# From Loop Groups

# To 2-Groups

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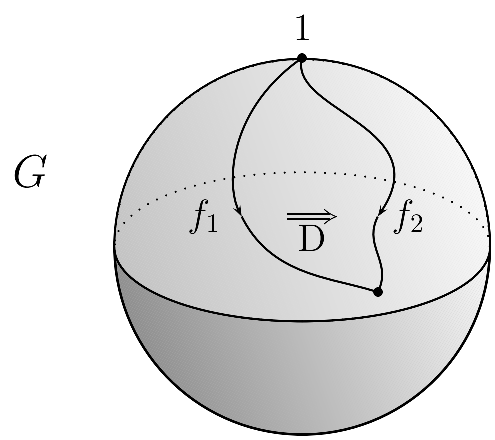
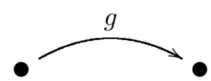



More details at:

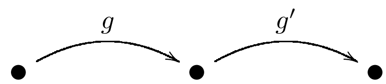
**http://math.ucr.edu/home/baez/2group/**

# Higher Gauge Theory

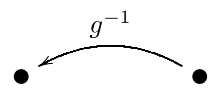
Ordinary gauge theory describes how point particles transform as we move them along 1-dimensional paths. It is natural to assign a group element to each path:



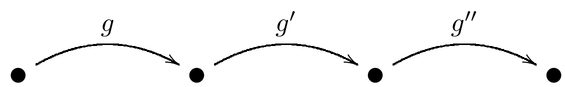

since composition of paths then corresponds to multiplication:




while reversing the direction of a path corresponds to taking inverses:

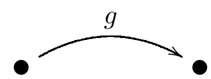



and the associative law makes this composite unambiguous:

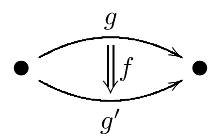



In short: *the topology dictates the algebra!*

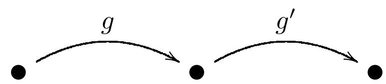
Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings. For this we must categorify the notion of a group! A '2-group' has objects:



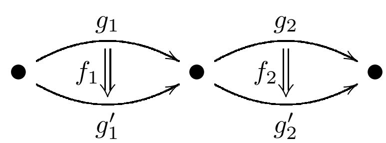

and also morphisms:



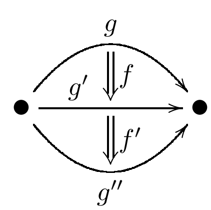

We can multiply objects:




multiply morphisms:




and also compose morphisms:




Various laws should hold....

In fact, we can make this precise and categorify the whole theory of Lie groups, Lie algebras, bundles, connections and curvature. But for now, let's just look at 2-groups and Lie 2-algebras.

# 2-Groups

A group is a monoid where every element has an inverse. Let's categorify this!

A **2-group** is a monoidal category where every object $x$ has a 'weak inverse':

$$x \otimes y \cong y \otimes x \cong \mathrm{I}$$

and every morphism $f$ has an inverse:

$$fg = gf = 1.$$

A **homomorphism** between 2-groups is a monoidal functors. A **2-homomorphism** is a monoidal natural transformation. The 2-groups $X$ and $X'$ are **equivalent** if there are homomorphisms

$$f \colon X \to X' \qquad \bar{f} \colon X' \to X$$

that are inverses up to 2-isomorphism:

$$f\bar{f} \cong 1, \qquad \bar{f}f \cong 1.$$

**Theorem.** 2-groups are classified up to equivalence by quadruples consisting of:

- a group $G$,
- an abelian group $H$,
- an action $\alpha$ of $G$ as automorphisms of $H$,
- an element $[a] \in H^3(G, H)$.

# Lie 2-Algebras

To categorify the concept of 'Lie algebra' we must first treat the concept of 'vector space':

A **2-vector space** $L$ is a category for which the set of objects and the set of morphisms are vector spaces, and all the category operations are linear.

We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

**Theorem.** The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.

The objects of the 2-vector space form $C_0$. The morphisms $f\colon 0 \to x$ form $C_1$, with $df = x$.

A **Lie 2-algebra** consists of:

- a 2-vector space $L$

equipped with:

- a functor called the **bracket**:

$$[\cdot,\cdot]\colon L \times L \to L,$$

  bilinear and skew-symmetric as a function of objects and morphisms,

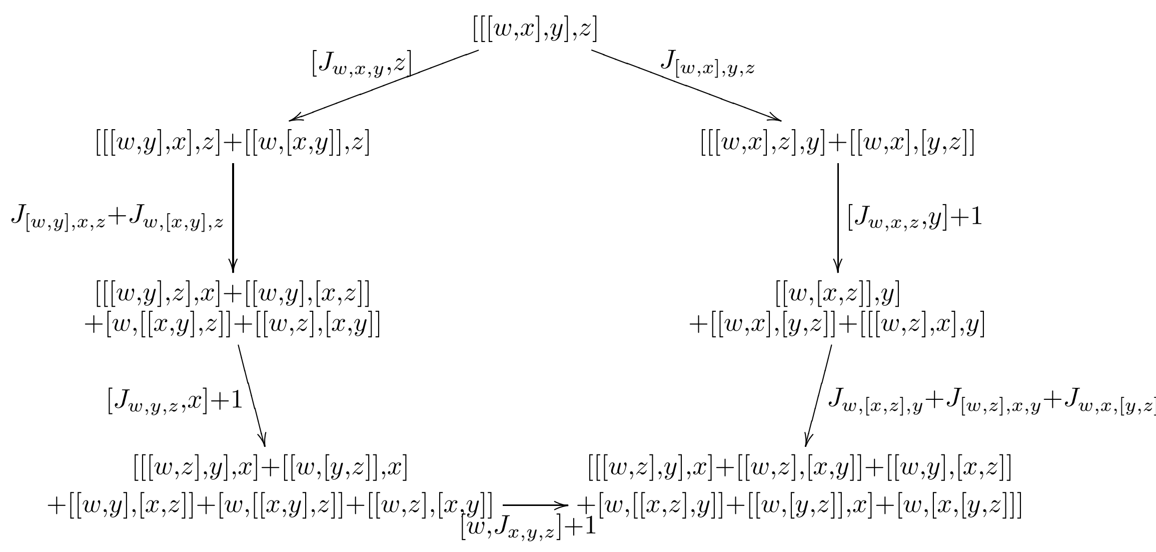
- a natural isomorphism called the **Jacobiator**:

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

  trilinear and antisymmetric as a function of the objects $x, y, z$,

such that:

- the **Jacobiator identity** holds: the following diagram commutes:



We can also define homomorphisms between Lie 2-algebras, and 2-homomorphisms between these. The Lie 2-algebras $L$ and $L'$ are **equivalent** if there are homomorphisms

$$f\colon L \to L' \qquad \bar{f}\colon L' \to L$$

that are inverses up to 2-isomorphism.

**Theorem.** Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra (= vector space) $\mathfrak{h}$,
- a representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

*Just like the classification of 2-groups, but with Lie algebra cohomology replacing group cohomology!*

Let's use this classification to find some interesting Lie 2-algebras. Then let's try to find the corresponding Lie 2-groups. A **Lie 2-group** is a 2-group where everything in sight is smooth.

# The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra having $\mathfrak{g}$ as objects we need:

- a vector space $\mathfrak{h}$,
- a representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Assume without loss of generality that $\rho$ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. By Whitehead's lemma, this only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

with a nontrivial 3-cocycle given by:

$$\nu(x, y, z) = \langle [x, y], z \rangle.$$

Using $k$ times this to define the Jacobiator, we get a Lie 2-algebra we call $\mathfrak{g}_k$.

In short: *every simple Lie algebra $\mathfrak{g}$ admits a 1-parameter deformation $\mathfrak{g}_k$ in the world of Lie 2-algebras!*

Do these Lie 2-algebras $\mathfrak{g}_k$ come from Lie 2-groups? We should use the relation between Lie group cohomology and Lie algebra cohomology. How is $H^3(G, \mathrm{U}(1))$ related to $H^3(\mathfrak{g}, \mathbb{R})$?

Suppose $G$ is a simply-connected compact simple Lie group whose Lie algebra is $\mathfrak{g}$. We have

$$H^3(G, \mathrm{U}(1)) \hookleftarrow \mathbb{Z} \hookrightarrow \mathbb{R} \cong H^3(\mathfrak{g}, \mathbb{R})$$

So, for $k \in \mathbb{Z}$ we get a 2-group $G_k$ with $G$ as objects and $\mathrm{U}(1)$ as the automorphisms of any object, with nontrivial associator when $k \neq 0$.

Can $G_k$ be made into a Lie 2-group?

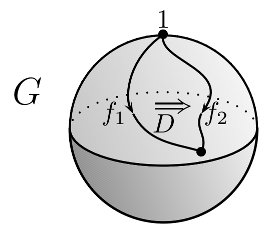
Here's the bad news:

**Theorem.** Unless $k = 0$, there is no way to give the 2-group $G_k$ the structure of a Lie 2-group for which the group $G$ of objects and the group $\mathrm{U}(1)$ of endomorphisms of any object are given their usual topology.

However, all is not lost. $\mathfrak{g}_k$ is *equivalent* to a Lie 2-algebra that *does* come from a Lie 2-group! However, this Lie 2-algebra is *infinite-dimensional!* This is where loop groups enter the game....

**Theorem.** For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.

An object of $\mathcal{P}_k G$ is a smooth path $f \colon [0, 2\pi] \to G$ starting at the identity. A morphism from $f_1$ to $f_2$ is an equivalence class of pairs $(D, \alpha)$ consisting of a disk $D$ going from $f_1$ to $f_2$ together with $\alpha \in \mathrm{U}(1)$:




For any two such pairs $(D_1, \alpha_1)$ and $(D_2, \alpha_2)$ there is a 3-ball $B$ whose boundary is $D_1 \cup D_2$, and the pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where $\nu$ is the left-invariant closed 3-form on $G$ with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that $\nu$ gives an integral cohomology class.

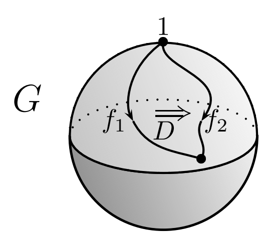
There's an obvious way to compose morphisms in $\mathcal{P}_k G$, and the resulting category inherits a Lie 2-group structure from the Lie group structure of $G$.

# The Role of Loop Groups

We can also describe the 2-group $\mathcal{P}_k G$ as follows:

- An object of $\mathcal{P}_k G$ is a smooth path in $G$ starting at the identity.
- Given objects $f_1, f_2 \in \mathcal{P}_k G$, a morphism
$$\widehat{\ell}\colon f_1 \to f_2$$
is an element $\widehat{\ell} \in \widehat{\Omega_k G}$ with
$$p(\widehat{\ell}) = f_2/f_1$$
where $\widehat{\Omega_k G}$ is the level-$k$ Kac–Moody central extension of the loop group $\Omega G$:
$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \xrightarrow{p} \Omega G \longrightarrow 1$$

Note: $p(\widehat{\ell})$ is a loop in $G$. We can get such a loop with
$$p(\widehat{\ell}) = f_2/f_1$$
from a disk $D$ like this:




An element $\widehat{\ell} \in \widehat{\Omega_k G}$ is an equivalence class of pairs $[D, \alpha]$ consisting of such a disk $D$ together with $\alpha \in \mathrm{U}(1)$.

# An Application to Topology

For any simply-connected compact simple Lie group $G$ there is a topological group $\widehat{G}$ obtained by killing the third homotopy group of $G$. When $G = \mathrm{Spin}(n)$, $\widehat{G}$ is called $\mathrm{String}(n)$.

**Theorem.** For any $k \in \mathbb{Z}$, the geometric realization of the nerve of $\mathcal{P}_k G$ is a topological group $|\mathcal{P}_k G|$. We have

$$\pi_3(|\mathcal{P}_k G|) \cong \mathbb{Z}/k\mathbb{Z}$$

When $k = \pm 1$,

$$|\mathcal{P}_k G| \simeq \widehat{G}.$$

*This, and the appearance of the Kac–Moody central extension of $\Omega G$, suggest that $\mathcal{P}_k G$ will be an especially interesting Lie 2-group for applications of higher gauge theory to string theory.*