maps are closed under composition, all constant maps between smooth spaces are smooth.

We make a disjoint union $X = \coprod_\alpha X_\alpha$ of smooth spaces into a smooth space as follows. We define a map $\phi\colon C \to X$ to be a plot in X if and only if there is a plot $\psi\colon C \to X_\alpha$ such that ϕ is ψ composed with the inclusion of X_α in X. One can check that X is indeed a smooth space and that it is the coproduct in C^∞ of the smooth spaces X_α.

We make a Cartesian product $X = \prod_\alpha X_\alpha$ of smooth spaces into a smooth space as follows. A map $\phi\colon C \to X$ is the same as a collection of maps $\phi_\alpha\colon C \to X_\alpha$, and we decree

ϕ to be a plot if and only if every map ϕ_α is a plot. One can check that X is indeed a smooth space and that it is the product in the category C^∞ of the smooth spaces X_α.

We make a subset X of a smooth space Y into a smooth space as follows. We define $\phi\colon C \to X$ to be a plot if and only its composite with the inclusion $i\colon X \to Y$ is a plot in Y. One can check that X is indeed a smooth space. Note also that the inclusion i is smooth.

We make a quotient space X of a smooth space Y into a smooth space as follows. Suppose $\sim$ is an equivalence relation on Y. As a set, X will be just the usual quotient $Y/\sim$. Let $j\colon Y \to X$ stand for the quotient map. We make X into a smooth space by decreeing that $\phi\colon C \to X$ is a plot iff there is an open cover of C by convex sets, $i_\alpha C_\alpha \to C$, such that for every α the map $\phi \circ i_\alpha$ is of the form $j \circ \phi_\alpha$ for some plot $\phi_\alpha\colon C_\alpha \to Y$. One can check that X is indeed a smooth space. Note also that the quotient map $j\colon Y \to X$ is smooth.

This issue arises when defining smooth spaces of thin homotopy classes of paths (Example 12) or parametrized bigons (Def. 35)...

Given smooth spaces X and Y, let $C^\infty(X,Y)$ be the set of smooth maps from X to Y. Given a convex set C, we say that a map $\phi\colon C \to C^\infty(X,Y)$ is a plot if and only if $\hat{\phi}\colon C \times X \to Y$ is smooth, where we define

$$\hat{\phi}(c,x) = \phi(c)(x)$$

for all $c \in C$ and $x \in X$. One can check that with this definition $C^\infty(X,Y)$ is a smooth space. One can also check that for any smooth space A, a map $f\colon A \times X \to Y$ is smooth if and only if the corresponding map $\hat{f}\colon A \to X \times Y$ is smooth. So, we have a one-to-one and onto map

$$\begin{array}{rcl} \hat{}\colon C^\infty(A \times X, Y) & \to & C^\infty(A, C^\infty(X,Y)) \\ f & \mapsto & \hat{f} \end{array}$$

and one can even check that this map is smooth, with a smooth inverse. CHECK.

We can summarize many of the facts observed so far in the following result, which actually goes a bit further:

Theorem 71. *The category* C^∞ *is complete, cocomplete, and cartesian closed.*

Proof. First let us show that C^∞ is complete, meaning that every diagram has a limit. Suppose we have any diagram $F\colon D \to \mathrm{C}^\infty$. We show this has a limit as follows. First we take the limit in Set of the underlying diagram of sets and obtain a set $\lim_D F$. We then make this into a smooth space by decreeing the set of plots $\phi\colon C \to \lim_D F$ to be $\lim_D C^\infty(C,F)$. One can check that this choice satisfies properties 1-3 in Definition Def. 69. Moreover, given a smooth space X, a map $f\colon X \to \lim_D F$ is the same as a collection of maps $f_\alpha\colon X \to F(d_\alpha)$ making the obvious triangles commute, where d_α are the objects of D. By definition f is smooth iff for any plot ϕ in X, $f \circ \phi$ is a plot in $\lim_D F$. This is equivalent to demanding that for any plot ϕ in X and any α, $f_\alpha \circ \phi$ is a plot in $F(d_\alpha)$. This in turn is equivalent to demanding that each map f_α is smooth. So, a smooth map $f\colon X \to \lim_D F$ is the same as a collection of smooth maps $f_\alpha\colon X \to F(d_\alpha)$ making the obvious triangles commute. So, $\lim_D F$ is indeed the desired limit.

For colimits we need sheafification...!!!
Then cartesian closedness!!! □

One can develop the whole theory of bundles, connections, differential forms and so on for smooth spaces. We only sketch the first few steps, focusing on what we need for this paper. We define a differential form on a smooth space in terms of its pullbacks along plots:

Definition 72. *A* p-**form** ω *on the smooth space* X *is an assignment of a smooth* p*-form* ω_ϕ *on* $\mathbb{R}^n$ *to each plot* $\phi\colon \mathbb{R}^n \to X$*, satisfying this* **pullback compatibility condition** *for any map* $f\colon \mathbb{R}^m \to \mathbb{R}^n$*:*

$$f^*\omega_\phi = \omega_{\phi \circ f}.$$

The space of p*-forms on* X *is denoted by* $\Omega^p(X)$*.*

Proposition 73. *Given a smooth map* $f\colon X \to Y$ *and* $\omega \in \Omega^p(Y)$ *there is a* p*-form* $f^*\omega \in \Omega^p(X)$ *given by*

$$(f^*\omega)_\phi = \omega_{\phi \circ f}$$

for every plot $\phi\colon \mathbb{R}^n \to X$*.*

We call $f^*\omega$ the **pullback of** ω **along** f. Given a differential form ω on X, the forms ω_ϕ defining it turn out to be just its pullbacks along plots:

Proposition 74. *Given a plot* $\phi\colon \mathbb{R}^n \to X$ *and* $\omega \in \Omega^p(X)$ *we have*

$$\phi^*\omega = \omega_\phi.$$

The above definition and results immediately generalize, for instance, to Lie-group-valued 0-forms $\Omega^0(X, G)$ and to Lie-algebra-valued p-forms $\Omega^p(X, \mathfrak{g})$. An element $F \in \Omega^0(X, G)$ is the same as a smooth map $F\colon X \to G$.

Given a smooth space X, we define the tangent space T_xX to consist of formal linear combinations of smooth maps $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$, modulo the space of formal linear combinations $\sum_i c_i\gamma_i$ with the property that

$$\sum_i c_i \frac{d}{dt} f(\gamma_i(t))|_{t=0} = 0$$

for all smooth functions $f\colon X \to \mathbb{R}$.

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References

[1] J. C. Baez and J. Dolan, Categorification, in *Higher Category Theory*, Contemp. Math. 230. American Mathematical Society, 1998. `math.QA/9802029`.

[2] J. C. Baez, Higher Yang-Mills theory. `hep-th/0206130`.

[3] J. C. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups, *Theory and Applications of Categories* **12** (2004) 423–491. `math.QA/0307200`.

[4] J. C. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory and Applications of Categories* **12** (2004) 492–528. `math.QA/0307263`.

[5] J. C. Baez and U. Schreiber, Higher gauge theory. `math.DG/0511710`.

[6] T. Bartels, Higher gauge theory: 2-bundles. `math.CT/0410328`.

[7] R. Attal, Combinatorics of non-abelian gerbes with connection and curvature. `math-ph/0203056`.

[8] H. Pfeiffer, Higher gauge theory and a non-Abelian generalization of 2-form electromagnetism, *Ann. Phys. (NY)* **308** (2003) 447. `hep-th/0304074`.

[9] F. Girelli and H. Pfeiffer, Higher gauge theory - differential versus integral formulation. `hep-th/0309173`.

[10] O. Alvarez, L. Ferreira, and J. Sánchez Guillén, A new approach to integrable theories in any dimension, *Nucl. Phys.* **B 529** (1998) 689 (1998). `hep-th/9710147`.

[11] U. Schreiber, Nonabelian 2-forms and loop space connections from 2d SCFT deformations. `hep-th/0407122`.

[12] O. Alvarez, 2004. (private communication).

[13] L. Breen and W. Messing, Differential geometry of gerbes. `math.AG/0106083`.

[14] P. Aschieri, L. Cantini, and B. Jurčo, Nonabelian bundle gerbes, their differential geometry and gauge theory, *Commun. Math. Phys.* **254** *(2005) 367–400* `hep-th/0312154`.

[15] P. Aschieri and B. Jurčo, Gerbes, M5-brane anomalies and E_8 gauge theory, *J. High Energy Phys.* **10** *(2004) 068*. `hep-th/0409200`.

[16] M. Murray, Bundle gerbes. `math.DG/9407015`.

[17] D. Stevenson, The Geometry of Bundle Gerbes, Ph.D. thesis, University of Adelaide, 2000. `math.DG/0004117`.

[18] M. Mackaay and R. Picken, Holonomy and parallel transport for abelian gerbes. `math.DG/0007053`.

[19] U. Schreiber, On deformations of 2d SCFTs, *J. High Energy Phys.* **06** *(2004) 058*. `hep-th/0401175`.

[20] E. Witten, Conformal Field Theory In Four and Six Dimensions, in *Topology, Geometry and Quantum Field Theory*, ed. U. Tillmann. Cambridge U. Press, 2004, pp. 405–419. Slides available at `http://www.maths.ox.ac.uk/notices/events/special/tgqfts/photos/witten/`.

[21] B. Eckmann and P. Hilton, Group-like structures in categories, *Math. Ann.* **145** (1962) 227–255.

[22] M. Forrester-Barker, Group objects and internal categories, `math.CT/0212065`.

[23] S. Mac Lane, *Categories for the Working Mathematician.* Springer, 1998.

[24] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization.* Birkhauser, 1993.

[25] S. J. A. Carey and M. Murray, *Holonomy on D-branes.* `hep-th/0204199`.

[26] M. Caicedo, I. Martín, and A. Restuccia, Gerbes and duality, *Ann. Phys. (NY)* **300** *(2002) 32.* `hep-th/0205002`.

[27] N. Hitchin, Lectures on special Lagrangian submanifolds. `math.DG/9907034`.

[28] A. Keurentjes, Flat connections from flat gerbes, *Fortschr. Phys.* **50** *(2002) 916.* `hep-th/0201072`.

[29] F. W. Lawvere, *Functorial Semantics of Algebraic Categories.* Ph.D. thesis, Columbia University, 1963. Available online at `http://www.tac.mta.ca/tac/reprints/articles/5/tr5abs.html`.

[30] C. Ehresmann, *Introduction to the theory of structured categories.* Technical Report, U. of Kansas at Lawrence, 1966.

[31] F. Borceux, *Handbook of Categorical Algebra 1: Basic Category Theory.* Cambridge U. Press, 1994.

[32] K. T. Chen, Iterated integrals, fundamental groups and covering spaces, *Trans. Amer. Math. Soc.* **206** (1975), 83–98.

[33] K. T. Chen, Iterated path integrals, *Bull. Amer. Math. Soc.* **83** (1977), 831–879.

[34] K. T. Chen, On differentiable spaces, in *Categories in Continuum Physics*, eds. F. W. Lawvere and S. Schanuel, Lecture Notes in Mathematics **1174**, Springer, Berlin, 1986, pp. 38-42.

[35] R. Brown and C. B. Spencer, $\mathcal{G}$-groupoids, crossed modules, and the classifying space of a topological group, *Proc. Kon. Akad. v. Wet.* **79** (1976), 296–302.

[36] R. Brown, *Groupoids and crossed objects in algebraic topology, Homology, Homotopy and Applications* **1** (1999), 1–78. Available online at `http://www.math.rutgers.edu/hha/volumes/1999/volume1-1.htm`.

[37] L. Crane and M. D. Sheppeard, 2-Categorical Poincaré representations and state sum applications. `math.QA/0306440`.

[38] L. Crane and D. Yetter, Measurable categories and 2-groups. `math.QA/0305176`.

[39] F. Thompson, Introducing quaternionic gerbes. `math.DG/0009201`.

[40] A. Pressley and G. Segal, *Loop Groups.* Oxford U. Press, 1986.

[41] J.C̃. Baez, A. Crans, D. Stevenson and U. Schreiber, From loop groups to 2-groups. `math.QA/0504123`.

[42] E. Witten, The index of the Dirac operator in loop space, in *Elliptic Curves and Modular Forms in Algebraic Topology*, ed. P. S. Landweber, Lecture Notes in Mathematics **1326**, Springer, Berlin, 1988, pp. 161–181.

[43] S. Stolz and P. Teichner, What is an elliptic object?, in *Topology, Geometry and Quantum Field Theory: Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal*, ed. U. Tillmann. Cambridge U. Press, Cambridge, 2004.

[44] A. Grothendieck, *Pursuing Stacks,* 1983. Available online at `http://www.math.jussieu.fr/∼leila/mathtexts.php`.

[45] J. Giraud, *Cohomologie Non-Abélienne.* Springer, 1971.

[46] I. Moerdijk, Introduction to the language of stacks and gerbes. `math.AT/0212266`.

[47] A. Caray, S. Johnson, M. Murray, D. Stevenson, and B. Wang, Bundle gerbes for Chern-Simons and Wess-Zumino-Witten theories. `math.DG/0410013`.

[48] G. Kelly and R. Street, Review of the elements of 2-categories, in Lecture Notes in Mathematics **420**, pp. 75-103. Springer Verlag, 1974.

[49] N. E. Steenrod, *The Topology of Fibre Bundles*, Princeton U. Press, 1951.

[50] E. Getzler, D. Jones, and S. Petrack, Differential forms on loop spaces and the cyclic bar complex, *Topology* **30** (1991), 339–371.

[51] A. Caetano and R. Picken, An axiomatic definition of holonomy, *Int. J. Math.* **5** (1993), 835–848.

[52] C. Hofman, Nonabelian 2-forms. `hep-th/0207017`.

[53] I. Chepelev, Non-abelian Wilson surfaces, *J. High Energy Phys.* **02** *(2002) 013.*

[54] D. Chatterjee, *On Gerbs*. PhD thesis, University of Cambridge, 1998. Available online at `http://www.ma.utexas.edu/~hausel/hitchin/hitchinstudents/chatterjee.pdf`.

[55] J. Polchinski, *String Theory.* Cambridge U. Press, 1998.

[56] R. Karp, F. Mansouri, and J. Rno, Product integral formalism and non-abelian Stokes theorem, *J. Math. Phys.* **40** *(1999) 6033.* `hep-th/9910173`.