Infinite-Dimensional Representations of 2-Groups

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Abstract

A '2-group' is a category equipped with a multiplication satisfying laws like those of a group. Just as groups have representations on vector spaces, 2-groups have representations on '2-vector spaces', which are categories analogous to vector spaces. Unfortunately, Lie 2-groups typically have few representations on the finite-dimensional 2-vector spaces introduced by Kapranov and Voevodsky. For this reason, Crane, Sheppeard and Yetter introduced certain infinite-dimensional 2-vector spaces called 'measurable categories' (since they are closely related to measurable fields of Hilbert spaces), and used these to study infinite-dimensional representations of certain Lie 2-groups. Here we continue this work. We begin with a detailed study of measurable categories. Then we give a geometrical description of the measurable representations, intertwiners and 2-intertwiners for any skeletal measurable 2-group. We study tensor products and direct sums for representations, and various concepts of subrepresentation. We describe direct sums of intertwiners, and sub-intertwiners—features not seen in ordinary group representation theory. We study irreducible and indecomposable representations and intertwiners. We also study 'irretractable' representations—another feature not seen in ordinary group representation theory. Finally, we argue that measurable categories equipped with some extra structure deserve to be considered 'separable 2-Hilbert spaces', and compare this idea to a tentative definition of 2-Hilbert spaces as representation categories of commutative von Neumann algebras.

Contents

1 Introduction

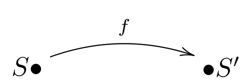
The goal of 'categorification' is to develop a richer version of existing mathematics by replacing sets with categories. This lets us exploit the following analogy:

set theory	category theory
elements	objects
equations between elements	isomorphisms between objects
sets	categories
functions	functors
equations between functions	natural isomorphisms between functors

Just as sets have elements, categories have objects. Just as there are functions between sets, there are functors between categories. The correct analogue of an equation between elements is not an equation between objects, but an isomorphism. More generally, the analog of an equation between functions is a natural isomorphism between functors.

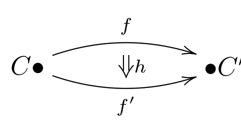
The word 'categorification' was first coined by Louis Crane [24] in the context of mathematical physics. Applications to this subject have always been among the most exciting [9], since categorification holds the promise of generalizing some of the special features of low-dimensional physics to higher dimensions. The reason is that categorification *boosts the dimension by one*.

To see this in the simplest possible way, note that we can draw sets as 0-dimensional dots and functions between sets as 1-dimensional arrows:



If we could draw all the sets in the world this way, and all the functions between them, we would have a picture of the category of all sets.

But there are many categories beside the category of sets, and when we study categories *en masse* we see an additional layer of structure. We can draw categories as dots, and functors between categories as arrows. But what about natural isomorphisms between functors, or more general natural transformations between functors? We can draw these as 2-dimensional surfaces:



So, the dimension of our picture has been boosted by one! Instead of merely a category of all categories, we say we have a '2-category'. If we could draw all the categories in the world this way, and all functors between them, and all natural transformations between those, we would have a picture of the 2-category of all categories.

This story continues indefinitely to higher and higher dimensions: categorification is a process than can be iterated. But our goal here lies in a different direction: we wish to take a specific branch of mathematics, the theory of infinite-dimensional group representations, and categorify that just once. This might seem like a purely formal exercise, but we shall see otherwise. In fact, the resulting theory has fascinating relations both to well-known topics within mathematics (fields of Hilbert spaces and Mackey's theory of induced group representations) and to interesting ideas in physics (spin foam models of quantum gravity, most notably the Crane–Sheppeard model).

1.1 2-Groups

To categorify group representation theory, we must first choose a way to categorify the basic notions involved: the notions of 'group' and 'vector space'. At present, categorifying mathematical definitions is not a completely straightforward exercise: it requires a bit of creativity and good taste. So, there is work to be done here.

By now, however, there is a fairly uncontroversial way to categorify the concept of 'group'. The resulting notion of '2-group' can be defined in various equivalent ways [8]. For example, we can think of a 2-group as a category equipped with a multiplication satisfying the usual axioms for a group. Since categorification involves replacing equations by natural isomorphisms, we should demand that the group axioms hold *up to natural isomorphism*. Then we should demand that these isomorphisms obey some laws of their own, called 'coherence laws'. This is where the creativity comes into play. Luckily, everyone agrees on the correct coherence laws for 2-groups.

However, to simplify our task in this paper, we shall only consider 'strict' 2-groups, where the axioms for a group hold as *equations*—not just up to natural isomorphisms. This lets us ignore the issue of coherence laws. Another advantage of strict 2-groups is that they are essentially the same as 'crossed modules' [35]. These were first introduced by Mac Lane and Whitehead as a generalization of the fundamental group of topological space [54]. Just as the fundamental group keeps track of all the 1-dimensional homotopy information of a connected space, the 'fundamental crossed module' keeps track of all its 1- and 2-dimensional homotopy information. As a result, crossed modules have been well studied: many examples, many constructions, and many general results are known [22]. This work makes it clear that strict 2-groups are a significant but still tractable generalization of groups.

Henceforth, we shall always use the term '2-group' to mean a strict 2-group. Suppose $\mathcal{G}$ is a 2-group of this kind. Since $\mathcal{G}$ is a category, it has objects and morphisms. The objects form a group under multiplication, so we can use them to describe symmetries. The new feature, where we go beyond traditional group theory, is the morphisms. For most of our more substantial results, we shall make a drastic simplifying assumption: we shall assume $\mathcal{G}$ is not only strict but also 'skeletal'. This means that there only exists a morphism from one object of $\mathcal{G}$ to another if these objects are actually equal. In other words, all the morphisms between objects of $\mathcal{G}$ are actually automorphisms. Since the objects of $\mathcal{G}$ describe symmetries, their automorphisms describe *symmetries of symmetries*.

The reader should not be fooled by the somewhat intimidating language. A skeletal 2-group is really a very simple thing. Using the theory of crossed modules, explained in Section 2.1.2, we shall see that a skeletal 2-group $\mathcal{G}$ consists of:

- a group G (the group of objects of $\mathcal{G}$),
- an abelian group H (the group of automorphisms of any object),
- a left action $\rhd$ of G as automorphisms of H.

A nice example is the 'Poincaré 2-group', first discovered by one of the authors [4]. But to understand this, and to prepare ourselves for the discussion of physics applications later in this introduction, let us first recall the ordinary Poincaré group.

In special relativity, we think of a point $\boldsymbol{x} = (t, x, y, z)$ in $\mathbb{R}^4$ as describing the time and location of an event. We equip $\mathbb{R}^4$ with a bilinear form, the so-called 'Minkowski metric':

$$\boldsymbol{x} \cdot \boldsymbol{x}' = tt' - xx' - yy' - zz'$$

which serves as substitute for the usual dot product on $\mathbb{R}^3$. With this extra structure, $\mathbb{R}^4$ is called 'Minkowski spacetime'. The group of all linear transformations

$$T\colon \mathbb{R}^4 \to \mathbb{R}^4$$

preserving the Minkowski metric is called $\mathrm{O}(3,1)$. The connected component of the identity in this group is called $\mathrm{SO}_0(3,1)$. This smaller group is generated by rotations in space together with transformations that mix time and space coordinates. Elements of $\mathrm{SO}_0(3,1)$ are called 'Lorentz transformations'. In special relativity, we think of Lorentz transformations as symmetries of spacetime. However, we also want to count translations of $\mathbb{R}^4$ as symmetries. To include these, we need to take the semidirect product

$$\mathrm{SO}_0(3,1) \ltimes \mathbb{R}^4,$$

and this is called the **Poincaré group**.

The Poincaré 2-group is built from the same ingredients, Lorentz transformation and translations, but in a different way. Now Lorentz transformations are treated as symmetries—that is, objects—while the translations are treated as symmetries of symmetries—that is, morphisms. More precisely, the **Poincaré 2-group** is defined to be the skeletal 2-group with:

- $G = \mathrm{SO}_0(3,1)$: the group of Lorentz transformations,
- $H = \mathbb{R}^4$: the group of translations of Minkowski space,
- the obvious action of $\mathrm{SO}_0(3,1)$ on $\mathbb{R}^4$.

As we shall see, the representations of this particular 2-group may have interesting applications to physics. For other examples of 2-groups, see our invitation to 'higher gauge theory' [7]. This is a generalization of gauge theory where 2-groups replace groups.

1.2 2-Vector spaces

Just as groups act on sets, 2-groups can act on categories. If a category is equipped with structure analogous to that of a vector space, we may call it a '2-vector space', and call a 2-group action preserving this structure a 'representation'. There is, however, quite a bit of experimentation underway when it comes to axiomatizing the notion of '2-vector space'. In this paper we investigate representations of 2-groups on infinite-dimensional 2-vector spaces, following a line of work initiated by Crane, Sheppeard and Yetter [26, 27, 73]. A quick review of the history will explain why this is a good idea.

To begin with, finite-dimensional 2-vector spaces were introduced by Kapranov and Voevodsky [44]. Their idea was to replace the 'ground field' $\mathbb{C}$ by the category Vect of finite-dimensional complex vector spaces, and exploit this analogy:

ordinary linear algebra	higher linear algebra
$\mathbb{C}$	Vect
$+$	$\oplus$
$\times$	$\otimes$
0	$\{0\}$
1	$\mathbb{C}$

Just as every finite-dimensional vector space is isomorphic to $\mathbb{C}^N$ for some N, every finite-dimensional Kapranov–Voevodsky 2-vector space is equivalent to Vect^N for some N. We can take this as a *definition* of these 2-vector spaces — but just as with ordinary vector spaces, there are also intrinsic characterizations which make this result into a theorem [58, 72].

Similarly, just as every linear map $T\colon \mathbb{C}^M \to \mathbb{C}^N$ is equal to one given by a $N \times M$ matrix of complex numbers, every linear map $T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ is isomorphic to one given by an $N \times M$ matrix of vector spaces. Matrix addition and multiplication work as usual, but with $\oplus$ and $\otimes$ replacing the usual addition and multiplication of complex numbers.

The really new feature of higher linear algebra is that we also have '2-maps' between linear maps. If we have linear maps $T, T'\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ given by $N \times M$ matrices of vector spaces $T_{n,m}$ and $T'_{n,m}$, then a 2-map $\alpha\colon T \Rightarrow T'$ is a matrix of linear operators $\alpha_{n,m}\colon T_{n,m} \to T'_{n,m}$. If we draw linear maps as arrows:

$$\mathrm{Vect}^M \xrightarrow{T} \mathrm{Vect}^N$$

then we should draw 2-maps as 2-dimensional surfaces, like this:

$$\mathrm{Vect}^M \underset{T'}{\overset{T}{\rightrightarrows}} \mathrm{Vect}^N \quad \Downarrow\alpha$$

So, compared to ordinary group representation theory, the key novelty of 2-group representation theory is that besides intertwining operators between representations, we also have '2-intertwiners', drawn as surfaces. This boosts the dimension of our diagrams by one, giving 2-group representation theory an intrinsically 2-dimensional character.

The study of representations of 2-groups on Kapranov–Voevodsky 2-vector spaces was initiated by Barrett and Mackaay [18], and continued by Elgueta [32]. They came to some upsetting conclusions. To understand these, we need to know a bit more about 2-vector spaces.

An object of Vect^N is an N-tuple of finite-dimensional vector spaces $(V_1, \ldots, V_N)$, so every object is a direct sum of certain special objects

$$e_i = (0, \ldots, \underbrace{\mathbb{C}}_{i\text{th place}}, \ldots, 0).$$

These objects e_i are analogous to the 'standard basis' of $\mathbb{C}^N$. However, unlike the case of $\mathbb{C}^N$, these objects e_i are essentially the *only* basis of Vect^N. More precisely, given any other basis e'_i, we have $e'_i \cong e_{\sigma(i)}$ for some permutation σ.

This fact has serious consequences for representation theory. A 2-group $\mathcal{G}$ has a group G of objects. Given a representation of $\mathcal{G}$ on Vect^N, each $g \in G$ maps the standard basis e_i to some new basis e'_i, and thus determines a permutation σ. So, we automatically get an action of G on the finite set $\{1, \ldots, N\}$.

If G is finite, it will typically have many actions on finite sets. So, we can expect that finite 2-groups have enough interesting representations on Kapranov–Voevodsky 2-vector spaces to yield an interesting theory. But there are many 'Lie 2-groups', such as the Poincaré 2-group, where the group of objects is a Lie group with few nontrivial actions on finite sets. Such 2-groups have few representations on Kapranov–Voevodsky 2-vector spaces.

This prompted the search for a 'less discrete' version of Kapranov–Voevodsky 2-vector spaces, where the finite index set $\{1, \ldots, N\}$ is replaced by something on which a Lie group can act in an interesting way. Crane, Sheppeard and Yetter [26, 27, 73] suggested replacing the index set by a measurable space X and replacing N-tuples of finite-dimensional vector spaces by 'measurable fields of Hilbert spaces' on X.

Measurable fields of Hilbert spaces have long been important for studying group representations [51], von Neumann algebras [29], and their applications to quantum physics [52, 71]. Roughly, a measurable field of Hilbert spaces on a measurable space X can be thought of as assigning a Hilbert space to each $x \in X$, in a way that varies measurably with x. There is also a well-known concept of 'measurable field of bounded operators' between measurable fields of Hilbert spaces over a fixed space X. These make measurable fields of Hilbert spaces over X into the objects of a category H^X. This is the prototypical example of what Crane, Sheppeard and Yetter call a 'measurable category'.

When X is finite, H^X is essentially just a Kapranov–Voevodsky 2-vector space. If X is finite and equipped with a measure, H^X acquires a kind of inner product, so it becomes a finite-dimensional '2-Hilbert space' [3]. When X is infinite, we should think of the measurable category H^X as some sort of *infinite-dimensional* 2-vector space. However, it lacks some features we expect from an infinite-dimensional 2-Hilbert space: in particular, there is no inner product of objects. We discuss this issue further in Section 5.

Most importantly, since Lie groups have many actions on measurable spaces, there is a rich supply of representations of Lie 2-groups on measurable categories. As we shall see, a representation of a 2-group $\mathcal{G}$ on the category H^X gives, in particular, an action of the group G of objects on the space X, just as a representation on Vect^N gave a group actions on an N-element set. These actions lead naturally to a geometric picture of the representation theory.

In fact, a measurable category H^X already has a considerable geometric flavor. To appreciate this, it helps to follow Mackey [52] and call a measurable field of Hilbert spaces on the measurable space X a 'measurable Hilbert space bundle' over X. Indeed, such a field $\mathcal{H}$ resembles a vector bundle in that it assigns a Hilbert space $\mathcal{H}_x$ to each point $x \in X$. The difference is that, since $\mathcal{H}$ lives in the world of measure theory rather than topology, we only require that each point x lie in a *measurable* subset of X over which $\mathcal{H}$ can be trivialized, and we only require the existence of *measurable* transition functions. As a result, we can always write X as a disjoint union of countably many measurable subsets on which $\mathcal{H}_x$ has constant dimension. In practice, we demand that this dimension be finite or countably infinite. Similarly, measurable fields of bounded operators may be viewed as measurable bundle maps. So, the measurable category H^X may be viewed as a measurable version of the category of Hilbert space bundles over X. In concrete examples, X is often a manifold or smooth algebraic variety, and measurable fields of Hilbert spaces often arise from bundles or coherent sheaves of Hilbert spaces over X.

1.3 Representations

The study of representations of skeletal 2-groups on measurable categories was begun by Crane and Yetter [27]. The special case of the Poincaré 2-group was studied by Crane and Sheppeard [26].

They noticed interesting connections to the orbit method in geometric quantization, and also to the theory of discrete subgroups of $\mathrm{SO}(3,1)$, known as 'Kleinian groups'. These observations suggest that Lie 2-group representations on measurable categories deserve a thorough and careful treatment.

This, then, is the goal of the present text. We give *geometric* descriptions of:

- a representation ρ of a skeletal 2-group $\mathcal{G}$ on a measurable category H^X,
- an intertwiner between such representations: $\rho \xrightarrow{\phi} \rho'$
- a 2-intertwiner between such intertwiners: $\rho \underset{\phi'}{\overset{\phi}{\Downarrow \alpha}} \rho'$.

We use the term 'intertwiner' as short for 'intertwining operator'. This is a commonly used term for a morphism between group representations; here we use it to mean a morphism between 2-group representations. But in addition to intertwiners, we have something really new: 2-intertwiners between interwiners! This extra layer of structure arises from categorification.

We define all these concepts in Sections 2 and 3. Instead of previewing the definitions here, we prefer to sketch the geometric picture that emerges in Section 4. So, we now assume $\mathcal{G}$ is a skeletal 2-group described by the data $(G, H, \rhd)$, as above. We also assume in what follows that all the spaces and maps involved are measurable. Under these assumptions we can describe representations of $\mathcal{G}$, as well as intertwiners and 2-intertwiners, in terms of familiar geometric constructions—but living in the category of measurable spaces, rather than smooth manifolds. Essentially—ignoring various technical issues which we discuss later—we obtain the following dictionary relating representation theory to geometry.

representation theory	geometry
a representation of $\mathcal{G}$ on H^X	a right action of G on X, and a map $X \to H^*$ making X a 'measurable G-equivariant bundle' over H^*
an intertwiner between representations on H^X and H^Y	a 'Hilbert G-bundle' over the pullback of G-equivariant bundles and a 'G-equivariant measurable family of measures' μ_y on X
a 2-intertwiner	a map of Hilbert G-bundles

This dictionary requires some explanation! First, H^* here is not quite the Pontrjagin dual of H, but rather the group, under pointwise multiplication, of measurable homomorphisms

$$\chi \colon H \to \mathbb{C}^\times$$

where $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers. However, this group H^* contains the Pontrjagin dual of H. It turns out that a measurable homomorphism like χ above, with our definition of measurable group, is automatically also continuous. Since $\mathbb{C}^\times \cong \mathrm{U}(1) \times \mathbb{R}$, we have

$$H^* = \widehat{H} \times \hom(H, \mathbb{R})$$

where $\widehat{H}$ is the Pontrjagin dual of H. One can consistently restrict to 'unitary' representations of $\mathcal{G}$, where we replace H^* by $\widehat{H}$ in the above table. In most of the paper, we shall have no reason to make this restriction, but it is often useful in examples, as we shall see below.

In any case, under some mild conditions on H, H^* is again a measurable space, and its group operations are measurable. The left action $\rhd$ of G on H naturally induces a right action of G on H^*, say $(\chi, g) \mapsto \chi_g$, given by

$$\chi_g(h) = \chi(g \rhd h).$$

This promotes H^* to a right G-space.

As indicated in the chart, a representation of $\mathcal{G}$ is simply a G-equivariant map $X \to H^*$, where X is a measurable G-space. Because of the measure-theoretic context, we are happy to call this a 'bundle' even with no implied local triviality in the topological sense. Indeed, most of the fibers may even be empty. Because of the G-equivariance, however, fibers are isomorphic along any given G-orbit in H^*.

This geometric pictures helps us understand irreducibility and related notions for 2-group representations. Recall that for ordinary groups, a representation is 'irreducible' if it has no subrepresentations other than the 0-dimensional representation and itself. It is 'indecomposable' if it has no direct summands other than the 0-dimensional representation and itself. Since every direct summand is a subrepresentation, every indecomposable representation is irreducible. The converse is generally false. However, it is true in some cases: for example, every *unitary* irreducible representation is indecomposable.

The situation with 2-groups is more subtle. The notions of subrepresentation and direct summand generalize to 2-group representations, but there is also an intermediate notion: a 'retract'. In fact this notion already exists for group representations. A group representation ρ' is a 'retract' of ρ if ρ' is a subrepresentation and there is also an intertwiner projecting down from ρ to this subrepresentation. So, we may say a representation is 'irretractable' if it has no retracts other than the 0-dimensional representation and itself. But for group representations, a retract turns out to be exactly the same thing as a direct summand, so there is no need for these additional notions.

However, we can generalize the concept of 'retract' to 2-group representations—and now things become more interesting! Now we have:

$$\text{direct summand} \Longrightarrow \text{retract} \Longrightarrow \text{subrepresentation}$$

and thus:

$$\text{irreducible} \Longrightarrow \text{irretractable} \Longrightarrow \text{indecomposable}$$

None of these implications are reversible, except *perhaps* every irretractable representation is irreducible. At present this question is unsettled.

Indecomposable and irretractable representations play important roles in our work. Each has a nice geometric picture. Suppose we have a representation of our skeletal 2-group $\mathcal{G}$ corresponding to a G-equivariant map $X \to H^*$. If the G-space X has more than a single orbit, then we can write it as a disjoint union of G-spaces $X = X' \cup X''$ and split the map $X \to H^*$ into a pair of maps. This amounts to writing our 2-group representation as a direct sum of representations. So, a representation on H^X is indecomposable if the G-action on X is transitive.

By equivariance, this implies that the image of the corresponding map $X \to H^*$ is a single orbit of H^*, and that the stabilizer of a point in X is a subgroup of the stabilizer of its image in H^*. In other words, the orbit in H^* is a quotient of X. It follows that indecomposable representations of $\mathcal{G}$ are classified up to equivalence by pairs consisting of:

- an orbit in H^*, and

- a subgroup of the stabilizer of a point in that orbit.

It turns out that a representation is irretractable if and only if it is indecomposable and the map $X \to H^*$ is injective. This of course means that X is isomorphic as a G-space to one of the orbits of H^*. Thus, irretractable representations are classified up to equivalence by G-orbits in H^*.

In the case of the Poincaré 2-group, this has an interesting interpretation. The group $H = \mathbb{R}^4$ has $H^* \cong \mathbb{C}^4$. So, a representation in general is given by a $\mathrm{SO}_0(3,1)$-equivariant map $p\colon X \to \mathbb{C}^4$, where $\mathrm{SO}_0(3,1)$ acts independently on the real and imaginary parts of a vector in $\mathbb{C}^4$. The representation is irretractable if the image of p is a single orbit. Restricting to the Pontrjagin dual $\widehat{H}$ amounts to choosing the orbit of some *real* vector, an element of $\mathbb{R}^4$. Thus 'unitary' irretractable representations are classified by the $\mathrm{SO}_0(3,1)$ orbits in $\mathbb{R}^4$, which are familiar objects from special relativity.

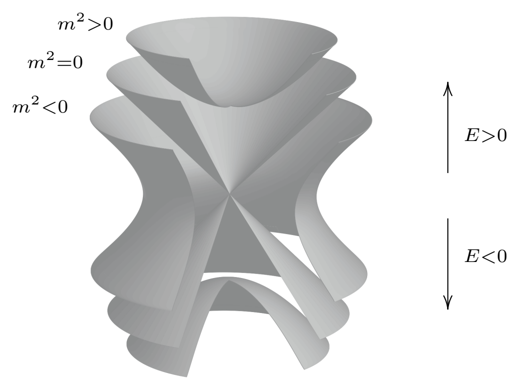
If we use $\boldsymbol{p} = (E, p_x, p_y, p_z)$ as our name for a point of $\mathbb{R}^4$, then any orbit is a connected component of the solution set of an equation of the form

$$\boldsymbol{p} \cdot \boldsymbol{p} = m^2$$

where the dot denotes the Minkowski metric. In other words:

$$E^2 - p_x^2 - p_y^2 - p_y^2 = m^2.$$

The variable names are the traditional ones in relativity: E stands for the energy of a particle, while p_x, p_y, p_z are the three components of its momentum, and the constant m is its mass. An orbit corresponding to a particular mass m describes the allowed values of energy and momentum for a particle of this mass. These orbits can be drawn explicitly if we suppress one dimension:



Though this picture is dimensionally reduced, it faithfully depicts all of the orbits in the 4-dimensional case. There are six types of orbits, thus giving us six types of irretractable representations of the Poincaré 2-group:

1. $E = 0$, $m = 0$: the trivial representation (orbit is a single point)
2. $E > 0$, $m = 0$: the 'positive energy massless' representation
3. $E < 0$, $m = 0$: the 'negative energy massless' representation
4. $E > 0$, $m > 0$: 'positive energy real mass' representations (one for each $m > 0$)
5. $E < 0$, $m > 0$: 'negative energy real mass' representations (one for each $m > 0$)

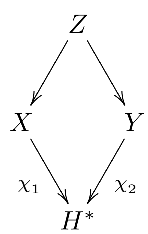
6. $m^2 < 0$: 'imaginary mass' or 'tachyon' representations (one for each $-im > 0$)

On the other hand, there are many more *indecomposable* representations, since these are classified by a choice of one of the above orbits together with a subgroup of the corresponding point stabilizer—SO(2), SO(3) or $\mathrm{SO}_0(2,1)$ depending on whether $m^2 = 0$, $m^2 > 0$, or $m^2 < 0$. These indecomposable representations were studied by Crane and Sheppeard [26], though they called them 'irreducible'.

To any reader familiar with the classification of irreducible unitary representations of the ordinary Poincaré *group*, the above story should seem familiar, but also a bit strange. It should seem familiar because these group representations are *partially* classified by SO(3, 1) orbits in Minkowski spacetime. The strange part is that for these group representations, some extra data is also needed. For example, a particle with positive mass and energy is characterized by both a mass $m > 0$ and a *spin*—an irreducible representation of SO(3) (or in a more detailed treatment, the double cover of this group). By switching to the Poincaré 2-group, we seem to have somehow lost the spin information.

This is not the case. In fact, as we now explain, the 'spin' information from the ordinary Poincaré group representation theory has simply been pushed up one categorical notch—we will find it in the intertwiners! In other words, the concept of spin shows up not in the classification of representations of the Poincaré 2-group, but in the classification of *morphisms* between representations. The reason, ultimately, is that Lorentz transformations and translations of $\mathbb{R}^4$ show up at different levels in the Poincaré 2-group: the Lorentz transformations as objects, and the translations as morphisms.

To see this in more detail, we need to understand the geometry of intertwiners. Suppose we have two representations, one on H^X and one on H^Y, given by equivariant bundles $\chi_1 \colon X \to H^*$ and $\chi_2 \colon Y \to H^*$. Looking again at the chart, the key geometric object is a Hilbert bundle over the pullback of χ_1 and χ_2. This pullback may be seen as a subspace Z of $Y \times X$:



$$Z = \{(y, x) \in Y \times X : \chi_2(y) = \chi_1(x)\}$$

It is easy to see that Z is a G-space under the diagonal action of G on $X \times Y$, and that the projections into X and Y are G-equivariant.

If H^X and H^Y are both indecomposable representations, then X and Y each lie over a single orbit of H^*. These orbits must be the same in order for the pullback Z, and hence the space of intertwiners, to be nontrivial. If H^X and H^Y are both irretractable, this implies that they are equivalent. Thus, given an irretractable representation represented by an orbit X in H^*, the self-intertwiners of this representation are classified by equivariant Hilbert space bundles over X.

Equivariant Hilbert bundles are the subject of Mackey's induced representation theory [49,51,52]. In general, a way to construct an equivariant bundle is to pick a point in the base space X and a Hilbert space that is a representation of the stabilizer of that point, and then use the action of G to 'translate' the Hilbert space along a G-orbit. Conversely, given an equivariant bundle, the fiber over a given point is a representation of the stabilizer of that point. Indeed, there is an equivalence of categories:

$$\begin{pmatrix} G\text{-equivariant vector bundles} \\ \text{over a homogeneous space } X \end{pmatrix} \simeq \begin{pmatrix} \text{representations of the} \\ \text{stabilizer of a point in } X \end{pmatrix}$$

Proving this is straightforward when we mean 'vector bundles' in the in the ordinary topological sense. But in Mackey's work, he generalized this correspondence to a measure-theoretic context—precisely the context that arises in the theory of 2-group representations we are considering here! The upshot for us is that self-intertwiners of an irretractable representation amount to representations of the stabilizer subgroup.

To illustrate this idea, let us return to the example of the Poincaré 2-group. Suppose we have a unitary irretractable representation of this 2-group. As we have seen, this is given by one of the orbits $X \subset \mathbb{R}^4$ of $\mathrm{SO}_0(3,1)$. Now, consider any self-intertwiner of this representation. This is given by a $\mathrm{SO}_0(3,1)$-invariant Hilbert space bundle over X. By induced representation theory, this amounts to the same thing as a representation of the stabilizer of any point $x \in X$. For a 'positive energy real mass' representation, for example, corresponding to an ordinary massive particle in special relativity, this stabilizer is SO(3), so self-intertwiners are essentially representations of SO(3).

In ordinary group representation theory, there is no notion of 'reducibility' for intertwiners. But here, because of the additional level of categorical structure, 2-group intertwiners in many ways more closely resemble group representations than group intertwiners. There is a natural concept of 'direct sum' of intertwiners, and this gives a notion of 'indecomposable' intertwiner. Similarly, the concept of 'sub-intertwiner' gives a notion of 'irreducible' intertwiner.

Returning yet again to the Poincaré 2-group example, consider the self-intertwiners of a positive energy real mass representation. We have just seen that these correspond to representations of SO(3). When is such a self-intertwiner irreducible? Unsurprisingly, the answer is: precisely when the corresponding representation of SO(3) is irreducible.

Because of the added layer of structure, we can also ask how a pair of intertwiners with the same source and target representations might be related by 2-intertwiner. As we shall see, intertwiners satisfy an analogue of Schur's lemma: a 2-intertwiner between *irreducible* intertwiners is either null or an isomorphism, and in the latter case is essentially unique. So, there is no interesting information in the self-2-intertwiners of an irreducible intertwiner.

We conclude with a small warning: in the foregoing description of the representation theory, we have for simplicity's sake glossed over certain subtle measure theoretic issues. Most of these issues make little difference in the case of the Poincaré 2-group, but may be important for general representations of an arbitrary measurable 2-group. For details, read the rest of the book!

1.4 Applications

Next we describe some applications to physics. Crane and Sheppeard [26] originally examined representations of the Poincaré 2-group as part of a plan to construct a physical theory of a specific sort. A very similar model is implicit in the work of two of the current authors on Feynman diagrams in quantum gravity [10]. Since proving this was one of our main motivations for studying the representations of Lie 2-groups, we would like to recall the ideas here.

A major problem in physics today is trying to extend quantum field theory, originally formulated for theories that neglect gravity, to theories that include gravity. Quantum field theories that neglect gravity, such as the Standard Model of particle physics, treat spacetime as flat. More precisely, they treat it as $\mathbb{R}^4$ with its Minkowski metric. The ordinary Poincaré group acts as symmetries here.

In quantum field theories, physical quantities are often computed with the help of 'Feynman diagrams'. The details can be found in any good book on quantum field theory—or, for that matter, Borcherds' review article for mathematicians [20]. However, from a very abstract perspective, a Feynman diagram can be seen as a graph with:

- edges labelled by irreducible representations of some group G, and

- vertices labelled by intertwiners,

where the intertwiner at any vertex goes from the trivial representation to the tensor product of all the representations labelling edges incident to that vertex. In the simplest theories, the group G is just the Poincaré group. In more complicated theories, such as the Standard Model, we use a larger group.

There is a way to evaluate Feynman diagrams and get complex numbers, called 'Feynman amplitudes'. Physically, we think of the group representations labelling Feynman diagram edges as *particles*. Indeed, we have already said a bit about how an irreducible representation of the Poincaré group can describe a particle with a given mass and spin. We think of the intertwiners as *interactions*: ways for the particles to collide and turn into other particles. So, a Feynman diagram describes a process involving particles. When we take the absolute value of its amplitude and square it, we obtain the probability for this process to occur.

Feynman diagrams are essentially one-dimensional structures, since they have vertices and edges. On the other hand, there is an approach to quantum gravity that uses closely analogous *two-dimensional* structures called 'spin foams' [5, 15, 38, 67]. The 2-dimensional analogue of a graph is called an '2-complex': it is a structure with vertices, edges *and faces*. In a spin foam, we label the vertices, edges and faces of a 2-complex with data of some sort. Like Feynman diagrams, spin foams should be thought of as describing physical processes—but now of a higher-dimensional sort. A spin foam model is a recipe for computing complex numbers from spin foams: their 'amplitudes'. As before, when we take the absolute value of these amplitude and square them, we obtain probabilities.

The first spin foam model, only later recognized as such, goes back to a famous 1968 paper by Ponzano and Regge [61]. This described *Riemannian* quantum gravity in *3-dimensional* spacetime—two drastic simplifications that are worth explaining.

First of all, gravity is much easier to deal with in 3d spacetime, since in this case, in the absence of matter, all solutions of Einstein's equations for general relativity look alike locally. More precisely, any spacetime obeying these equations can be locally identified, after a suitable coordinate transformation, with $\mathbb{R}^3$ equipped with its Minkowski metric

$$\boldsymbol{x} \cdot \boldsymbol{x}' = tt' - xx' - yy'.$$

This is very different from the physically realistic 4d case, where gravitational waves can propagate through the vacuum, giving a plethora of locally distinct solutions. Physicists say that 3d gravity lacks 'local degrees of freedom'. This makes it much easier to study—but it retains some of the conceptual and technical challenges of the 4d problem.

Second of all, in 'Riemannian quantum gravity', we investigate a simplified world where time is just the same as space. In 4d spacetime, this involves replacing Minkowski spacetime with 4d Euclidean space—that is, $\mathbb{R}^4$ with the inner product

$$\boldsymbol{x} \cdot \boldsymbol{x}' = tt' + xx' + yy' + zz'.$$

While physically quite unrealistic, this switch simplifies some of the math. The reason, ultimately, is that the group of Lorentz transformations, $\mathrm{SO}_0(3,1)$, is noncompact, while the rotation group $\mathrm{SO}(4)$ is compact. A compact Lie group has a countable set of irreducible unitary representations instead of a continuum, and this makes some calculations easier. For example, certain integrals become sums.

Ponzano and Regge found that after making both these simplifications, they could write down an elegant theory of quantum gravity, now called the Ponzano–Regge model. Their theory is deeply related to representations of the 3-dimensional rotation group, $\mathrm{SO}(3)$. In modern terms, the idea is to start with a 3-manifold equipped with a triangulation Δ. Then we form the Poincaré dual of Δ and look at its 2-skeleton K. In simple terms, K is the 2-complex with:

- one vertex for each tetrahedron in Δ,
- one edge for each triangle in Δ,
- one face for each edge of Δ.

We call such a thing a '2-complex'. Note that a 2-complex is precisely the sort of structure that, when suitably labelled, gives a spin foam! To obtain a spin foam, we:

- label each face of K with an irreducible representation of SO(3), and
- label each edge of K with an intertwiner.

There is a way to compute an amplitude for such a spin foam, and we can use these amplitudes to answer physically interesting questions about 3d Riemannian quantum gravity.

The Ponzano–Regge model served as an inpiration for many further developments. In 1997, Barrett and Crane proposed a similar model for *4-dimensional* Riemannian quantum gravity [15]. More or less simultaneously, the general concept of 'spin foam model' was formulated [5]. Shortly thereafter, spin foam models of 4d Lorentzian quantum gravity were proposed, closely modelled after the Barrett-Crane model [28, 62]. Later, 'improved' models were developed by Freidel and Krasnov [38] and Engle, Pereira, Rovelli and Livine [33]. These newer models are beginning to show signs of correctly predicting some phenomena we expect from a realistic theory of quantum gravity. However, this is work in progress, whose ultimate success is far from certain.

One fundamental challenge is to incorporate *matter* in a spin foam model of quantum gravity. Indeed, any theory that fails to do this is at best a warmup for a truly realistic theory. Recently, a lot of progress has been made on incorporating matter in the Ponzano–Regge model. Here is where spin foams meet Feynman diagrams!

The idea is to compute Feynman amplitudes using a slight generalization of the Ponzano–Regge model which lets us include matter [14]. This model takes the gravitational interactions of particles into account. As a consistency check, we want the 'no-gravity limit' of this model to reduce to the standard recipe for computing Feynman amplitudes in quantum field theory—or more precisely its analogue with Euclidean $\mathbb{R}^3$ replacing 4d Minkowski spacetime. And indeed, this was shown to be true [63, 64, 65].

This raised the hope that the same sort of strategy can work in 4-dimensional quantum gravity. It was natural to start with the 'no-gravity limit', and ask if the usual Feynman amplitudes for quantum field theory in flat 4d spacetime can be computed using a spin foam model. If we could do this, the result would not be a theory of quantum gravity, but it would provide a radical new formulation of quantum field theory, in which Minkowski spacetime is replaced by an inherently quantum-mechanical spacetime built from spin foams. If a formulation exists, it may help us develop models describing quantum gravity and matter in 4 dimensions.

Recent work by [10] gives precisely such a formulation, at least in the 4-dimensional *Riemannian* case. In other words, this work gives a spin foam model for computing Feynman amplitudes for quantum field theories, not on Minkowski spacetime, but rather on 4-dimensional Euclidean space. Feynman diagrams for such theories are built using representations, not of the Poincaré group, but of the **Euclidean group**:

$$\mathrm{SO}(4) \ltimes \mathbb{R}^4.$$

More recently still, it was seen that this new model is a close relative of the Crane–Sheppeard model [11, 13]! The only difference is that where the Crane–Sheppeard model uses the Poincaré 2-group, the new model uses the **Euclidean 2-group**, a skeletal 2-group for which:

- $G = \mathrm{SO}(4)$: the group of rotations of 4d Euclidean space,
- $H = \mathbb{R}^4$: the group of translations 4d Euclidean space,
- the obvious action of SO(4) on $\mathbb{R}^4$.

The representation theory of the Euclidean 2-group is very much like that of the Poincaré 2-group, but with concentric spheres replacing the hyperboloids

$$E^2 - p_x^2 - p_y^2 - p_y^2 = m^2.$$

So, we can now guess the meaning of the Crane–Sheppeard model: it should give a new way to compute Feynman integrals for ordinary quantum field theories on 4d Minkowski spacetime. To conclude, let us just say a word about how this model actually works.

It helps to go back to the Ponzano–Regge model. We can describe this directly in terms of a 3-manifold with triangulation Δ, instead of the Poincaré dual picture. In these terms, each spin foam corresponds to a way to:

- label each edge of Δ with an irreducible representation of SO(3), and
- label each triangle of Δ with an intertwiner.

The Ponzano–Regge model gives a way to compute an amplitude for any such labelling.

The Crane–Sheppeard model does a similar thing one dimension up. Suppose we take a 4-manifold with a triangulation Δ. Then we may:

- label each edge of Δ with an irretractable representation of the Poincaré 2-group,
- label each triangle of Δ with an irreducible intertwiner, and
- label each tetrahedron of Δ with a 2-intertwiner.

The Crane–Sheppeard model gives a way to compute an amplitude for any such labelling.

1.5 Plan of the paper

Above we describe a 2-group as a category equipped with a multiplication and inverses. While this is correct, another equivalent approach turns out to be more useful for our purposes here. Just as a group can be thought of as a category that has one object and for which all morphisms are invertible, a 2-group can be thought of as a 2-category that has one object and for which all morphisms and 2-morphisms are invertible. In Section 2 we recall the definition of a 2-category and explain how to think of a 2-group as a 2-category of this sort. We also describe how to construct 2-groups from crossed modules, and vice versa. We conclude by defining the 2-category $\mathbf{2Rep}(\mathcal{G})$ of representations of a fixed 2-group $\mathcal{G}$ in a fixed 2-category $\mathcal{C}$.

In Section 3 we explain measurable categories. We first recall Kapranov and Voevodsky's 2-vector spaces, and then introduce the necessary analysis to present Yetter's results on measurable categories. To do this, we need to construct the 2-category **Meas** of measurable categories. The problem is that we do not yet know an intrinsic characterization of measurable categories. At present, a measurable category is simply defined as one that is 'C^*-equivalent' to a category of measurable fields of Hilbert spaces. So, it is a substantial task to construct the 2-category **Meas**. As a warmup, we carry out a similar construction of the 2-category of Kapranov–Voevodsky 2-vector spaces (for which an intrinsic characterization is known, making a simpler approach possible).

Working in this picture, we study the representations of 2-groups on measurable categories in Section 4. We present a detailed study of equivalence, direct sums, tensor products, reducibility, decomposability, and retractability for representations and 1-intertwiners. While our work is hugely indebted to that of Crane, Sheppeard, and Yetter, we confront many issues they did not discuss. Some of these arise from the fact that they implicitly consider representations of discrete 2-groups, while we treat *measurable* representations of *measurable* 2-groups—for example, Lie 2-groups. The representations of a Lie group viewed as a discrete group are vastly more pathological than its measurable representations. Indeed, this is already true for $\mathbb{R}$, which has enormous numbers of nonmeasurable 1-dimensional representations if we assume the axiom of choice, but none if we assume the axiom of determinacy. The same phenomenon occurs for Lie 2-groups. So, it is important to treat them as measurable 2-groups, and focus on their measurable representations.

In Section 5, we conclude by sketching some directions for future research. We argue that a measurable category H^X becomes a 'separable 2-Hilbert space' when the measurable space X is equipped with a σ-finite measure. We also sketch how this approach to separable 2-Hilbert spaces should fit into a more general approach to 2-Hilbert spaces based on von Neumann algebras.

Finally, Appendix A contains some results from analysis that we need. **Nota Bene:** in this paper, we always use 'measurable space' to mean 'standard Borel space': that is, a set X with a σ-algebra of subsets generated by the open subsets for some complete separable metric on X. Similarly, we use 'measurable group' to mean 'lcsc group': that is, a topological group for which the topology is locally compact Hausdorff and second countable. We also assume all our measures are σ-finite and positive. These background assumptions give a fairly convenient framework for the analysis in this paper.

Acknowledgments

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2 Representations of 2-groups

2.1 From groups to 2-groups

2.1.1 2-groups as 2-categories

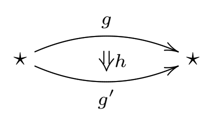
We have said that a 2-group is a category equipped with product and inverse operations satisfying the usual group axioms. However, a more powerful approach is to think of a 2-group as a special sort of 2-category.

To understand this, first note that a group G can be thought of as a category with a single object $\star$, morphisms labeled by elements of G, and composition defined by multiplication in G:

$$\star \xrightarrow{g_1} \star \xrightarrow{g_2} \star \quad = \quad \star \xrightarrow{g_2 g_1} \star$$

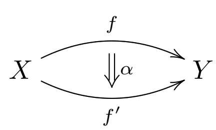
In fact, one can define a group to be a category with a single object and all morphisms invertible. The object $\star$ can be thought of as an object whose symmetry group is G.

In a 2-group, we add an additional layer of structure to this picture, to capture the idea of *symmetries between symmetries*. So, in addition to having a single object $\star$ and its automorphisms, we have isomorphisms *between* automorphisms of $\star$:

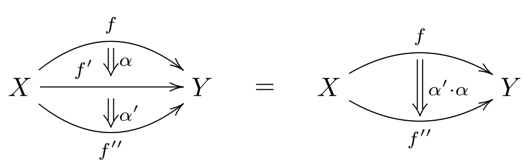


These 'morphisms between morphisms' are called *2-morphisms*.

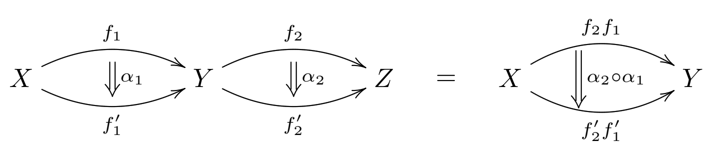
To make this precise, we should recall that a 2-category consists of:

- objects: $X, Y, Z, \ldots$
- morphisms: $X \xrightarrow{f} Y$
- 2-morphisms: 

Morphisms can be composed as in a category, and 2-morphisms can be composed in two distinct ways: vertically:



and horizontally:



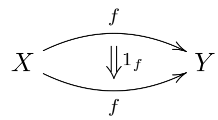
A few simple axioms must hold for this to be a 2-category:

- Composition of morphisms must be associative, and every object X must have a morphism

$$X \xrightarrow{1_x} X$$

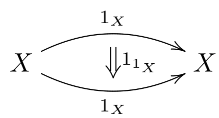
serving as an identity for composition, just as in an ordinary category.

- Vertical composition must be associative, and every morphism $X \xrightarrow{f} Y$ must have a 2-morphism



serving as an identity for vertical composition.

- Horizontal composition must be associative, and the 2-morphism

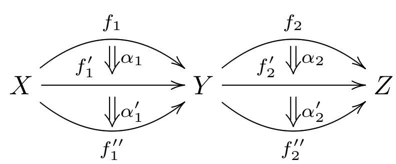


must serve as an identity for horizontal composition.

- Vertical composition and horizontal composition of 2-morphisms must satisfy the following **exchange law**:

$$(\alpha_2' \cdot \alpha_2) \circ (\alpha_1' \cdot \alpha_1) = (\alpha_2' \circ \alpha_1') \cdot (\alpha_2 \circ \alpha_1) \tag{1}$$

so that diagrams of the form



define unambiguous 2-morphisms.

For more details, see the references [45, 53].

We can now define a 2-group:

Definition 1 *A* **2-group** *is a 2-category with a unique object such that all morphisms and 2-morphisms are invertible.*

In fact it is enough for all 2-morphisms to have 'vertical' inverses; given that morphisms are invertible it then follows that 2-morphisms have horizontal inverses. Experts will realize that we are defining a 'strict' 2-group [8]; we will never use any other sort.

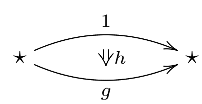
The 2-categorical approach to 2-groups is a powerful conceptual tool. However, for explicit calculations it is often useful to treat 2-groups as 'crossed modules'.

2.1.2 Crossed modules

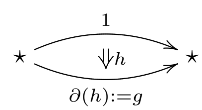
Given a 2-group $\mathcal{G}$, we can extract from it four pieces of information which form something called a 'crossed module'. Conversely, any crossed module gives a 2-group. In fact, 2-groups and crossed modules are just different ways of describing the same concept. While less elegant than 2-groups, crossed modules are good for computation, and also good for constructing examples.

Let $\mathcal{G}$ be a 2-group. From this we can extract:

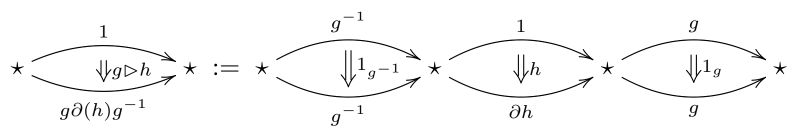
- the group G consisting of all morphisms of $\mathcal{G}$: $\star \xrightarrow{g} \star$
- the group H consisting of all 2-morphisms whose source is the identity morphism:



- the homomorphism $\partial\colon H \to G$ assigning to each 2-morphism $h \in H$ its target:



- the action $\rhd$ of G as automorphisms of H given by 'horizontal conjugation':



It is easy to check that the homomorphism $\partial\colon H \to G$ is compatible with $\rhd$ in the following two ways:

$$\partial(g \rhd h) = g\partial(h)g^{-1} \tag{2}$$

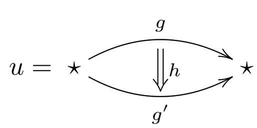
$$\partial(h) \rhd h' = hh'h^{-1}. \tag{3}$$

Such a system $(G, H, \rhd, \partial)$ satisfying equations (2) and (3) is called a **crossed module**.

We can recover the 2-group $\mathcal{G}$ from its crossed module $(G, H, \rhd, \partial)$, using a process we now describe. In fact, every crossed module gives a 2-group via this process [35].

Given a crossed module $(G, H, \rhd, \partial)$, we construct a 2-group $\mathcal{G}$ with:

- one object: $\star$
- elements of G as morphisms: $\star \xrightarrow{g} \star$
- pairs $u = (g, h) \in G \times H$ as 2-morphisms, where (g, h) is a 2-morphism from g to $\partial(h)g$. We draw such a pair as:

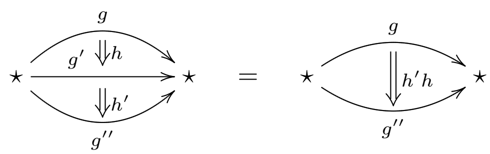


  where $g' = \partial(h)g$.

Composition of morphisms and vertical composition of 2-morphisms are defined using multiplication in G and H, respectively:

$$\star \xrightarrow{g_1} \star \xrightarrow{g_2} \star \quad = \quad \star \xrightarrow{g_2 g_1} \star$$

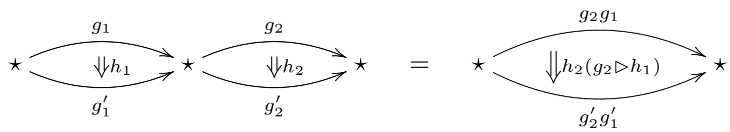
and



with $g' = \partial(h)g$ and $g'' = \partial(h')\partial(h)g = \partial(h'h)g$. In other words, suppose we have 2-morphisms $u = (g, h)$ and $u' = (g', h')$. If $g' = \partial(h)g$, they are vertically composable, and their vertical composite is given by:

$$u' \cdot u = (g', h') \cdot (g, h) = (g, h'h) \tag{4}$$

They are always horizontally composable, and we define their horizontal composite by:



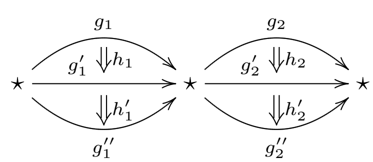
So, horizontal composition makes the set of 2-morphisms into a group, namely the semidirect product $G \ltimes H$ with multiplication:

$$(g_2, h_2) \circ (g_1, h_1) \equiv (g_2 g_1, h_2(g_2 \rhd h_1)) \tag{5}$$

One can check that the exchange law

$$(u_2' \cdot u_2) \circ (u_1' \cdot u_1) = (u_2' \circ u_1') \cdot (u_2 \circ u_1) \tag{6}$$

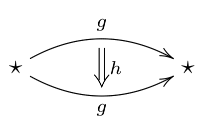
holds for 2-morphisms $u_i = (g_i, h_i)$ and $u_i' = (g_i', h_i')$, so that the diagram



gives a well-defined 2-morphism.

To see an easy example of a 2-group, start with a group G acting as automorphisms of a group H. If we take $\rhd$ to be this action and let $\partial \colon H \to G$ be the trivial homomorphism, we can easily check that the crossed module axioms (2) and (3) hold *if H is abelian*. So, if H is abelian, we obtain a 2-group with G as its group of objects and $G \ltimes H$ as its group of morphisms, where the semidirect product is defined using the action $\rhd$.

Since ∂ is trivial in this example, any 2-morphism $u = (g, h)$ goes from g to itself:



So, this type of 2-group has only 2-*auto*morphisms, and each morphism has precisely one 2-automorphism for each element of H.

A 2-group with trivial ∂ is called **skeletal**, and one can easily see that every skeletal 2-group is of the form just described. An important point is that for a skeletal 2-group, the group H is necessarily abelian. While we derived this using (3) above, the real reason is the Eckmann–Hilton argument [30].

An important example of a skeletal 2-group is the 'Poincaré 2-group' coming from the semidirect product $SO(3,1) \ltimes \mathbb{R}^4$ in precisely the way just described [4].

2.2 From group representations to 2-group representations

2.2.1 Representing groups

In the ordinary theory of groups, a group G may be represented on a vector space. In the language of categories, such a representation is nothing but a *functor* $\rho\colon G \to \mathrm{Vect}$, where G is seen as category with one object $*$, and Vect is the category of vector spaces and linear operators. To see this, note that such a functor must send the object $* \in G$ to some vector space $\rho(*) = V \in \mathrm{Vect}$. It must also send each morphism $\star \xrightarrow{g} \star$ in G—or in other words, each *element* of our group—to a linear map

$$V \xrightarrow{\rho(g)} V$$

Saying that ρ is a functor then means that it preserves identities and composition:

$$\rho(1) = \mathbb{1}_V$$

$$\rho(gh) = \rho(g)\rho(h)$$

for all group elements g, h.

In this language, an intertwining operator between group representations—or 'intertwiner', for short—is nothing but a *natural transformation*. To see this, suppose that $\rho_1, \rho_2\colon G \to \mathrm{Vect}$ are functors and $\phi\colon \rho_1 \Rightarrow \rho_2$ is a natural transformation. Such a transformation must give for each object $\star \in G$ a linear operator from $\rho_1(*) = V_1$ to $\rho_2(*) = V_2$. But G is a category with one object, so we have a single operator $\phi\colon V_1 \to V_2$. Saying that the transformation is 'natural' then means that this square commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array} \qquad (7)$$

for each group element g. This says simply that

$$\rho_2(g)\phi = \phi\rho_1(g) \qquad (8)$$

for all $g \in G$. So, ϕ is an intertwiner in the usual sense.

Why bother with the categorical viewpoint on on representation theory? One reason is that it lets us generalize the concepts of group representation and intertwiner:

Definition 2 *If G is a group and C is any category, a* **representation** *of G in C is a functor ρ from G to C, where G is seen as a category with one object. Given representations ρ_1 and ρ_2 of G in C, an* **intertwiner** *$\phi\colon \rho \to \rho'$ is a natural transformation from ρ to ρ'.*

In ordinary representation theory we take $C = \mathrm{Vect}$; but we can also, for example, work with the category of sets $C = \mathrm{Set}$, so that a representation of G in C picks out a set together with an action of G on this set.

Quite generally, there is a category $\mathbf{Rep}(G)$ whose objects are representations of G in C, and whose morphisms are the intertwiners. Composition of intertwiners is defined by composing natural transformations. We define two representations $\rho_1, \rho_2\colon G \to C$ to be **equivalent** if there exists an intertwiner between them which has an inverse. In other words, ρ_1 and ρ_2 are equivalent if there is a natural *isomorphism* between them.

In the next section we shall see that the representation theory of 2-groups amounts to taking all these ideas and 'boosting the dimension by one', using 2-categories everywhere instead of categories.

2.2.2 Representing 2-groups

Just as groups are typically represented in the category of vector spaces, 2-groups may be represented in some 2-category of '2-vector spaces'. However, just as for group representations, the definition of a 2-group representation does not depend on the particular target 2-category we wish to represent our 2-groups in. We therefore present the definition in its abstract form here, before describing precisely what sort of 2-vector spaces we will use, in Section 3.

We have seen that a representation of a group G in a category C is a functor $\rho\colon G \to C$ between categories. Similarly, a representation of a 2-group will be a '2-functor' between 2-categories. As with group representations, we have intertwiners between 2-group representations, which in the language of 2-categories are 'pseudonatural transformations'. But the extra layer of categorical structure implies that in 2-group representation theory we also have '2-intertwiners' going between intertwiners. These are defined to be 'modifications' between pseudonatural transformations.

The reader can learn the general notions of '2-functor', 'pseudonatural transformation' and 'modification' from the review article by Kelly and Street [45]. However, to make this paper self-contained, we describe these concepts below in the special cases that we actually need.

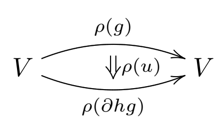
Definition 3 *If $\mathcal{G}$ is a 2-group and $\mathcal{C}$ is any 2-category, then a* **representation** *of $\mathcal{G}$ in $\mathcal{C}$ is a 2-functor ρ from $\mathcal{G}$ to $\mathcal{C}$.*

Let us describe what such a 2-functor amounts to. Suppose a 2-group $\mathcal{G}$ is given by the crossed module $(G, H, \partial, \rhd)$, so that G is the group of morphisms of $\mathcal{G}$, and $G \ltimes H$ is the group of 2-morphisms, as described in section 2.1.2. Then a representation $\rho\colon \mathcal{G} \to \mathcal{C}$ is specified by:

- an object V of $\mathcal{C}$, associated to the single object of the 2-group: $\rho(\star) = V$
- for each morphism $g \in G$, a morphism in $\mathcal{C}$ from V to itself:

$$V \xrightarrow{\rho(g)} V$$

- for each 2-morphism $u = (g, h)$, a 2-morphism in $\mathcal{C}$



That ρ is a 2-functor means these correspondences preserve identities and all three composition operations: composition of morphisms, and horizontal and vertical composition of 2-morphisms. In the case of a 2-group, preserving identities follows from preserving composition. So, we only need require:

- for all morphisms g, g':
$$\rho(g'g) = \rho(g')\,\rho(g) \tag{9}$$

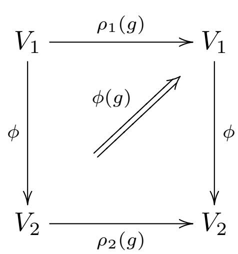
- for all vertically composable 2-morphisms u and u':
$$\rho(u' \cdot u) = \rho(u') \cdot \rho(u) \tag{10}$$

- for all 2-morphisms u, u':
$$\rho(u' \circ u) = \rho(u') \circ \rho(u) \tag{11}$$

Here the compositions laws in $\mathcal{G}$ and $\mathcal{C}$ have been denoted the same way, to avoid an overabundance of notations.

Definition 4 *Given a 2-group $\mathcal{G}$, any 2-category $\mathcal{C}$, and representations ρ_1, ρ_2 of $\mathcal{G}$ in $\mathcal{C}$, an* **intertwiner** $\phi\colon \rho_1 \to \rho_2$ *is a pseudonatural transformation from ρ_1 to ρ_2.*

This is analogous to the usual representation theory of groups, where an intertwiner is a natural transformation between functors. As before, an intertwiner involves a morphism $\phi\colon V_1 \to V_2$ in $\mathcal{C}$. However, as usual when passing from categories to 2-categories, this morphism is only required to satisfy the commutation relations (8) *up to 2-isomorphism*. In other words, whereas before the diagram (7) commuted, so that the morphisms $\rho_2(g)\phi$ and $\phi\rho_1(g)$ were *equal*, here we only require that there is a specified invertible 2-morphism $\phi(g)$ from one to the other. (An invertible 2-morphism is called a '2-isomorphism'.) The commutative square (7) for intertwiners is thus generalized to:



(12)

We say the commutativity of the diagram (7) has been 'weakened'.

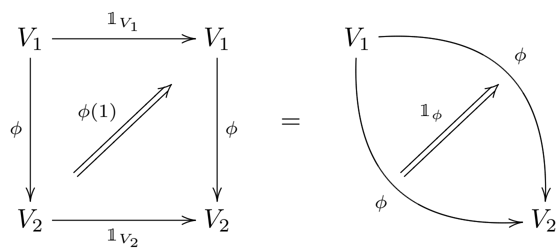
In short, a intertwiner from ρ_1 to ρ_2 is really a pair consisting of a morphism $\phi\colon V_1 \to V_2$ together with a family of 2-isomorphisms

$$\phi(g)\colon \rho_2(g)\,\phi \xrightarrow{\sim} \phi\,\rho_1(g) \tag{13}$$

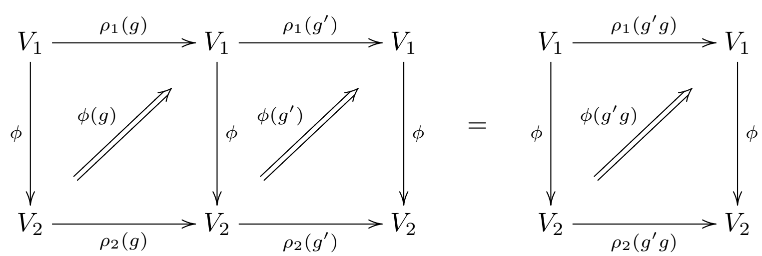
one for each $g \in G$. These data must satisfy some additional conditions in order to be 'pseudonatural':

- ϕ should be compatible with the identity $1 \in G$:
$$\phi(1) = \mathbb{1}_\phi \tag{14}$$

where $\mathbb{1}_\phi \colon \phi \to \phi$ is the identity 2-morphism. Diagrammatically:

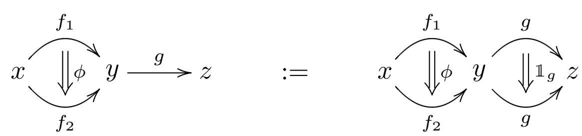


- ϕ should be compatible with composition of morphisms in G. Intuitively, this means we should be able to glue $\phi(g)$ and $\phi(g')$ together in the most obvious way, and obtain $\phi(g'g)$:

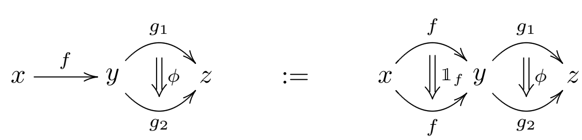


(15)

To make sense of this equation we need the concept of 'whiskering', which we now explain. Suppose in any 2-category we have morphisms $f_1, f_2 \colon x \to y$, a 2-morphism $\phi \colon f_1 \Rightarrow f_2$, and a morphism $g \colon y \to z$. Then we can **whisker** ϕ by g by taking the horizontal composite $\mathbb{1}_g \circ \phi$, defining:



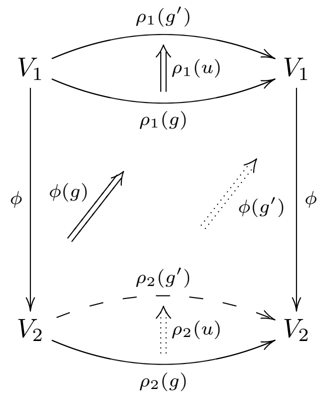
We can also whisker on the other side:



To define the 2-morphism given by the diagram on the left-hand side of (15), we whisker $\phi(g)$ on one side by $\rho_2(g')$, whisker $\phi(g')$ on the other side by $\rho_1(g)$, and then vertically compose the resulting 2-morphisms. So, the equation in (15) is a diagrammatic way of writing:

$$\left[\phi(g') \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \left[\mathbb{1}_{\rho_2(g')} \circ \phi(g)\right] = \phi(g'g) \tag{16}$$

- Finally, the intertwiner ϕ should satisfy a higher-dimensional analogue of diagram (7), so that it 'intertwines' the 2-morphisms $\rho_1(u)$ and $\rho_2(u)$ where $u = (g, h)$ is a 2-morphism in the 2-group. So, we demand that the following "pillow" diagram commute for all $g \in G$ and $h \in H$:



(17)

where we have introduced $g' = \partial(h)g$. In other words:

$$[\mathbb{1}_\phi \circ \rho_1(u)] \cdot \phi(g) = \phi(g') \cdot [\rho_2(u) \circ \mathbb{1}_\phi] \tag{18}$$

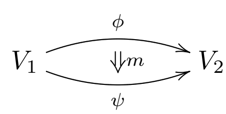
where we have again used whiskering to glue together the 2-morphisms on the front and top, and similarly the bottom and back.

Now a word about notation is required. While an intertwiner from ρ_1 to ρ_2 is really a pair consisting of a morphism $\phi\colon V_1 \to V_2$ and a family of 2-morphisms $\phi(g)$, for efficiency we refer to an intertwiner simply as ϕ, and denote it by $\phi\colon \rho_1 \to \rho_2$. This should not cause any confusion.

So far, we have described representation of 2-groups as *2-functors* and intertwiners as *pseudonatural transformations*. As mentioned earlier, there are also things going between pseudonatural transformations, called *modifications*. The following definition should thus come as no surprise:

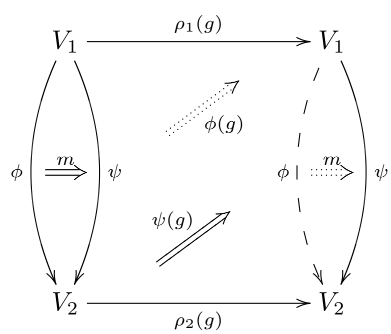
Definition 5 *Given a 2-group $\mathcal{G}$, a 2-category $\mathcal{C}$, representations ρ_1 and ρ_2 of G in $\mathcal{C}$, and intertwiners $\phi, \psi\colon \rho \to \rho'$, a* **2-intertwiner** *$m\colon \phi \Rightarrow \psi$ is a modification from ϕ to ψ.*

Let us say what modifications amount to in this case. A modification $m\colon \phi \Rightarrow \psi$ is a 2-morphism



(19)

in $\mathcal{C}$ such that the following pillow diagram:



(20)

commutes. Equating the front and left with the back and right, this means precisely that:

$$\psi(g) \cdot \left[\mathbb{1}_{\rho_2(g)} \circ m\right] = \left[m \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \phi(g) \tag{21}$$

where we have again used whiskering to attach the morphisms $\rho_i(g)$ to the 2-morphism m.

It is helpful to compare this diagram with the condition shown in (17). One important difference is that in that case, we had a "pillow" for each element $g \in G$ and $h \in H$, whereas here we have one only for each $g \in G$. For a intertwiner, the pillow involves 2-morphisms between the maps given by representations. Here the condition states that we have a fixed 2-morphism m between morphisms I and J between representation spaces, making the given diagram commute for each g. This is what representation theory of ordinary groups would lead us to expect from an intertwiner.

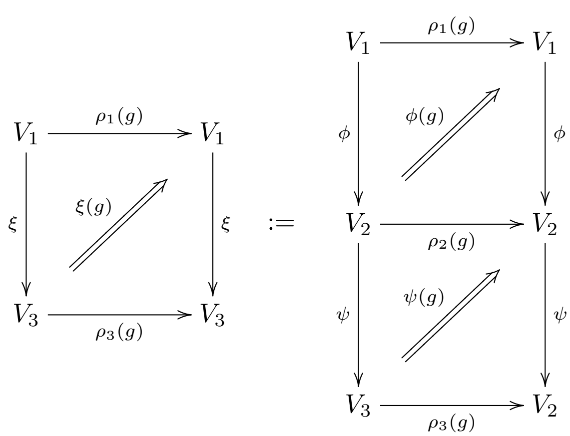
2.2.3 The 2-category of representations

Just as any group G gives a category $\mathbf{Rep}(G)$ with representations as objects and intertwiners as morphisms, any 2-group $\mathcal{G}$ gives a 2-category $\mathbf{2Rep}(\mathcal{G})$ with representations as objects, intertwiners as morphisms, 2-intertwiners as 2-morphisms. It is worth describing the structure of this 2-category explicitly. In particular, let us describe the rules for composing intertwiners and for vertically and horizontally composing 2-intertwiners:

- First, given a composable pair of intertwiners:

$$\rho_1 \xrightarrow{\phi} \rho_2 \xrightarrow{\psi} \rho_3$$

we wish to define their composite, which will be an intertwiner from ρ_1 to ρ_3. Recall that this intertwiner is a pair consisting of a morphism $\xi\colon V_1 \to V_3$ in $\mathcal{C}$ together with a family of 2-morphisms $\xi(g)$. We define ξ to be the composite $\psi\phi$, and for any $g \in G$ we define $\xi(g)$ by gluing together the diagrams (12) for $\phi(g)$ and $\psi(g)$ in the obvious way:



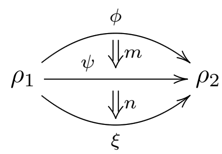
(22)

The diagram on the left hand side is once again evaluated with the help of whiskering: we whisker $\phi(g)$ on one side by ψ and $\psi(g)$ on the other side by ϕ, then vertically compose the resulting 2-morphisms. In summary:

$$\xi = \psi\phi, \qquad \xi(g) = [\mathbb{1}_\psi \circ \phi(g)] \cdot [\psi(g) \circ \mathbb{1}_\phi] \tag{23}$$

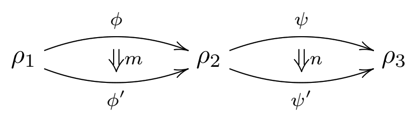
By some calculations best done using diagrams, one can check that these formulas define an intertwiner: relations (12), (14), (15) and (17) follow from the corresponding relations for ψ and ϕ.

- Next, suppose we have a vertically composable pair of 2-intertwiners:



  Then the 2-intertwiners m and n can be vertically composed using vertical composition in $\mathcal{C}$. With some further calculations one one check that the relation (21) for $n \cdot m \colon \phi \Rightarrow \xi$ follows from the corresponding relations for m and n.

- Finally, consider a horizontally composable pair of 2-intertwiners:



  Then m and n can be composed using horizontal composition in $\mathcal{C}$. With more calculations, one can check that the result $n \circ m$ defines a 2-intertwiner: it satisfies relation (21) because n and m satisfy the corresponding relations.

All the calculations required above are well-known in 2-category theory [45]. Quite generally, these calculations show that for *any* 2-categories $\mathcal{X}$ and $\mathcal{Y}$, there is a 2-category with:

- 2-functors $\rho \colon \mathcal{X} \to \mathcal{Y}$ as objects,
- pseudonatural transformations between these as morphisms,
- modifications between these as 2-morphisms.

We are just considering the case $\mathcal{X} = \mathcal{G}$, $\mathcal{Y} = \mathcal{C}$.

We conclude our description of $\mathbf{2Rep}(\mathcal{G})$ by discussing invertibility for intertwiners and 2-intertwiners; this will allow us to introduce natural equivalence relations for representations and intertwiners.

We first need to fill a small gap in our description of the 2-category $\mathbf{2Rep}(\mathcal{G})$: we need to describe the identity morphisms and 2-morphisms. Every representation ρ, with representation space V, has its **identity intertwiner** given by the identity morphism $\mathbb{1}_V \colon V \to V$ in $\mathcal{C}$, together with for each g the identity 2-morphism

$$\mathbb{1}_{\rho(g)} \colon \rho(g)\mathbb{1}_V \xrightarrow{\sim} \mathbb{1}_V \rho(g)$$

Also, every intertwiner ϕ has its **identity 2-intertwiner**, given by the identity 2-morphism $\mathbb{1}_\phi$ in $\mathcal{C}$.

We define a 2-intertwiner $m \colon \phi \Rightarrow \psi$ to be **invertible** (for vertical composition) if there exists $n \colon \psi \Rightarrow \phi$ such that

$$n \cdot m = \mathbb{1}_\phi \quad \text{and} \quad m \cdot n = \mathbb{1}_\psi$$

Similarly, we define a intertwiner $\phi\colon \rho_1 \to \rho_2$ to be **strictly invertible** if there exists an intertwiner $\psi\colon \rho_2 \to \rho_1$ with

$$\psi\phi = \mathbb{1}_{\rho_1} \quad \text{and} \quad \phi\psi = \mathbb{1}_{\rho_2} \tag{24}$$

However, it is better to relax the notion of invertibility for intertwiners by requiring that the equalities (24) hold only *up to invertible 2-intertwiners.* In this case we say that ϕ is **weakly invertible**, or simply **invertible**.

As for ordinary groups, we often consider equivalence classes of representations, rather than representations themselves:

Definition 6 *We say that two representations ρ_1 and ρ_2 of a 2-group are* **equivalent***, and write $\rho_1 \simeq \rho_2$, when there exists a weakly invertible intertwiner between them.*

In the representation theory of 2-groups, however, where an extra layer of categorical structure is added, it is also natural to consider equivalence classes of intertwiners:

Definition 7 *We say two intertwiners $\psi, \phi\colon \rho_1 \to \rho_2$ are* **equivalent***, and write $\phi \simeq \psi$, when there exists an invertible 2-intertwiner between them.*

Sometimes it is useful to relax this notion of equivalence to include pairs of intertwiners that are not strictly parallel. Namely, we call intertwiners $\phi\colon \rho_1 \to \rho_2$ and $\psi\colon \rho_1' \to \rho_2'$ 'equivalent' if there are invertible intertwiners $\rho_i \to \rho_i'$ such that

$$\rho_1 \xrightarrow{\phi} \rho_2 \xrightarrow{\sim} \rho_2' \qquad \text{and} \qquad \rho_1 \xrightarrow{\sim} \rho_1' \xrightarrow{\psi} \rho_2'$$

are equivalent, in the sense of the previous definition.

A major task of 2-group representation theory is to classify the representations and intertwiners up to equivalence. Of course, one can only do this concretely after choosing a 2-category in which to represent a given 2-group. We turn to this task next.

3 Measurable categories

We have described the passage from groups to 2-groups, and from representations to 2-representations. Having presented these definitions in a fairly abstract form, our next objective is to describe a suitable target 2-category for representations of 2-groups. Just as ordinary groups are typically represented on vector spaces, 2-groups can be represented on higher analogues called '2-vector spaces'. The idea of a 2-vector space can be formalized in several ways. In this section we describe the general idea of 2-vector spaces, then focus on a particular formalism: the 2-category **Meas** defined by Yetter [73].

3.1 From vector spaces to 2-vector spaces

To understand 2-vector spaces, it is helpful first to remember the naive point of view on linear algebra that vectors are lists of numbers, operators are matrices. Namely, any finite dimensional complex vector space is isomorphic to $\mathbb{C}^N$ for some natural number N, and a linear map

$$T\colon \mathbb{C}^M \to \mathbb{C}^N$$

is an $N \times M$ matrix of complex numbers $T_{n,m}$, where $n \in \{1, \ldots, N\}$, $m \in \{1, \ldots, M\}$. Composition of operators is accomplished by matrix multiplication:

$$(UT)_{k,m} = \sum_{n=1}^{N} U_{k,n} T_{n,m}$$

for $T\colon \mathbb{C}^M \to \mathbb{C}^N$ and $U\colon \mathbb{C}^N \to \mathbb{C}^K$.

As a setting for doing linear algebra, we can form a category whose objects are just the sets $\mathbb{C}^N$ and whose morphisms are $N \times M$ matrices. This category is smaller than the category Vect of *all* finite dimensional vector spaces, but it is *equivalent* to Vect. This is why one can accomplish the same things with matrices as with abstract linear maps—an oft used fact in practical computations.

Kapranov and Voevodsky [44] observed that we can 'categorify' this naive version of the category of vector spaces and define a 2-category of '2-vector spaces'. When we categorify a concept, we replace sets with categories. In this case, we replace the set $\mathbb{C}$ of complex numbers, along with its usual product and sum operations, by the category Vect of complex vector spaces, with its tensor product and direct sum. Thus a '2-vector' is a list, not of numbers, but of vector spaces. Since we can define maps between such lists they form, not just a set, but a category: a '2-vector space'. A morphism between 2-vector spaces is a matrix, not of numbers, but of vector spaces. We also get another layer of structure: *2-morphisms*. These are matrices of linear maps.

More precisely, there is a 2-category denoted **2Vect** defined as follows:

Objects

The objects of **2Vect** are the categories

$$\mathrm{Vect}^0, \mathrm{Vect}^1, \mathrm{Vect}^2, \mathrm{Vect}^3, \ldots$$

where Vect^N denotes the N-fold cartesian product. Note in particular that the zero-dimensional 2-vector space Vect^0 has just one object and one morphism.

Morphisms

Given 2-vector spaces Vect^M and Vect^N, a morphism

$$T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$$

is given by an $N \times M$ matrix of complex vector spaces $T_{n,m}$, where $n \in \{1, \ldots, N\}$, $m \in \{1, \ldots, M\}$. Composition is accomplished by matrix multiplication, as in ordinary linear algebra, but using tensor product and direct sum:

$$(UT)_{k,m} = \bigoplus_{n=1}^{N} U_{k,n} \otimes T_{n,m} \tag{25}$$

for $T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ and $U\colon \mathrm{Vect}^N \to \mathrm{Vect}^K$.

2-Morphisms

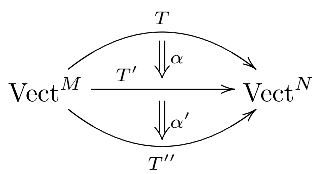
Given morphisms $T, T'\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$, a 2-morphism α between these:

$$\mathrm{Vect}^M \underset{T'}{\overset{T}{\rightrightarrows}} \mathrm{Vect}^N \quad (\Downarrow \alpha)$$

is an $N \times M$ matrix of linear maps of vector spaces, with components

$$\alpha_{n,m} \colon T_{n,m} \to T'_{n,m}.$$

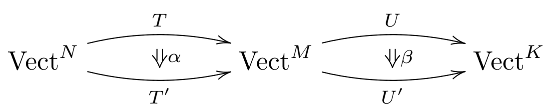
Such 2-morphisms can be composed *vertically*:



simply by composing componentwise the linear maps:

$$(\alpha' \cdot \alpha)_{n,m} = {\alpha'}_{n,m} \alpha_{n,m}. \tag{26}$$

They can also be composed *horizontally*:



analogously with (25), by using 'matrix multiplication' with respect to tensor product and direct sum of maps:

$$(\beta \circ \alpha)_{k,m} = \bigoplus_{n=1}^{N} \beta_{k,n} \otimes \alpha_{n,m}. \tag{27}$$

While simple in spirit, this definition of **2Vect** is problematic for a couple of reasons. First, composition of morphisms is not strictly associative, since the direct sum and tensor product of vector spaces satisfy the associative and distributive laws only up to isomorphism, and these laws are used in proving the associativity of matrix multiplication. So, **2Vect** as just defined is not a 2-category, but only a 'weak' 2-category, or 'bicategory'. These are a bit more complicated, but luckily any bicategory is equivalent, in a precise sense, to some 2-category. The next section gives a concrete description of a such a 2-category. (See also the work of Elgueta [31].)

The above definition of **2Vect** is also somewhat naive, since it categorifies a naive version of Vect where the only vector spaces are those of the form $\mathbb{C}^N$. A more sophisticated approach involves 'abstract' 2-vector spaces. One can define these axiomatically by listing properties of a category that guarantee that it is equivalent to Vect^N (see Def. 2.12 in [58], and also [72]). A cruder way to accomplish the same effect is to *define* an abstract 2-vector space to be a category equivalent to Vect^N. We take this approach in the next section, because we do not yet know an axiomatic approach to measurable categories, and we wish to prepare the reader for our discussion of those.

3.2 Categorical perspective on 2-vector spaces

In this section we give a definition of **2Vect** which involves treating it as a sub-2-category of the 2-category **Cat**, in which objects, morphisms, and 2-morphisms are categories, functors, and natural transformations, respectively. This approach addresses both problems mentioned at the end of the last subsection. Similar ideas will be very useful in our study of measurable categories in the sections to come.

In this approach the objects of **2Vect** are 'linear categories' that are 'linearly equivalent' to Vect^N for some N. The morphisms are 'linear functors' between such categories, and the 2-morphisms are natural transformations.

Let us define the three quoted terms. First, a **linear category** is a category where for each pair of objects x and y, the set of morphisms from x to y is equipped with the structure of a finite-dimensional complex vector space, and composition of morphisms is a bilinear operation. For example, Vect^N is a linear category.

Second, a functor $F\colon \mathsf{V} \to \mathsf{V}'$ between linear categories is a **linear equivalence** if it is an equivalence that maps morphisms to morphisms in a linear way. We define a **2-vector space** to be a linear category that is linearly equivalent to Vect^N for some N. For example, given a category V and an equivalence $F\colon \mathsf{V} \to \mathrm{Vect}^N$, we can use this equivalence to equip V with the structure of a linear category; then F becomes a linear equivalence and V becomes a 2-vector space.

Third, note that any $N \times M$ matrix of vector spaces $T_{n,m}$ gives a functor $T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ as follows. For an object $V \in \mathrm{Vect}^M$, we define $TV \in \mathrm{Vect}^N$ by

$$(TV)_n = \bigoplus_{m=1}^{M} T_{n,m} \otimes V_m.$$

For a morphism ϕ in Vect^M, we define $T\phi$ by:

$$(T\phi)_n = \bigoplus_{m=1}^{M} \mathbb{1}_{T_{n,m}} \otimes \phi_m$$

where $\mathbb{1}_{T_{n,m}}$ denotes the identity map on the vector space $T_{n,m}$. It is straightforward to check that these operations define a functor. We call such a functor from Vect^N to Vect^M a **matrix functor**. More generally, given 2-vector spaces V and V', we define a **linear functor** from V to V' to be any functor naturally isomorphic to a composite

$$\mathsf{V} \xrightarrow{F} \mathrm{Vect}^M \xrightarrow{T} \mathrm{Vect}^N \xrightarrow{G} \mathsf{V}'$$

where T is a matrix functor and F, G are linear equivalences.

These definitions may seem complicated, but unlike the naive definitions in the previous section, they give a 2-category:

Theorem 8 *There is a sub-2-category* **2Vect** *of* **Cat** *where the objects are 2-vector spaces, the morphisms are linear functors, and the 2-morphisms are natural transformations.*

The proof of this result will serve as the pattern for a similar argument for measurable categories. We break it into a series of lemmas. It is easy to see that identity functors and identity natural transformations are linear. It is obvious that natural transformations are closed under vertical and horizontal composition. So, we only need to check that linear functors are closed under composition. This is Lemma 12.

Lemma 9 *A composite of matrix functors is naturally isomorphic to a matrix functor.*

Proof: Suppose $T\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$ and $U\colon \mathrm{Vect}^N \to \mathrm{Vect}^K$ are matrix functors. Their composite UT applied to an object $V \in \mathrm{Vect}^M$ gives an object UTV with components

$$(UTV)_k = \bigoplus_{n=1}^{N} U_{k,n} \otimes \left(\bigoplus_{m=1}^{M} T_{n,m} \otimes V_m \right)$$

but this is naturally isomorphic to

$$\bigoplus_{m=1}^{M}\left(\bigoplus_{n=1}^{N} U_{k,n}\otimes T_{n,m}\right)\otimes V_m$$

so UT is naturally isomorphic to the matrix functor defined by formula (25). ∎

Lemma 10 *If $F\colon \mathrm{Vect}^N \to \mathrm{Vect}^M$ is a linear equivalence, then $N = M$ and F is a linear functor.*

Proof: Let e_i be the standard basis for Vect^N:

$$e_i = (0,\ldots,\underbrace{\mathbb{C}}_{i\text{th place}},\ldots,0).$$

Since an equivalence maps indecomposable objects to indecomposable objects, we have $F(e_i) \cong e_{\sigma(i)}$ for some function σ. This function must be a permutation, since F has a weak inverse. Let $\tilde{F}$ be the matrix functor corresponding to the permutation matrix associated to σ. One can check that F is naturally isomorphic to $\tilde{F}$, hence a linear functor. Checking this makes crucial use of the fact that F be a *linear* equivalence: for example, taking the complex conjugate of a vector space defines an equivalence $K\colon \mathrm{Vect} \to \mathrm{Vect}$ that is not a matrix functor. We leave the details to the reader. ∎

Lemma 11 *If $T\colon \mathsf{V} \to \mathsf{V}'$ is a linear functor and $F\colon \mathsf{V} \to \mathrm{Vect}^M$, $G\colon \mathrm{Vect}^N \to \mathsf{V}'$ are* arbitrary *linear equivalences, then T is naturally isomorphic to the composite*

$$\mathsf{V} \xrightarrow{F} \mathrm{Vect}^M \xrightarrow{\tilde{T}} \mathrm{Vect}^N \xrightarrow{G} \mathsf{V}'$$

for some matrix functor $\tilde{T}$.

Proof: Since T is linear we know there exist linear equivalences $F'\colon \mathsf{V} \to \mathrm{Vect}^{M'}$ and $G'\colon \mathrm{Vect}^{N'} \to \mathsf{V}'$ such that T is naturally isomorphic to the composite

$$\mathsf{V} \xrightarrow{F'} \mathrm{Vect}^{M'} \xrightarrow{\tilde{T}'} \mathrm{Vect}^{N'} \xrightarrow{G'} \mathsf{V}'$$

for some matrix functor $\tilde{T}'$. We have $M' = M$ and $N' = N$ by Lemma 10. So, let $\tilde{T}$ be the composite

$$\mathrm{Vect}^M \xrightarrow{\bar{F}} \mathsf{V} \xrightarrow{F'} \mathrm{Vect}^M \xrightarrow{\tilde{T}'} \mathrm{Vect}^N \xrightarrow{G'} \mathsf{V}' \xrightarrow{\bar{G}} \mathrm{Vect}^N$$

where $\bar{F}$ and $\bar{G}$ are weak inverses for F and G. Since $F'\bar{F}\colon \mathrm{Vect}^M \to \mathrm{Vect}^M$ and $\bar{G}G'\colon \mathrm{Vect}^N \to \mathrm{Vect}^N$ are linear equivalences, they are naturally isomorphic to matrix functors by Lemma 10. Since $\tilde{T}$ is a composite of functors that are naturally isomorphic to matrix functors, $\tilde{T}$ itself is naturally isomorphic to a matrix functor by Lemma 9. Note that the composite

$$\mathsf{V} \xrightarrow{F} \mathrm{Vect}^M \xrightarrow{\tilde{T}} \mathrm{Vect}^N \xrightarrow{G} \mathsf{V}'$$

is naturally isomorphic to T. Since F and G are linear equivalences and $\tilde{T}$ is naturally isomorphic to a matrix functor, it follows that T is a linear functor. ∎

Lemma 12 *A composite of linear functors is linear.*

Proof: Suppose we have a composable pair of linear functors $T\colon \mathsf{V} \to \mathsf{V}'$ and $U\colon \mathsf{V}' \to \mathsf{V}''$. By definition, T is naturally isomorphic to a composite

$$\mathsf{V} \xrightarrow{F} \mathrm{Vect}^L \xrightarrow{\tilde{T}} \mathrm{Vect}^M \xrightarrow{G} \mathsf{V}'$$

where $\tilde{T}$ is a matrix functor, and F and G are linear equivalences. By Lemma 11, U is naturally isomorphic to a composite

$$\mathsf{V}' \xrightarrow{\bar{G}} \mathrm{Vect}^M \xrightarrow{\tilde{U}} \mathrm{Vect}^N \xrightarrow{H} \mathsf{V}''$$

where $\tilde{U}$ is a matrix functor, $\bar{G}$ is a weak inverse for G, and H is a linear equivalence. The composite UT is thus naturally isomorphic to

$$\mathsf{V} \xrightarrow{F} \mathrm{Vect}^L \xrightarrow{\tilde{U}\tilde{T}} \mathrm{Vect}^N \xrightarrow{H} \mathsf{V}''$$

Since $\tilde{U}\tilde{T}$ is naturally isomorphic to a matrix functor by Lemma 9, it follows that UT is a linear functor. ■

These results justify the naive recipe for composing 1-morphisms using matrix multiplication, namely equation (25). First, Lemma 9 shows that the composite of matrix functors is naturally isomorphic to their matrix product as given by equation (25). More generally, given any linear functors $T\colon \mathrm{Vect}^L \to \mathrm{Vect}^M$ and $U\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$, we can choose matrix functors naturally isomorphic to these, and the composite UT will be naturally isomorphic to the matrix product of these matrix functors. Finally, we can reduce the job of composing linear functors between *arbitrary* 2-vector spaces to matrix multiplication by choosing linear equivalences between these 2-vector spaces and some of the form Vect^N.

Similar results hold for natural transformations. Any $N \times M$ matrix of linear operators $\alpha_{n,m}\colon T_{n,m} \to T'_{n,m}$ determines a natural transformation between the matrix functors $T, T'\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$. This natural transformation gives, for each object $V \in \mathrm{Vect}^M$, a morphism $\alpha_V : TV \to T'V$ with components

$$(\alpha_V)_n \colon \bigoplus_{m=1}^{M} T_{n,m} \otimes V_m \to \bigoplus_{m=1}^{M} T'_{n,m} \otimes V_m$$

given by

$$(\alpha_V)_n = \bigoplus_{m=1}^{M} \alpha_{n,m} \otimes 1\!\!1_{V_m}.$$

We call a natural transformation of this sort a **matrix natural transformation**. However:

Theorem 13 *Any natural transformation between matrix functors is a matrix natural transformation.*

Proof: Given matrix functors $T, T'\colon \mathrm{Vect}^M \to \mathrm{Vect}^N$, a natural transformation $\alpha\colon T \Rightarrow T'$ gives for each basis object $e_m \in \mathrm{Vect}^M$ a morphism in Vect^N with components

$$(\alpha_{e_m})_n \colon T_{n,m} \otimes \mathbb{C} \to T'_{n,m} \otimes \mathbb{C}.$$

Using the natural isomorphism between a vector space and that vector space tensored with $\mathbb{C}$, these can be reinterpreted as operators

$$\alpha_{n,m} \colon T_{n,m} \to T'_{n,m}.$$

These operators define a matrix natural transformation from T to T', and one can check using naturality that this equals α. ∎

One can check that vertical composition of matrix natural transformations is given by the matrix formula of the previous section, namely formula (26). Similarly, the horizontal composite of matrix natural transformations is 'essentially' given by formula (27). So, while these matrix formulas are a bit naive, they are useful tools when properly interpreted.

3.3 From 2-vector spaces to measurable categories

In the previous sections, we saw the 2-category **2Vect** of Kapranov–Voevodsky 2-vector spaces as a categorification of Vect, the category of finite-dimensional vector spaces. While one can certainly study representations of 2-groups in **2Vect** [18,32], our goal is to describe representations of 2-groups in something more akin to infinite-dimensional 2-*Hilbert* spaces. Such objects should be roughly like 'Hilb^X', where Hilb is the category of Hilbert spaces and X may now be an infinite index set. In fact, for our purposes, X should have at least the structure of a measurable space. This allows one to categorify Hilbert spaces $L^2(X,\mu)$ in such a way that measurable functions are replaced by 'measurable fields of Hilbert spaces', and integrals of functions are replaced by 'direct integrals' of such fields.

We can construct a chart like the one in the introduction, outlining the basic strategy for categorification:

ordinary L^2 spaces	higher L^2 spaces
$\mathbb{C}$	Hilb
$+$	$\oplus$
$\times$	$\otimes$
0	$\{0\}$
1	$\mathbb{C}$
measurable functions	measurable fields of Hilbert spaces
$\int$ (integral)	$\int^\oplus$ (direct integral)

Various alternatives spring from this basic idea. In this section and the following one, we provide a concrete description of one possible categorification of L^2 spaces: 'measurable categories' as defined by Yetter [73], which provide a foundation for earlier work by Crane, Sheppeard, and Yetter [26,27].

Measurable categories do not provide a full-fledged categorification of the concept of Hilbert space, so they do not deserve to be called '2-Hilbert spaces'. Indeed, *finite-dimensional* 2-Hilbert spaces are well understood [3, 17], and they have a bit more structure than measurable categories with a finite basis of objects. Namely, we can take the 'inner product' of two objects in such a 2-Hilbert space and get a Hilbert space. We expect something similar in an infinite-dimensional 2-Hilbert space, and it happens in many interesting examples, but the definition of measurable category lacks this feature. So, our work here can be seen as a stepping-stone towards a theory of unitary representations of 2-groups on infinite-dimensional 2-Hilbert spaces. See Section 5 for a bit more on this issue.

The goal of this section is to construct a 2-category of measurable categories, denoted **Meas**. This requires some work, in part because we do not have an intrinsic characterization of measurable categories. We also give concrete practical formulas for composing morphisms and 2-morphisms in **Meas**. This will equip the reader with the tools necessary for calculations in the representation theory developed in Section 4. But first we need some preliminaries in analysis. For basic results and standing assumptions the reader may also turn to Appendix A.

3.3.1 Measurable fields and direct integrals

We present here some essential analytic tools: measurable fields of Hilbert spaces and operators, their measure-classes and direct integrals, and measurable families of measures.

We have explained the categorical motivation for generalizing *functions* on a measurable space to '*fields of Hilbert spaces*' on a measurable space. But one cannot simply assign an arbitrary Hilbert space to each point in a measurable space X and expect to perform operations that make good analytic sense. Fortunately, 'measurable fields' of Hilbert spaces have been studied in detail—see especially the book by Dixmier [29]. Algebraists may view these as representations of abelian von Neumann algebras on Hilbert spaces, as explained by Dixmier and also Arveson [2, Chap. 2.2]. Geometers may instead prefer to view them as 'measurable bundles of Hilbert spaces', following the treatment of Mackey [52]. Measurable fields of Hilbert spaces have also been studied from a category-theoretic perspective by Yetter [73].

It will be convenient to impose some simplifying assumptions. Our measurable spaces will all be 'standard Borel spaces' and our measures will always be σ-finite and positive. Standard Borel spaces can be characterized in several ways:

Lemma 14 *Let $(X, \mathcal{B})$ be a measurable space, i.e. a set X equipped with a σ-algebra of subsets $\mathcal{B}$. Then the following are equivalent:*

1. *X can be given the structure of a separable complete metric space in such a way that $\mathcal{B}$ is the σ-algebra of Borel subsets of X.*
2. *X can be given the structure of a second-countable, locally compact Hausdorff space in such a way that $\mathcal{B}$ is the σ-algebra of Borel subsets of X.*
3. *$(X, \mathcal{B})$ is isomorphic to one of the following:*
   - *a finite set with its σ-algebra of all subsets;*
   - *a countably infinite set with its σ-algebra of all subsets;*
   - *$[0, 1]$ with its σ-algebra of Borel subsets.*

   A measurable space satisfying any of these equivalent conditions is called a **standard Borel space**.

Proof: It is clear that 3) implies 2). To see that 2) implies 1), we need to check that every second-countable locally compact Hausdorff space X can be made into a separable complete metric space. For this, note that the one-point compactification of X, say X^+, is a second-countable compact Hausdorff space, which admits a metric by Urysohn's metrization theorem. Since X^+ is compact this metric is complete. Finally, any open subset of separable complete metric space can be given a new metric giving it the same topology, where the new metric is separable and complete [21, Chap. IX, §6.1, Prop. 2]. Finally, that 1) implies 3) follows from two classical results of Kuratowski. Namely: two standard Borel spaces (defined using condition 1) are isomorphic if and only if they have the same cardinality, and any uncountable standard Borel space has the cardinality of the continuum [59, Chap. I, Thms. 2.8 and 2.13]. ∎

The following definitions will be handy:

Definition 15 *By a* **measurable space** *we mean a standard Borel space* $(X, \mathcal{B})$*. We call sets in* $\mathcal{B}$ **measurable***. Given spaces* X *and* Y*, a map* $f\colon X \to Y$ *is* **measurable** *if* $f^{-1}(S)$ *is measurable whenever* $S \subseteq Y$ *is measurable.*

Definition 16 *By a* **measure** *on a measurable space* $(X, \mathcal{B})$ *we mean a* σ*-finite measure, i.e. a countably additive map* $\mu\colon \mathcal{B} \to [0, +\infty]$ *for which* X *is a countable union of* $S_i \in \mathcal{B}$ *with* $\mu(S_i) < \infty$*.*

A key idea is that a measurable field of Hilbert spaces should know what its 'measurable sections' are. That is, there should be preferred ways of selecting one vector from the Hilbert space at each point; these preferred sections should satisfy some properties, given below, to guarantee reasonable measure-theoretic behavior:

Definition 17 *Let* X *be a measurable space. A* **measurable field of Hilbert spaces** $\mathcal{H}$ *on* X *is an assignment of a Hilbert space* $\mathcal{H}_x$ *to each* $x \in X$*, together with a subspace* $\mathcal{M}_\mathcal{H} \subseteq \prod_x \mathcal{H}_x$ *called the* **measurable sections** *of* $\mathcal{H}$*, satisfying the properties:*

- $\forall \xi \in \mathcal{M}_\mathcal{H}$*, the function* $x \mapsto \|\xi_x\|_{\mathcal{H}_x}$ *is measurable.*
- *For any* $\eta \in \prod_x \mathcal{H}_x$ *such that* $x \mapsto \langle \eta_x, \xi_x \rangle_{\mathcal{H}_x}$ *is measurable for all* $\xi \in \mathcal{M}_\mathcal{H}$*, we have* $\eta \in \mathcal{M}_\mathcal{H}$*.*
- *There is a sequence* $\xi_i \in \mathcal{M}_\mathcal{H}$ *such that* $\{(\xi_i)_x\}_{i=1}^\infty$ *is dense in* $\mathcal{H}_x$ *for all* $x \in X$*.*

Definition 18 *Let* $\mathcal{H}$ *and* $\mathcal{H}'$ *be measurable fields of Hilbert spaces on* X*. A* **measurable field of bounded linear operators** $\phi\colon \mathcal{H} \to \mathcal{H}'$ *on* X *is an* X*-indexed family of bounded operators* $\phi_x\colon \mathcal{H}_x \to \mathcal{H}'_x$ *such that* $\xi \in \mathcal{M}_\mathcal{H}$ *implies* $\phi(\xi) \in \mathcal{M}_{\mathcal{H}'}$*, where* $\phi(\xi)_x := \phi_x(\xi_x)$*.*

Given a positive measure μ on X, measurable fields can be integrated. The integral of a function gives an element of $\mathbb{C}$; the integral of a field of Hilbert spaces gives an object of Hilb. Formally, we have the following definition:

Definition 19 *Let* $\mathcal{H}$ *be a measurable field of Hilbert spaces on a measurable space* X*; let* $\langle \cdot, \cdot \rangle_x$ *denote the inner product in* $\mathcal{H}_x$*, and* $\| \cdot \|_x$ *the induced norm. The* **direct integral**

$$\int_X^\oplus d\mu(x)\, \mathcal{H}_x$$

of $\mathcal{H}$ *with respect to the measure* μ *is the Hilbert space of all* μ*-a.e. equivalence classes of measurable* L^2 *sections of* $\mathcal{H}$*, that is, sections* $\psi \in \mathcal{M}_\mathcal{H}$ *such that*

$$\int_X d\mu(x)\, \|\psi_x\|_x^2 < \infty,$$

with inner product given by

$$\langle \psi, \psi' \rangle = \int_X d\mu(x)\, \langle \psi_x, \psi'_x \rangle_x.$$

for $\psi, \psi' \in \int_X^\oplus d\mu\, \mathcal{H}$*.*

That the inner product is well defined for L^2 sections follows by polarization. Of course, for $\int_X^\oplus d\mu\, \mathcal{H}$ to be a Hilbert space as claimed in the definition, one must also check that it is Cauchy-complete with respect to the induced norm. This is indeed the case [29, Part II Ch. 1 Prop. 5]. We often denote an element of the direct integral of $\mathcal{H}$ by

$$\int_X^\oplus d\mu(x)\, \psi_x$$

where $\psi_x \in \mathcal{H}_x$ is defined up to μ-*a.e.* equality.

We also have a corresponding notion of direct integral for fields of linear operators:

Definition 20 *Suppose $\phi\colon \mathcal{H} \to \mathcal{H}'$ is a μ-essentially bounded measurable field of linear operators on X. The* **direct integral** *of ϕ is the linear operator acting pointwise on sections:*

$$\begin{array}{rccc} \int_X^\oplus d\mu(x)\, \phi_x\colon & \int_X^\oplus d\mu(x)\, \mathcal{H}_x & \to & \int_X^\oplus d\mu(x)\, \mathcal{H}'_x \\ & \int_X^\oplus d\mu(x)\, \psi_x & \mapsto & \int_X^\oplus d\mu(x)\, \phi_x(\psi_x) \end{array}$$

Note requiring that the field be μ-essentially bounded—*i.e.* that the operator norms $\|\phi_x\|$ have a common bound for μ-almost every x—guarantees that the image lies in the direct integral of $\mathcal{H}'$, since

$$\int_X d\mu(x)\, \|\phi_x(\psi_x)\|^2_{\mathcal{H}'_x} \;\le\; \operatorname{ess\,sup}_{x'} \|\phi_{x'}\|^2 \int d\mu(x)\, \|\psi_x\|^2_{\mathcal{H}_x} \;<\; \infty.$$

Notice that direct integrals indeed generalize direct sums: in the case where X is a finite set and μ is counting measure, direct integrals of Hilbert spaces and operators simply reduce to direct sums.

In ordinary integration theory, one typically identifies functions that coincide almost everywhere with respect to the relevant measure. This is also useful for the measurable fields defined above, for the same reasons. To make '*a.e.*-equivalence of measurable fields' precise, we first need a notion of 'restriction'.

If $A \subseteq X$ is a measurable set, any measurable field $\mathcal{H}$ of Hilbert spaces on X induces a field $\mathcal{H}|_A$ on A, called the **restriction** of $\mathcal{H}$ to A. The restricted field is constructed in the obvious way: we let $(\mathcal{H}|_A)_x = \mathcal{H}_x$ for each $x \in A$, and define the measurable sections to be the restrictions of measurable sections on X: $\mathcal{M}_{\mathcal{H}|_A} = \{\psi|_A : \psi \in \mathcal{M}_\mathcal{H}\}$. It is straightforward to check that $(\mathcal{H}|_A, \mathcal{M}_{\mathcal{H}|_A})$ indeed defines a measurable field. The first and third axioms in the definition are obvious. To check the second, pick $\eta \in \prod_{x\in A} \mathcal{H}_x$ such that $x \mapsto \langle \eta_x, \xi_x\rangle$ is a measurable function on A for every $\xi \in \mathcal{M}|_{\mathcal{H}|_A}$. Extend η to $\tilde\eta \in \prod_{x\in X} \mathcal{H}_x$ by setting

$$\tilde\eta_x = \begin{cases} \eta_x & x \in A \\ 0 & x \notin A. \end{cases}$$

Then, use the fact that $\mathcal{H}$ obeys the second axiom.

Similarly, if $\phi\colon \mathcal{H} \to \mathcal{K}$ is a field of linear operators, its **restriction** to a measurable subset $A \subseteq X$ is the obvious A-indexed family of operators $\phi|_A\colon \mathcal{H}|_A \to \mathcal{K}|_A$ given by $(\phi|_A)_x = \phi_x$ for each x in A. It is easy to check that $\xi \in \mathcal{M}_{\mathcal{H}|_A}$ implies $\phi(\xi) \in \mathcal{M}_{\mathcal{K}|_A}$, so $\phi|_A$ defines a measurable field on A.

We say two measurable fields of Hilbert spaces on X are **μ-almost everwhere equivalent** if they have equal restrictions to some measurable $A \subseteq X$ with $\mu(X - A) = 0$. This is obviously an equivalence relation, and an equivalence class is called a **μ-class of measurable fields**. Two fields in the same μ-class have canonically isomorphic direct integrals, so the direct integral of a μ-class makes sense.

Equivalence classes of measurable fields of linear operators work similarly, but with one subtlety. First suppose we have two measurable fields of Hilbert spaces T_x and U_x on X, and a measurable field of operators $\alpha_x\colon T_x \to U_x$. Given a measure μ, one can clearly identify two such α if they coincide outside a set of μ-measure 0, thus defining a notion of **μ-class of fields of operators** from T to U. So far T and U are fixed, but now we wish to take equivalence classes of them as well. In fact, it is often useful to pass to t-classes of T and u-classes of U, where t and u are in general *different* measures on X. We then ask what sort of measure μ must be for the μ-class of α to pass to a well defined map

$$[\alpha_x]_\mu\colon [T_x]_t \to [U_x]_u,$$

where brackets denote the relevant classes. This works if and only if each t-null set and each u-null set is also μ-null. Thus we require

$$\mu \ll t \quad \text{and} \quad \mu \ll u, \tag{28}$$

where '$\ll$' denotes absolute continuity of measures. Given a measure μ satisfying these properties, it makes sense to speak of the **μ-class of fields of operators** from a t-class of fields of Hilbert spaces to a u-class of fields of Hilbert spaces. In practice, one would like to pick μ to be maximal with respect to the required properties (28), so that μ-*a.e.* equivalence is the transitive closure of u-*a.e.* and t-*a.e.* equivalences.

In fact, if t and u are both σ-finite measures, there is a natural choice for which measure μ to take in the above construction: the 'geometric mean measure' $\sqrt{tu}$ of the measures t and u. The notion of geometric mean measure is discussed in Appendix A.2, but the basic idea is as follows. If t is absolutely continuous with respect to u, denoted $t \ll u$, then we have the Radon–Nikodym derivative $\frac{dt}{du}$. More generally, even when t is not absolutely continuous with respect to u, we will use the notation

$$\frac{dt}{du} := \frac{dt^u}{du}$$

where t^u is the absolutely continuous part of the Lebesgue decomposition of t with respect to u. An important fact, proved in Appendix A.2, is that

$$\sqrt{\frac{dt}{du}}du = \sqrt{\frac{du}{dt}}dt,$$

so we can define the **geometric mean measure**, denoted $\sqrt{dtdu}$ or simply $\sqrt{tu}$, using either of these expressions.

Every set of t-measure or u-measure zero also has $\sqrt{tu}$-measure zero. That is,

$$\sqrt{tu} \ll t \quad \text{and} \quad \sqrt{tu} \ll u.$$

In fact, every $\sqrt{tu}$-null set is the union of a t-null set and a u-null set, as we show in Appendix A.2. This means $\sqrt{tu}$ is a measure that is maximal with respect to (28).

Recall that we are assuming our measures are σ-finite. Using this, one can show that

$$\frac{dt}{du}\frac{du}{dt} = 1 \qquad \sqrt{tu}\text{-}a.e. \tag{29}$$

This rule, obvious when the two measures are equivalent, is proved in Appendix A.2.

We shall need one more type of 'field', which may be thought of as 'measurable fields of *measures*'. In general, these involve two measure spaces: they are certain families μ_y of measures on a measurable space X, indexed by elements of a measurable space Y. We first introduce the notion of fibered measure distribution [73]:

Definition 21 *Suppose X and Y are measurable spaces and every one-point set of Y is measurable. Then a* ***Y-fibered measure distribution on $Y \times X$*** *is a Y-indexed family of measures $\bar{\mu}_y$ on $Y \times X$ satisfying the properties:*

- *$\bar{\mu}_y$ is supported on $\{y\} \times X$: that is, $\bar{\mu}_y((Y - \{y\}) \times X) = 0$*
- *For every measurable $A \subseteq Y \times X$, the function $y \mapsto \bar{\mu}_y(A)$ is measurable*
- *The family is uniformly finite: that is, there exists a constant M such that for all $y \in Y$, $\bar{\mu}_y(X) < M$.*

Any fibered measure distribution gives rise to a Y-indexed family of measures on X:

Definition 22 *Given measurable spaces X and Y, μ_y is a* ***Y-indexed measurable family of measures on X*** *if it is induced by a Y-fibered measure distribution $\bar{\mu}_y$ on $Y \times X$; that is, if*

$$\mu_y(A) = \bar{\mu}_y(Y \times A)$$

for every measurable $A \subseteq X$.

Notice that, if $\bar{\mu}_y$ is the fibered measure distribution associated to the measurable family μ_y, we have

$$\bar{\mu}_y = \delta_y \otimes \mu_y \tag{30}$$

as measures on $Y \times X$, where for each $y \in Y$, δ_y is the Dirac measure concentrated at y.

By itself, a fibered measure distribution $\bar{\mu}_y$ on $Y \times X$ is *not* a measure on $Y \times X$. However, taken together with a suitable measure ν on Y, it may yield a measure λ on $Y \times X$:

$$\lambda = \int_Y d\nu\, (\delta_y \otimes \mu_y) \tag{31}$$

Because this measure λ is obtained from μ_y by *integration* with respect to ν, the measurable family μ_y is also called the **disintegration** of λ with respect to ν. It is often the *dis*integration problem one is interested in: given a measure λ on a product space and a measure ν on one of the factors, can λ be written as an integral of some measurable family of measures on the other factor, as in (31)? Conditions for the disintegration problem to have a solution are given by the 'disintegration theorem':

Theorem 23 (Disintegration Theorem) *Suppose X and Y are measurable spaces. Then a measure λ on $Y \times X$ has a disintegration μ_y with respect to the measure ν on Y if and only if $\nu(U) = 0$ implies $\lambda(U \times X) = 0$ for every measurable $U \subseteq Y$. When this is the case, the measures μ_y are determined uniquely for ν-almost every y.*

Proof: Graf and Mauldin [40] state a theorem due to Maharam [55] that easily implies a stronger version of this result: namely, that the conclusions hold whenever X and Y are Lusin spaces. Recall that a topological space space homeomorphic to separable complete metric space is called a **Polish space**, while more generally a **Lusin space** is a topological space that is the image of a Polish space under a continuous bijection. By Lemma 14, every measurable space we consider — i.e., every standard Borel space—is isomorphic to some Polish space equipped with its σ-algebra of Borel sets. ∎

3.3.2 The 2-category of measurable categories: Meas

We are now in a position to give a definition of the 2-category **Meas** introduced in the work of Crane and Yetter [27,73]. The aim of this section is essentially practical: we give concrete descriptions of the objects, morphisms, and 2-morphisms of **Meas**, and formulae for the composition laws. These formulae will be analogous to those presented in the finite-dimensional case in Section 3.1, which the current section parallels.

Before diving into the technical details, let us sketch the basic idea behind the 2-category **Meas**:

- The objects of **Meas** are 'measurable categories', which are categories somewhat analogous to Hilbert spaces. The most important sort of example is the category H^X whose objects are measurable fields of Hilbert spaces on the measurable space X, and whose morphisms are measurable fields of bounded operators. If X is a finite set with n elements, then $H^X \cong \mathrm{Hilb}^n$. So, H^X generalizes Hilb^n to situations where X is a measurable space instead of a finite set.

- The morphisms of **Meas** are 'measurable functors'. The most important examples are 'matrix functors' $T\colon H^X \to H^Y$. Such a functor is constructed using a field of Hilbert spaces on $X \times Y$, which we also denote by T. When X and Y are finite sets, such a field is simply a matrix of Hilbert spaces. But in general, to construct a matrix functor $T\colon H^X \to H^Y$ we also need a Y-indexed measure on X.

- The 2-morphisms of **Meas** are 'measurable natural transformations'. The most important examples are 'matrix natural transformations' $\alpha\colon T \to T'$ between matrix functors $T, T'\colon H^X \to H^Y$. Such a natural transformation is constructed using a uniformly bounded field of linear operators $\alpha_{y,x}\colon T_{y,x} \to T'_{y,x}$.

Here we have sketchily described the most important objects, morphisms and 2-morphisms in **Meas**. However, following our treatment of **2Vect** in Section 3.2, we need to make **Meas** bigger to obtain a 2-category instead of a bicategory. To do this, we include as objects of **Meas** certain categories that are *equivalent* to categories of the form H^X, and include as morphisms certain functors that are *naturally isomorphic* to matrix functors.

Objects

Given a measurable space X, there is a category H^X with:

- measurable fields of Hilbert spaces on X as objects;
- bounded measurable fields of linear operators on X as morphisms.

Objects of the 2-category **Meas** are 'measurable categories'—that is, 'C^*-categories' that are 'C^*-equivalent' to H^X for some X. Let us make this precise:

Definition 24 *A* **Banach category** *is a category C enriched over Banach spaces, meaning that for any pair of objects $x, y \in C$, the set of morphisms from x to y is equipped with the structure of a Banach space, composition is bilinear, and*

$$\|fg\| \leq \|f\|\|g\|$$

for every pair of composable morphisms f, g in C.

Definition 25 *A* **Banach $*$-category** *is a Banach category in which each morphism $f\colon x \to y$ has an associated morphism $f^*\colon y \to x$, such that:*

- *each map* $\hom(x,y) \to \hom(y,x)$ *given by* $f \mapsto f^*$ *is conjugate linear;*
- $(gf)^* = f^*g^*$, $1_x^* = 1_x$, *and* $f^{**} = f$, *for every object* x *and pair of composable morphisms* f, g;
- *for any morphism* $f\colon x \to y$, *there exists a morphism* $g\colon x \to x$ *such that* $f^*f = g^*g$;
- $f^*f = 0$ *if and only if* $f = 0$.

Definition 26 *A* ***C*-category*** *is a Banach* $*$*-category such that for each morphism* $f\colon x \to y$,

$$\|f^*f\| = \|f\|^2.$$

Note that for each object x in a C^*-category, its endomorphisms form a C^*-algebra. Note also that for any measurable space X, H^X is a C^*-category, where the norm of any bounded measurable field of operators $\phi\colon \mathcal{H} \to \mathcal{K}$ is

$$\|\phi\| = \sup_{x \in X} \|\phi_x\|$$

and we define the $*$ operation pointwise:

$$(\phi^*)_x = (\phi_x)^*$$

where the right-hand side is the Hilbert space adjoint of the operator ϕ_x.

Definition 27 *A functor* $F\colon C \to C'$ *between* C^**-categories is a* ***C*-functor*** *if it maps morphisms to morphisms in a linear way, and satisfies*

$$F(f^*) = F(f)^*$$

for every morphism f *in* C.

Using the fact that a $*$-homomorphism between unital C^*-algebras is automatically norm-decreasing, we can show that any C^*-functor satisfies

$$\|F(f)\| \leq \|f\|.$$

Definition 28 *Given* C^**-categories* C *and* C', *a natural transformation* $\alpha\colon F \Rightarrow F'$ *between functors* $F, F'\colon C \to C'$ *is* **bounded** *if for some constant* K *we have*

$$\|\alpha_x\| \leq K$$

for all $x \in C$. *If there is a bounded natural isomorphism between functors between* C^**-categories, we say they are* **boundedly naturally isomorphic**.

Definition 29 *A* C^**-functor* $F\colon C \to C'$ *is a* ***C*-equivalence*** *if there is a* C^**-functor* $\bar{F}\colon C' \to C$ *such that* $\bar{F}F$ *and* $F\bar{F}$ *are boundedly naturally isomorphic to identity functors.*

Definition 30 *A* **measurable category** *is a* C^**-category that is* C^**-equivalent to* H^X *for some measurable space* X.

Morphisms

The morphisms of **Meas** are 'measurable functors'. The most important measurable functors are the 'matrix functors', so we begin with these. Given two objects H^X and H^Y in **Meas**, we can construct a functor

$$H^X \xrightarrow{T,t} H^Y$$

from the following data:

- a uniformly finite Y-indexed measurable family t_y of measures on X,
- a t-class of measurable fields of Hilbert spaces T on $Y \times X$, such that t is concentrated on the support of T; that is, for each $y \in Y$, $t_y(\{x \in X : T_{y,x} = 0\}) = 0$.

Here by ***t*-class** we mean a t_y-class for each y, as defined in the previous section.

For brevity, we will sometimes denote the functor constructed from these data simply by T. This functor maps any object $\mathcal{H} \in H^X$—a measurable field of Hilbert spaces on X—to the object $T\mathcal{H} \in H^Y$ given by

$$(T\mathcal{H})_y = \int_X^{\oplus} dt_y \, T_{y,x} \otimes \mathcal{H}_x.$$

Similarly, it maps any morphism $\phi\colon \mathcal{H} \to \mathcal{H}'$ to the morphism $T\phi\colon T\mathcal{H} \to T\mathcal{H}'$ given by the direct integral of operators

$$(T\phi)_y = \int_X^{\oplus} dt_y \, \mathbb{1}_{T_{y,x}} \otimes \phi_x$$

where $\mathbb{1}_{T_{y,x}}$ denotes the identity operator on $T_{y,x}$. Note that T is a C^*-functor.

Definition 31 *Given measurable spaces X and Y, a functor $T\colon H^X \to H^Y$ of the above sort is called a* **matrix functor***.*

Starting from matrix functors, we can define measurable functors in general:

Definition 32 *Given objects $\mathsf{H}, \mathsf{H}' \in$ **Meas**, a* **measurable functor** *from H to H' is a C^*-functor that is boundedly naturally isomorphic to a composite*

$$\mathsf{H} \xrightarrow{F} H^X \xrightarrow{T} H^Y \xrightarrow{G} \mathsf{H}'$$

where T is a matrix functor and the first and last functors are C^*-equivalences.

In Section 3.3.3 we use results of Yetter to show that the composite of measurable functors is measurable. A key step is showing that the composite of two matrix functors:

$$H^X \xrightarrow{T,t} H^Y \xrightarrow{U,u} H^Z$$

is boundedly naturally isomorphic to a matrix functor

$$H^X \xrightarrow{UT,ut} H^Z.$$

Let us sketch how this step goes, since we will need explicit formulas for UT and ut. Picking any object $\mathcal{H} \in H^X$, we have

$$\begin{aligned}(UT\mathcal{H})_z &= \int_Y^{\oplus} du_z\, U_{z,y} \otimes (T\mathcal{H})_y \\ &= \int_Y^{\oplus} du_z\, U_{z,y} \otimes \left(\int_X^{\oplus} dt_y T_{y,x} \otimes \mathcal{H}_x \right)\end{aligned}$$

To express this in terms of a matrix functor, we will write it as direct integral over X with respect to a Z-indexed family of measures on X denoted ut, defined by:

$$(ut)_z = \int_Y du_z(y)\, t_y. \tag{32}$$

To do this we use the disintegration theorem, Thm. 23, to obtain a field of measures $k_{z,x}$ such that

$$\int_X d(ut)_z(x)\, (k_{z,x} \otimes \delta_x) = \int_Y du_z(y)\, (\delta_y \otimes t_y). \tag{33}$$

as measures on $Y \times X$. That is, $k_{z,x}$ and t_y are, respectively, the X- and Y-disintegrations of the same measure on $X \times Y$, with respect to the measures $(ut)_z$ on X and u_z on Y. The measures $k_{y,x}$ are determined uniquely for all z and $(ut)_z$-almost every x. With these definitions, it follows that there is a bounded natural isomorphism

$$(UT\mathcal{H})_z \cong \int_X^{\oplus} d(ut)_z \left(\int_Y^{\oplus} dk_{z,x} U_{z,y} \otimes T_{y,x} \right) \otimes \mathcal{H}_x \tag{34}$$

$$= \int_X^{\oplus} d(ut)_z (UT)_{z,x} \otimes \mathcal{H}_x \tag{35}$$

where

$$(UT)_{z,x} = \int_Y^{\oplus} dk_{z,x}(y)\, U_{z,y} \otimes T_{y,z}, \tag{36}$$

This formula for UT is analogous to (25). We refer to Yetter [73] for proofs that the family of measures ut and the field of Hilbert spaces UT are measurable, and hence define a matrix functor.

It is often convenient to use an alternative form of (33) in terms of integrals of functions: for every measurable function F on $Y \times X$ and for all $z \in Z$,

$$\int_X d(ut)_z(x) \int_Y dk_{z,x}(y)\, F(y,x) = \int_Y du_z(y) \int_X dt_y(x)\, F(y,x). \tag{37}$$

This can be thought of as a sort of 'Fubini theorem', since it lets us change the order of integration, but here the measure on one factor in the product is parameterized by the other factor.

Besides composition of morphisms in **Meas**, we also need identity morphisms. Given an object H^X, to show its identity functor $1_X \colon H^X \to H^X$ is a matrix functor we need an X-indexed family of measures on X, and a field of Hilbert spaces on $X \times X$. Denote the coordinates of $X \times X$ by (x', x). The family of measures assigns to each $x' \in X$ the unit Dirac measure concentrated at the point x':

$$\delta_{x'}(A) = \begin{cases} 1 & \text{if} \quad x' \in A \\ 0 & \text{otherwise} \end{cases} \qquad \text{for every measurable set } A \subseteq X$$

The field of Hilbert spaces on $X \times X$ is the constant field $(\mathbb{1}_X)_{x',x} = \mathbb{C}$. It is simple to check that this acts as both left and right identity for composition. Let us check that it is a right identity by forming this composite:

$$H^X \xrightarrow{\mathbb{1}_X,\delta} H^X \xrightarrow{T,t} H^Y$$

One can check that the composite measure is:

$$(t\delta)_y = \int_X^{\oplus} dt_y(x')\delta'_x = t_y,$$

and hence, using (37),

$$k_{y,x} = \delta_x.$$

We can then calculate the field of operators:

$$(T\mathbb{1}_X)_{y,x} \cong \int_X^{\oplus} d\delta_x(x')\, T_{y,x'} \otimes \mathbb{C} = T_{y,x}.$$

2-Morphisms

The 2-morphisms in **Meas** are 'measurable natural transformations'. The most important of these are the 'matrix natural transformations'. Given two matrix functors (T,t) and (T',t'), we can construct a natural transformation between them from a $\sqrt{tt'}$-class of bounded measurable fields of linear operators

$$\alpha_{y,x} \colon T_{y,x} \longrightarrow T'_{y,x}$$

on $Y \times X$. Here by a $\boldsymbol{\sqrt{tt'}}$**-class**, we mean a $\sqrt{t_y t'_y}$-class for each y, where the $\sqrt{t_y t'_y}$ is the geometric mean of the measures t_y and t'_y. By **bounded**, we mean $\alpha_{y,x}$ have a common bound for all y and $\sqrt{t_y t'_y}$-almost every x.

We denote the natural transformation constructed from these data simply by α. This natural transformation assigns to each object $\mathcal{H} \in H^X$ the morphism $\alpha_{\mathcal{H}} \colon T\mathcal{H} \to T'\mathcal{H}$ in H^Y with components:

$$\begin{array}{rccc} (\alpha_{\mathcal{H}})_y \colon & \displaystyle\int_X^{\oplus} dt_y\; T_{y,x} \otimes \mathcal{H}_x & \to & \displaystyle\int_X^{\oplus} dt'_y\; T'_{y,x} \otimes \mathcal{H}_x \\ & \int_X^{\oplus} dt_y\; \psi_{y,x} & \mapsto & \int_X^{\oplus} dt'_y\; [\tilde{\alpha}_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x}) \end{array}$$

where $\tilde{\alpha}$ is the rescaled field

$$\tilde{\alpha} = \sqrt{\frac{dt_y}{dt'_y}}\, \alpha. \tag{38}$$

To check that $\alpha_{\mathcal{H}}$ is well defined, pick $\psi \in T\mathcal{H}$ and compute

$$\begin{aligned} \int_X dt'_y \|[\tilde{\alpha}_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x})\|^2 &= \int_X dt^c_y \|[\alpha_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x})\|^2 \\ &\le \operatorname*{ess\,sup}_{x'} \|\alpha_{y,x'}\|^2 \int_X^{\oplus} dt_y \|\psi_{y,x}\|^2 < \infty \end{aligned}$$

where t^c_y is the absolutely continuous part of the Lebesgue decomposition of t_y with respect to t'_y; note that, since t^c_y is equivalent to $\sqrt{t_y t'_y}$, the field α is essentially bounded with respect to t^c_y. This inequality shows that the image $(\alpha_{\mathcal{H}})_y(\psi)$ belongs to $(T'\mathcal{H})_y$, and that $\alpha_{\mathcal{H}}$ is a field of bounded linear maps, as required. Note also that the direct integral defining the image does not depend on the chosen representative of α.

To check that α is natural it suffices to choose a morphism $\phi\colon \mathcal{H} \to \mathcal{H}'$ in H^X and show that the naturality square

$$\begin{array}{ccc} T\mathcal{H} & \xrightarrow{T\phi} & T\mathcal{H}' \\ {\scriptstyle \alpha_H}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{H}'}} \\ T'\mathcal{H} & \xrightarrow[T'\phi]{} & T'\mathcal{H}' \end{array}$$

commutes; that is,

$$\alpha_{\mathcal{H}'}\,(T\phi) = (T'\phi)\,\alpha_{\mathcal{H}}.$$

To check this, apply the operator on the left to $\psi \in T\mathcal{H}$ and calculate:

$$\begin{aligned} (\alpha_{\mathcal{H}'})_y(T\phi)_y(\psi_y) &= (\alpha_{\mathcal{H}'})_y \left(\int_X^{\oplus} dt_y(x)[\mathbb{1}_{T_{y,x}} \otimes \phi_x](\psi_y) \right) \\ &= \int_X^{\oplus} dt'_y(x)[\tilde{\alpha}_{y,x} \otimes \mathbb{1}_{H'_x}][\mathbb{1}_{T_{y,x}} \otimes \phi_x](\psi_y) \\ &= \int_X^{\oplus} dt'_y(x)[\mathbb{1}_{T'_{y,x}} \otimes \phi_x][\alpha_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_y) = (T'\phi)_y(\alpha_{\mathcal{H}})_y(\psi_y). \end{aligned}$$

Definition 33 *Given measurable spaces X and Y and matrix functors $T, T'\colon H^X \to H^Y$, a natural transformation $\alpha\colon T \Rightarrow T'$ of the above sort is called a* **matrix natural transformation***.*

However, in analogy to Thm. 13, we have:

Theorem 34 *Given measurable spaces X and Y and matrix functors $T, T'\colon H^X \to H^Y$, every bounded natural transformation $\alpha\colon T \Rightarrow T'$ is a matrix natural transformation, and conversely.*

Proof: The converse is easy. So, suppose $T, T'\colon H^X \to H^Y$ are matrix natural transformations and $\alpha\colon T \Rightarrow T'$ is a bounded natural transformation. Denote by t and t' the families of measures of the two matrix functors. We will show that α is a matrix natural transformation in three steps. We begin by assuming that for each $y \in Y$, $t_y = t'_y$; we then extend the result to the case where the measures are only equivalent $t_y \sim t'_y$; then finally we treat the general case.

Assume first $t = t'$. Let $\mathcal{J}$ be the measurable field of Hilbert spaces on X with

$$\mathcal{J}_x = \mathbb{C} \qquad \text{for all } x \in X.$$

Then $T\mathcal{J}$ and $T'\mathcal{J}$ are measurable fields of Hilbert spaces on Y with canonical isomorphisms

$$(T\mathcal{J})_y \cong \int_X^{\oplus} dt_y\; T_{x,y}, \qquad (T'\mathcal{J})_y \cong \int_X^{\oplus} dt_y\; T'_{x,y} \tag{39}$$

Using these, we may think of $\alpha_{\mathcal{J}}$ as a measurable field of operators on Y with

$$(\alpha_{\mathcal{J}})_y\colon \int_X^{\oplus} dt_y\; T_{x,y} \to \int_X^{\oplus} dt_y\; T'_{x,y}.$$

We now show that for any fixed $y \in Y$ there is a bounded measurable field of operators on X, say

$$\alpha_{y,x} \colon T_{x,y} \to T'_{x,y},$$

with the property that

$$(\alpha_{\mathcal{J}})_y \colon \int_X^{\oplus} dt_y \; \psi_{y,x} \mapsto \int_X^{\oplus} dt_y \; \alpha_{y,x}(\psi_{y,x}) \tag{40}$$

for any measurable field of vectors $\psi_{y,x} \in T_{y,x}$. For this, note that any measurable bounded function f on X defines a morphism

$$f \colon \mathcal{J} \to \mathcal{J}$$

in H^X, mapping a vector field ψ_x to $f(x)\psi_x$. The functors T and T' map f to the some morphisms

$$T_f \colon T\mathcal{J} \to T\mathcal{J} \qquad \text{and} \qquad T'_f \colon T'\mathcal{J} \to T'\mathcal{J}$$

in H^Y. Using the canonical isomorphisms (39), we may think of T_f as a measurable field of **multiplication operators** on Y with

$$\begin{array}{rccc} (T_f)_y \colon & \int_X^{\oplus} dt_y \; T_{y,x} & \to & \displaystyle\int_X^{\oplus} dt_y \; T_{y,x} \\ & \int_X^{\oplus} dt_y \; \psi_{y,x} & \mapsto & \int_X^{\oplus} dt_y \; f(x)\psi_{y,x} \end{array}$$

and similarly for T'_f. The naturality of α implies that the square

$$\begin{array}{ccc} T\mathcal{J} & \xrightarrow{T_f} & T\mathcal{J} \\ {\scriptstyle \alpha_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{J}}} \\ T'\mathcal{J} & \xrightarrow[T'_f]{} & T'\mathcal{J} \end{array}$$

commutes; unraveling this condition it follows that, for each $y \in Y$,

$$(\alpha_{\mathcal{J}})_y \; (T_f)_y = (T'_f)_y \; (\alpha_{\mathcal{J}})_y.$$

Now we use this result:

Lemma 35 *Suppose X is a measurable space and μ is a measure on X Suppose T and T' are measurable fields of Hilbert spaces on X and*

$$\alpha \colon \int_X^{\oplus} d\mu \, T_x \to \int_X^{\oplus} d\mu \, T'_x$$

is a bounded linear operator such that

$$\alpha \, T_f = T'_f \, \beta$$

for every $f \in L^\infty(X, \mu)$, where T_f and T'_f are multiplication operators as above. Then there exists a uniformly bounded measurable field of operators

$$\alpha_x \colon T_x \to T'_x$$

such that

$$\alpha\colon \int_X^{\oplus} d\mu\ \psi_x \mapsto \int_X^{\oplus} d\mu\ \alpha_x(\psi_x).$$

Proof: This can be found in Dixmier's book [29, Part II Chap. 2 Thm. 1]. □

It follows that for any $y \in Y$ there is a uniformly bounded measurable field of operators on X, say

$$\alpha_{y,x}\colon T_{x,y} \to T'_{x,y},$$

satisfying Eq. 40.

Next note that as we let y vary, $\alpha_{y,x}$ defines a uniformly bounded measurable field of operators on $X \times Y$. The uniform boundedness follows from the fact that for all y,

$$\operatorname*{ess\,sup}_x \|\alpha_{y,x}\| = \|(\alpha_{\mathcal{J}})_y\| \leq K$$

since α is a bounded natural transformation. The measurability follows from the fact that $(\alpha_J)_y$ is a measurable field of bounded operators on Y.

To conclude, we use this measurable field $\alpha_{y,x}$ to prove that α is a matrix natural transformation. For this, we must show that for *any* measurable field $\mathcal{H}$ of Hilbert spaces on X, we have

$$(\alpha_{\mathcal{H}})_y\colon \int_X^{\oplus} dt_y\ \psi_{y,x} \mapsto \int_X^{\oplus} dt_y\ [\alpha_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x})$$

To prove this, first we consider the case where $\mathcal{K}$ is a constant field of Hilbert spaces:

$$\mathcal{K}_x = K \qquad \text{for all } x \in X,$$

for some Hilbert space K of countably infinite dimension. We handle this case by choosing an orthonormal basis $e_j \in K$ and using this to define inclusions

$$i_j\colon \mathcal{J} \to \mathcal{K}, \qquad \psi_x \mapsto \psi_x e_j$$

The naturality of α implies that the square

$$\begin{array}{ccc} T\mathcal{J} & \xrightarrow{Ti_j} & T\mathcal{K} \\ \Big\downarrow{\scriptstyle \alpha_{\mathcal{J}}} & & \Big\downarrow{\scriptstyle \alpha_{\mathcal{K}}} \\ T'\mathcal{J} & \xrightarrow[T'i_j]{} & T'\mathcal{K} \end{array}$$

commutes; it follows that

$$(\alpha_{\mathcal{K}})_y\ (Ti_j)_y = (T'i_j)_y\ (\alpha_{\mathcal{J}})_y.$$

Since we already know $\alpha_{\mathcal{J}}$ is given by Eq. 40, writing any vector field in $\mathcal{K}$ in terms of the orthonormal basis e_j, we obtain that

$$(\alpha_{\mathcal{K}})_y\colon \int_X^{\oplus} dt_y\ \psi_{y,x} \mapsto \int_X^{\oplus} dt_y\ [\alpha_{y,x} \otimes \mathbb{1}_K](\psi_{y,x}) \tag{41}$$

Next, we use the fact that every measurable field $\mathcal{H}$ of Hilbert spaces is isomorphic to a direct summand of $\mathcal{K}$ [29, Part II, Chap. 1, Prop. 1]. So, we have a projection

$$p\colon \mathcal{K} \to \mathcal{H}.$$

The naturality of α implies that the square

$$\begin{array}{ccc} T\mathcal{K} & \xrightarrow{Tp} & T\mathcal{H} \\ \downarrow{\scriptstyle \alpha_{\mathcal{K}}} & & \downarrow{\scriptstyle \alpha_{\mathcal{H}}} \\ T'\mathcal{J} & \xrightarrow[T'p]{} & T'\mathcal{H} \end{array}$$

commutes; it follows that

$$(\alpha_{\mathcal{H}})_y \ (Tp)_y = (T'p)_y \ (\alpha_{\mathcal{K}})_y.$$

Since we already know $\alpha_{\mathcal{K}}$ is given by Eq. 41, using the fact that any vector field in $\mathcal{H}$ is the image by p of a vector field in $\mathcal{K}$, we obtain that

$$(\alpha_{\mathcal{H}})_y\colon \int_X^{\oplus} dt_y \ \psi_{y,x} \mapsto \int_X^{\oplus} dt_y \ [\alpha_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x})$$

We have assumed so far that the matrix functors T, T' are constructed from the same family of measures $t = t'$. Next, let us relax this hypothesis and suppose that for each $y \in Y$, we have $t_y \sim t'_y$. Let $\tilde{T}'$ be the matrix functor constructed from the family of measures t and the field of Hilbert space T'. The bounded measurable field of identity operators $\mathbb{1}_{T'_{y,x}}$ defines a matrix natural transformation

$$r_{t,t'}\colon T \Rightarrow \tilde{T}'.$$

This natural transformation assigns to any object $\mathcal{H} \in H^X$ a morphism $r_{t,t'\mathcal{H}}\colon T\mathcal{H} \to \tilde{T}'\mathcal{H}$ with components:

$$(r_{t,t'\mathcal{H}})_y\colon \int^{\oplus} dt'_y \psi_{y,x} \mapsto \int^{\oplus} dt_y \sqrt{\frac{dt'_y}{dt_y}} \psi_{y,x}$$

Moreover, by equivalence of the measures, $r_{t,t'}$ is a natural isomorphism and $r^{-1}_{t,t'} = r_{t',t}$.

Suppose $\alpha\colon T \Rightarrow T'$ is a bounded natural transformation. The composite $r_{t,t'}\alpha\colon T \to \tilde{T}'$ is a bounded natural transformation between matrix functors constructed from the same families of measures t. According to the result shown above, we know that this composite is a matrix measurable transformation, defined by some measurable field of operators

$$\alpha_{y,x}\colon T_{y,x} \to T'_{y,x}$$

Writing $\alpha = r_{t',t}(r_{t,t'}\alpha)$, we conclude that α acts on each object $\mathcal{H} \in H^X$ as

$$(\alpha_{\mathcal{H}})_y\colon \int_X^{\oplus} dt_y \ \psi_{y,x} \mapsto \int_X^{\oplus} dt_y \ [\tilde{\alpha}_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x})$$

where $\tilde{\alpha}$ is the rescaled field

$$\tilde{\alpha} = \sqrt{\frac{dt_y}{dt'_y}} \alpha$$

This shows that α is a matrix natural transformation.

Finally, to prove the theorem in its full generality, we consider the Lebesgue decomposition of the measures t_y and t'_y with respect to each other (see Appendix A.1):

$$t = t^{t'} + \overline{t^{t'}}, \qquad t^{t'} \ll t' \quad \overline{t^{t'}} \perp t'$$

and likewise,

$$t' = t'^{t} + \overline{t'^{t}}, \qquad t'^{t} \ll t \quad \overline{t'^{t}} \perp t$$

where the subscript y indexing the measures is dropped for clarity. Prop.107 shows that $t_y^{t'} \perp \overline{t_y^{t'}}$ and $t'^{t}_y \perp \overline{t'^{t}_y}$. Moreover, Prop.108 shows that $t_y^{t'} \sim t'^{t}_y$. Consequently, for each $y \in Y$, there are disjoint measurable sets A_y, B_y and B'_y such that $t_y^{t'}$ and t'^{t}_y are supported on A_y, that is,

$$t_y^{t'}(S) = t_y^{t'}(S \cap A_y) \qquad t'^{t}_y(S) = t'^{t}_y(S \cap A_y),$$

for all measurable sets S; and such that $\overline{t_y^{t'}}$ is supported on B_y, and $\overline{t'^{t}_y}$ is supported on B'_y.

Let $\tilde{T}$ be the matrix functor constructed from the family of measures $t^{t'}$ and the field of Hilbert spaces $T_{y,x}$; let $\tilde{T'}$ be the matrix functor constructed from the the family of measures t'^{t} and the field of Hilbert spaces $T'_{y,x}$. The bounded measurable field of identity operators $\mathbb{1}_{T_{y,x}}$ define matrix natural transformations:

$$i \colon \tilde{T} \Rightarrow T, \quad p \colon T \Rightarrow \tilde{T}$$

Given any object $\mathcal{H} \in H^X$, we get a morphism $i_{\mathcal{H}} \colon \tilde{T}\mathcal{H} \to T\mathcal{H}$, whose components act as inclusions:

$$(i_{\mathcal{H}})_y \colon \int^{\oplus} dt_y^{t'}\, \psi_{y,x} \mapsto \int^{\oplus} dt_y\, \chi_{A_y}(x)\, \psi_{y,x}$$

where χ_A is the characteristic function of the set $A \subset X$:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

We also get a morphism $p_{\mathcal{H}} \colon T\mathcal{H} \to \tilde{T}\mathcal{H}$, whose components act as projections:

$$(p_{\mathcal{H}})_y \colon \int^{\oplus} dt'_y\, \psi_{y,x} \mapsto \int^{\oplus} dt_y'^{t}\, \psi_{y,x}.$$

Likewise, the bounded measurable field of identity operators $\mathbb{1}_{T'_{y,x}}$ define an inclusion and a projection:

$$i' \colon \tilde{T'} \Rightarrow T', \quad p' \colon T' \Rightarrow \tilde{T'}$$

Suppose $\alpha \colon T \Rightarrow T'$ is a bounded natural transformation. The composite $p'\alpha i \colon \tilde{T} \Rightarrow \tilde{T'}$ is then a bounded natural transformation between matrix functors constructed from equivalent families of measures. According to the result shown above, we know that this composite is a matrix natural tranformation, defined by some measurable field of operators

$$\alpha_{y,x} \colon T_{y,x} \to T'_{y,x}$$

We will show below the equality of natural transformations:

$$\alpha = i'[p'\alpha i]p \tag{42}$$

This equality leads to our final result. Indeed, for any $\mathcal{H} \in H^X$ and each $y \in Y$, it yields:

$$(\alpha_{\mathcal{H}})_y \colon \int^{\oplus} dt_y \, \psi_{y,x} \mapsto \int^{\oplus} dt'_y \, \chi_{A_y}(x) \sqrt{\frac{dt_y^{t'}}{dt_y'^{t}}} \, [\alpha_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x})$$

and we conclude using the fact that, for all y and t'_y-almost all x,

$$\chi_{A_y}(x) \sqrt{\frac{dt_y^{t'}}{dt_y'^{t}}} = \sqrt{\frac{dt_y^{t'}}{dt'_y}}.$$

The equality (42) follows from naturality of α. In fact, naturality implies that, for any morphism $\phi \colon \mathcal{H} \to \mathcal{H}$, the square

$$\begin{array}{ccc} T\mathcal{H} & \xrightarrow{T\phi} & T\mathcal{H} \\ {\scriptstyle \alpha_{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{H}}} \\ T'\mathcal{H} & \xrightarrow[T'\phi]{} & T'\mathcal{H} \end{array}$$

commutes. It follows that, for each $y \in Y$,

$$(\alpha_{\mathcal{H}})_y (T\phi)_y = (T'\phi)_y (\alpha_{\mathcal{H}})_y$$

Let us fix $y \in Y$. We apply naturality to the morphism

$$\chi_{B_y} \colon H \to H$$

mapping any vector field ψ_x to the vector field $\chi_{B_y}(x)\psi_x$. Its image by the functor T' defines a projection operator

$$(T'\chi_{B_y})_y \equiv T'_{B_y} = \int^{\oplus}_{B_y} dt'_y \, \mathbb{1}_{T'_{y,x}} \otimes \mathbb{1}_{\mathcal{H}_x}$$

Since B_y is a t'_y-null set, this operator acts trivially on $T'\mathcal{H}$. It then follows from naturality that

$$(\alpha_{\mathcal{H}})_y T_{B_y} = T'_{B_y} (\alpha_{\mathcal{H}})_y = 0. \tag{43}$$

Likewise, applying naturality to the morphism

$$\chi_{B'_y} \colon H \to H$$

leads to

$$0 = (\alpha_{\mathcal{H}})_y T_{B'_y} = T'_{B'_y} (\alpha_{\mathcal{H}})_y \tag{44}$$

We now use the following decompositions of the identities operators on the Hilbert spaces $(T\mathcal{H})_y$ and $(T'\mathcal{H})_y$ into direct sums of projections:

$$\mathbb{1}_{(T\mathcal{H})_y} = T_{A_y} \oplus T_{B_y}, \qquad \mathbb{1}_{(T'\mathcal{H})_y} = T'_{A_y} \oplus T'_{B'_y}$$

to write:

$$(\alpha_{\mathcal{H}})_y = [T'_{A_y} \oplus T'_{B'_y}] (\alpha_{\mathcal{H}})_y [T_{A_y} \oplus T_{B_y}]$$

Together with (43) and (44), it yields:

$$(\alpha_{\mathcal{H}})_y = T'_{A_y}(\alpha_{\mathcal{H}})_y T_{A_y}$$

To conclude, observe that

$$T_{A_y} = (ip_{\mathcal{H}})_y, \qquad T'_{A_y} = (i'p'_{\mathcal{H}})_y$$

We finally obtain:

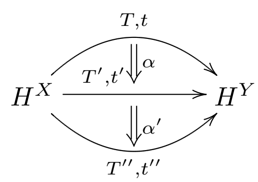
$$(\alpha_{\mathcal{H}})_y = (i'p'_{\mathcal{H}})_y(\alpha_{\mathcal{H}})_y(ip_{\mathcal{H}})_y$$

which shows our equality (42). This completes the proof of the theorem. ∎

This allows an easy definition for the 2-morphisms in **Meas**:

Definition 36 *A* **measurable natural transformation** *is a bounded natural transformation between measurable functors.*

For our work it will be useful to have explicit formulas for composition of matrix natural transformations. So, let us compute the vertical composite of two matrix natural transformations α and α':



For any object $\mathcal{H} \in H^X$, we get morphisms $\alpha_{\mathcal{H}}$ and $\alpha'_{\mathcal{H}}$ in H^Y. Their composite is easy to calculate:

$$\begin{array}{rccc} (\alpha_{\mathcal{H}'})(\alpha_{\mathcal{H}})_y \colon & \displaystyle\int_X^{\oplus} dt_y \; T_{y,x} \otimes \mathcal{H}_x & \rightarrow & \displaystyle\int_X^{\oplus} dt''_y \; T''_{y,x} \otimes \mathcal{H}_x \\ & \int_X^{\oplus} dt_y \; \psi_{y,x} & \mapsto & \int_X^{\oplus} dt''_y \; [(\tilde{\alpha}'_{y,x}\tilde{\alpha}_{y,x}) \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{y,x}) \end{array}$$

So, the composite is a measurable natural transformation $\alpha' \cdot \alpha$ with:

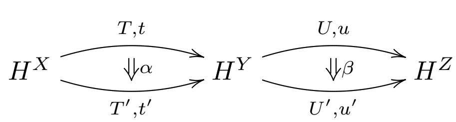
$$(\widetilde{\alpha' \cdot \alpha})_{y,x} = \tilde{\alpha}'_{y,x}\tilde{\alpha}_{y,x}. \tag{45}$$

For some calculations it will be useful to have this equation written explicitly in terms of the original fields α and α', rather than their rescalings:

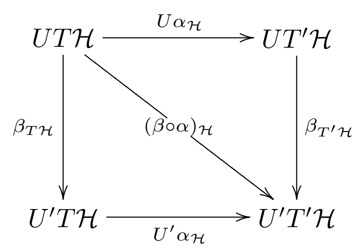
$$(\alpha' \cdot \alpha)_{y,x} = \sqrt{\frac{dt''_y}{dt_y}} \sqrt{\frac{dt'_y}{dt''_y}} \sqrt{\frac{dt_y}{dt'_y}} \; \alpha'_{y,x}\alpha_{y,x} \tag{46}$$

This equality defines the composite field almost everywhere for the geometric mean measure $\sqrt{t_y t''_y}$.

Next, let us compute the horizontal composite of two matrix natural transformations:



Recall that the horizontal composite $\beta \circ \alpha$ is defined so that



commutes. Let us pick an element $\psi \in UT\mathcal{H}$, which can be written in the form

$$\psi_z = \int_X^{\oplus} d(ut)_z \, \psi_{z,x}, \quad \text{with} \quad \psi_{z,x} = \int_Y^{\oplus} dk_{z,x} \, \psi_{z,y,x}$$

by definition of the composite field UT. Note that, thanks to Eq. (37) which defines the family of measures $k_{z,x}$, the section ψ_z can also be written as

$$\psi_z = \int_Y^{\oplus} du_z \, \psi_{z,y}, \quad \text{with} \quad \psi_{z,y} = \int_X^{\oplus} dt_y \, \psi_{z,y,x}$$

Having introduced all these notations, we now evaluate the image of ψ under the morphism $(\beta \circ \alpha)_{\mathcal{H}}$:

$$\begin{aligned}
((\beta \circ \alpha)_{\mathcal{H}})_z(\psi_z) &= (U'\alpha_{\mathcal{H}})_z \circ (\beta_{T\mathcal{H}})_z(\psi_z) \\
&= \left(\int_Y^{\oplus} du'_z \, \mathbb{1}_{U'_{z,y}} \otimes (\alpha_{\mathcal{H}})_y \right) \left(\int_Y^{\oplus} du'_z \, [\tilde{\beta}_{z,y} \otimes \mathbb{1}_{(T\mathcal{H})_y}](\psi_{z,y}) \right) \\
&= \int_Y^{\oplus} du'_z \, [\tilde{\beta}_{z,y} \otimes (\alpha_{\mathcal{H}})_y](\psi_{z,y}) \\
&= \int_Y^{\oplus} du'_z \int_X^{\oplus} dt'_y \, [\tilde{\beta}_{z,y} \otimes \tilde{\alpha}_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{z,y,x})
\end{aligned}$$

Applying the disintegration theorem, we can rewrite this last direct integral as an integral over X with respect to the measure

$$(u't')_z = \int_Y du'_z(y) \, t'_y$$

We obtain

$$\begin{aligned}
((\beta \circ \alpha)_{\mathcal{H}})_z(\psi_z) &= \int_X^{\oplus} d(u't')_z \int_Y^{\oplus} dk'_{z,x} \, [\tilde{\beta}_{z,y} \otimes \tilde{\alpha}_{y,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{z,y,x}) \\
&= \int_X^{\oplus} d(u't')_z [(\widetilde{\beta \circ \alpha})_{z,x} \otimes \mathbb{1}_{\mathcal{H}_x}](\psi_{z,x})
\end{aligned}$$

where

$$(\widetilde{\beta \circ \alpha})_{z,x}(\psi_{z,x}) = \int_Y^{\oplus} dk'_{z,x} \, [\tilde{\beta}_{z,y} \otimes \tilde{\alpha}_{y,x}](\psi_{z,y,x}). \tag{47}$$

Equivalently, in terms of the original fields α and β:

$$(\beta \circ \alpha)_{z,x}(\psi_{z,x}) = \sqrt{\frac{d(u't')_z}{d(ut)_z}} \int_Y^{\oplus} dk'_{z,x} \, [\sqrt{\frac{du_z}{du'_z}} \sqrt{\frac{dt_y}{dt'_y}} \beta_{z,y} \otimes \alpha_{y,x}](\psi_{z,y,x}) \tag{48}$$

A special case is worth mentioning. When the source and target morphisms of α and β coincide, we have $k = k'$, and the horizontal composition formula above simply says $(\beta \circ \alpha)_{z,x}$ is a direct integral of the fields of operators $\beta_{z,y} \otimes \alpha_{y,x}$.

Besides composition of 2-morphisms in **Meas** we also need identity 2-morphsms. Given a matrix functor $T\colon H^X \to H^Y$, its identity 2-morphism $\mathbb{1}_T\colon T \Rightarrow T$ is, up t-*a.e.*–equivalence, given by the field of identity operators:

$$(\mathbb{1}_T)_{y,x} = \mathbb{1}_{T_{y,x}}\colon T_{y,x} \longrightarrow T_{y,x}.$$

This acts as an identity for the vertical composition; the identity 2-morphism of an identity morphism, $\mathbb{1}_{\mathbb{1}_X}$, acts as an identity for horizontal composition as well.

In calculations, it is often convenient to be able to describe a 2-morphism either by α or its rescaling $\tilde{\alpha}$. The relationship between these two descriptions is given by the following:

Lemma 37 *The fields $\alpha_{y,x}$ and $\alpha'_{y,x}$ are $\sqrt{t_y t'_y}$-equivalent if and only if their rescalings $\tilde{\alpha}_{y,x}$ and $\tilde{\alpha}'_{y,x}$ are t'_y-equivalent.*

Proof: For each y, let A_y and $\tilde{A}_y$ be the subsets of X on which $\alpha \neq \alpha'$, and $\tilde{\alpha} \neq \tilde{\alpha}'$, respectively. Observe that $\tilde{A}_y$ is the intersection of A_y with the set of x for which the rescaling factor is non-zero:

$$\tilde{A}_y = A_y \cap \left\{ x : \sqrt{\tfrac{dt_y}{dt'_y}}(x) \neq 0 \right\}.$$

Supposing first that $\alpha_{y,x}$ and $\alpha'_{y,x}$ are $\sqrt{t_y t'_y}$-equivalent, we have $\sqrt{t_y t'_y}\,(A_y) = 0$, so by the definition of the geometric mean measure

$$\sqrt{t_y t'_y}\,(A_y) = \int_{A_y} dt'_y \sqrt{\frac{dt_y}{dt'_y}} = 0.$$

Thus the rescaling factor vanishes for t'_y-almost every $x \in A_y$; that is, $\tilde{A}_y$ has t'_y-measure zero. Conversely, if $t'_y(\tilde{A}_y) = 0$, we have:

$$\sqrt{t_y t'_y}\,(A_y) = \sqrt{t_y t'_y}\,(\tilde{A}_y) + \sqrt{t_y t'_y}\,(A_y - \tilde{A}_y).$$

The first term on the right vanishes because $\sqrt{t_y t'_y} \ll t'_y$, while the second vanishes since $\sqrt{\frac{dt_y}{dt'_y}} = 0$ on $A_y - \tilde{A}_y$. So, the rescaling $\alpha \mapsto \tilde{\alpha}$ induces a one-to-one correspondence between $\sqrt{tt'}$-classes of fields α and t'-classes of rescaled fields $\tilde{\alpha}$. ■

3.3.3 Construction of Meas as a 2-category

Theorem 38 *There is a sub-2-category* **Meas** *of* **Cat** *where the objects are measurable categories, the morphisms are measurable functors, and the 2-morphisms are measurable natural transformations.*

In Section 3.3.2 we showed that for any measurable space X, the identity $\mathbb{1}_X\colon H^X \to H^X$ is a matrix functor. It follows that the identity on any measurable category is a measurable functor. Similarly, in Section 3.3.2 we showed that for any matrix functor T, the identity $1_T\colon T \Rightarrow T$ is a matrix natural transformation. This implies that the identity on any measurable functor is a measurable natural transformation. To prove that the composite of measurable functors is measurable, we will use the sequence of lemmas below. Since measurable natural transformations are just bounded natural transformations between measurable functors, by Thm. 34, it will then easily follow that measurable natural transformations are closed under vertical and horizontal composition.

Lemma 39 *A composite of matrix functors is boundedly naturally isomorphic to a matrix functor.*

Proof: This was proved by Yetter [73, Thm. 45], and we have sketched his argument in Section 3.3.2. Yetter did not emphasize that the natural isomorphism is bounded, but one can see from equation (34) that it is. ■

Lemma 40 *If $F\colon H^X \to H^Y$ is a C^*-equivalence, then there is a measurable bijection between X and Y, and F is a measurable functor.*

Proof: This was proved by Yetter [73, Thm. 40]. In fact, Yetter failed to require that F be linear on morphisms, which is necessary for this result. Careful examination of his proof shows that it can be repaired if we include this extra condition, which holds automatically for a C^*-equivalence. ■

Lemma 41 *If $T\colon \mathsf{H} \to \mathsf{H}'$ is a measurable functor and $F\colon \mathsf{H} \to H^X$, $G\colon H^Y \to \mathsf{H}'$ are* arbitrary *C^*-equivalences, then T is naturally isomorphic to the composite*

$$\mathsf{H} \xrightarrow{F} H^X \xrightarrow{\tilde{T}} H^Y \xrightarrow{G} \mathsf{H}'$$

for some matrix functor $\tilde{T}$.

Proof: The proof is analogous to the proof of Lemma 11. Since T is measurable we know there exist C^*-equivalences $F'\colon \mathsf{H} \to H^{X'}$, $G'\colon H^{Y'} \to \mathsf{H}'$ such that T is boundedly naturally isomorphic to the composite

$$\mathsf{H} \xrightarrow{F'} H^{X'} \xrightarrow{\tilde{T}'} H^{Y'} \xrightarrow{G'} \mathsf{H}'$$

for some matrix functor $\tilde{T}'$. By Lemma 40 we may assume $X' = X$ and $Y' = Y$. So, let $\tilde{T}$ be the composite

$$H^X \xrightarrow{\bar{F}} \mathsf{H} \xrightarrow{F'} H^X \xrightarrow{\tilde{T}'} H^Y \xrightarrow{G'} \mathsf{H}' \xrightarrow{\bar{G}} H^Y$$

where the weak inverses $\bar{F}$ and $\bar{G}$ are chosen using the fact that F and G are C^*-equivalences. Since $F'\bar{F}\colon H^X \to H^X$ and $\bar{G}G'\colon H^Y \to H^Y$ are C^*-equivalences, they are matrix functors by Lemma 40. It follows that $\tilde{T}$ is a composite of three matrix functors, hence boundedly naturally isomorphic to a matrix functor by Lemma 39. Moreover, the composite

$$\mathsf{H} \xrightarrow{F} H^X \xrightarrow{\tilde{T}} H^Y \xrightarrow{G} \mathsf{H}'$$

is boundedly naturally isomorphic to T. Since F and G are C^*-equivalences and $\tilde{T}$ is boundedly naturally isomorphic to a matrix functor, it follows that T is a measurable functor. ■

Lemma 42 *A composite of measurable functors is measurable.*

Proof: The proof is analogous to the proof of Lemma 12. Suppose we have a composable pair of measurable functors $T\colon \mathsf{H} \to \mathsf{H}'$ and $U\colon \mathsf{H}' \to \mathsf{H}''$. By definition, T is boundedly naturally isomorphic to a composite

$$\mathsf{H} \xrightarrow{F} H^X \xrightarrow{\tilde{T}} H^Y \xrightarrow{G} \mathsf{H}'$$

where $\tilde{T}$ is a matrix functor and F and G are C^*-equivalences. By Lemma 41, U is naturally isomorphic to a composite

$$\mathsf{H}' \xrightarrow{\bar{G}} H^Y \xrightarrow{\tilde{U}} H^X \xrightarrow{H} \mathsf{H}''$$

where $\tilde{U}$ is a matrix functor, $\bar{G}$ is the chosen weak inverse for G, and H is a C^*-equivalence. The composite UT is thus boundedly naturally isomorphic to

$$\mathsf{H} \xrightarrow{F} H^X \xrightarrow{\tilde{U}\tilde{T}} H^Z \xrightarrow{H} \mathsf{H}''$$

Since $\tilde{U}\tilde{T}$ is a matrix functor by Lemma 39, it follows that UT is a measurable functor. ∎

4 Representations on measurable categories

We have reviewed, in Section 2, an abstract framework for studying representations of 2-groups in an arbitrary target 2-category. In Section 3 we have given an explicit construction of the 2-category **Meas** of measurable categories. So, our task in the rest of this work is to see what the abstract theory amounts to concretely when we use **Meas** as our target 2-category.

We begin with a summary.

4.1 Main results

We saw a preview of our main results in the Introduction, where we described the following geometric picture of the representation theory of a skeletal 2-group:

representation theory of a skeletal 2-group $\mathcal{G} = (G, H, \rhd)$	geometry
a representation of $\mathcal{G}$ on H^X	a right action of G on X, and a map $X \to H^*$ making X a 'measurable G-equivariant bundle' over H^*
an intertwiner between representations on H^X and H^Y	a 'Hilbert G-bundle' over the pullback of G-equivariant bundles and a 'G-equivariant measurable family of measures' μ_y on X
a 2-intertwiner	a map of Hilbert G-bundles

We are now in a position to explain this correspondence between 2-group reprsentations and geometry in more detail.

Representations

Consider a representation $\rho\colon \mathcal{G} \to$ **Meas** on a measurable category H^X. An essential step in understanding such a representation is understanding what the measurable automorphisms of the category H^X look like. In Section 4.2, we show that any automorphism of H^X is 2-isomorphic to one induced by pullback along some measurable automorphism $f\colon X \to X$. Such an automorphism, which we denote $H^f\colon H^X \to H^X$, acts on fields of Hilbert spaces and linear maps on X simply by pulling them back along f.

In Thm. 49, we show that if ρ is a representation on H^X such that for each $g \in G$, $\rho(g) = H^{f_g}$ for some f_g, then ρ is determined by two pieces of geometric data we can extract from it:

- a right action $\lhd$ of G as measurable transformations of the measurable space X,
- a map $\chi\colon X \to H^*$ that is G-equivariant, i.e.:

$$\chi(x \lhd g) = \chi(x)_g \tag{49}$$

  for all $x \in X$ and $g \in G$.

Roughly, these data can be characterized geometrically by saying that the map $\chi\colon X \to H^*$ is a **equivariant fiber bundle** over the character group $H^* = \hom(H, \mathbb{C}^\times)$:

$$\begin{array}{c} X \quad . \\ \Big\downarrow{\scriptstyle \chi} \\ H^* \end{array}$$

This rough statement becomes precisely true—in the measurable category—when we impose some extra conditions. In particular, we want all of the spaces involved to be appropriate sorts of measurable spaces, and we want all relevant maps, including the maps defining group actions, to be measurable maps. We thus ultimately build these requirements into our definitions of *measurable 2-group* (see Def. 52) and *measurable representation* (see Def. 53). In the rest of this summary of results we consider only measurable representations of measurable 2-groups.

We show in Thm. 70 that two measurable representations are 'measurably equivalent' if and only if the corresponding equivariant bundles are isomorphic. Two representations on H^X are equivalent, by definition, if they are related by a pair of intertwiners that are weak inverses of each other, and they are 'measurably equivalent' if these intertwiners are 'measurable' in a suitable sense. We discuss general measurable intertwiners and their geometry below; for now it suffices to know that *invertible* measurable intertwiners between measurable representations correspond to invertible measurable bundle maps:

$$\begin{array}{ccc} X & \xrightarrow{\sim} & Y \\ & {\scriptstyle \chi_1}\searrow \quad \swarrow{\scriptstyle \chi_2} & \\ & H^* & \end{array}$$

So, equivalence of representations corresponds geometrically to isomorphism of bundles.

We say that a representation is 'indecomposable' if it is not equivalent to a '2-sum' of nontrivial representations, where a '2-sum' is a categorified version of the direct sum of ordinary group representations. We say a representation is 'irreducible' if, roughly speaking, it does not contain any subrepresentations other than itself and the trivial representation. Irreducible representations are automatically indecomposable, but not necessarily vice versa. An (*a priori*) intermediate notion is that of an 'irretractable representation'—a representation ρ such that if any composite of intertwiners of the form

$$\rho' \longrightarrow \rho \longrightarrow \rho'$$

is equivalent to the identity intertwiner on ρ', then ρ' is either trivial or equivalent to ρ. While for ordinary group representations irretractable representations are the same as indecomposable ones, this is not true for 2-group representations in **Meas**. We thus classify both the irretractable and indecomposable 2-group representations in **Meas**. The irreducible ones remain more challenging: in particular, we do not know if every irretractable representation is irreducible.

In Thm. 85 we show that a measurable representation of $\mathcal{G}$ on H^X is indecomposable if and only if G acts transitively on X. The study of indecomposable representations, and hence irreducible and irretractable representations as special cases, is thus rooted in Klein's geometry of homogeneous spaces. Recall that for any point $x^o \in X$, the **stabilizer** of x^o is the subgroup $S \subseteq G$ consisting of group elements g with $x^o \lhd g = x^o$. By a standard argument, we have

$$X \cong G/S.$$

Then, let $\chi^o = \chi(x^o)$. By equation (49), the image of $\chi\colon X \to H^*$ is a single G-orbit in H^*, and S is contained in the stabilizer S^* of χ^o. This shows that an indecomposable representation essentially amounts to an equivariant map of homogeneous spaces $\chi\colon G/S \to G/S^*$, where S^* is the stabilizer of some point in H^*, and $S \subseteq S^*$. In other words, indecomposable representations are classified up to equivalence by a choice of G-orbit in H^*, along with a subgroup S of the stabilizer of a point χ^o in the orbit.

In Thm. 87, we show an indecomposable representation ρ is irretractable if and only if S is *equal* to the stabilizer of χ^o; irretractable representations are thus classified up to equivalence by G-orbits in H^*.

Intertwiners

Next we turn to the main results concerning intertwiners. To state these, we first need some concepts from measure theory. Let X be a measurable space. Recall that two measures μ and ν on X are **equivalent**, or in the same **measure class**, if they have the same null sets. Next, suppose G acts on X as measurable transformations. Given a measure μ on X, for each g we define the 'transformed' measure μ^g by setting

$$\mu^g(A) := \mu(A \lhd g^{-1}). \tag{50}$$

The measure is **invariant** if $\mu^g = \mu$ for every g. If μ^g and μ are only equivalent, we say that μ is **quasi-invariant**. It is well-known that if G is a separable, locally compact topological group, acting measurably and transitively on X, then there exist nontrivial quasi-invariant measures on X, and moreover, all such measures belong to the same measure class (see Appendix A.4 for further details).

Next, let X and Y be two G-spaces. We may consider Y-indexed families μ_y of measures on X. Such a family is **equivariant**[1] under the action of G if for all g, $\mu_{y\lhd g}$ is *equivalent* to μ_y^g.

With these definitions we can now give a concrete description of intertwiners. Suppose ρ_1 and ρ_2 are measurable representations of a skeletal 2-group $\mathcal{G}$ on measurable categories H^X and H^Y, respectively, with corresponding equivariant bundles χ_1 and χ_2:

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle\chi_1}\searrow \quad \swarrow{\scriptstyle\chi_2} & \\ & H^* & \end{array}$$

Then an intertwiner $\phi\colon \rho_1 \to \rho_2$ is specified, up to equivalence, by:

- an equivariant Y-indexed family of measures μ_y on X, with each μ_y supported on $\chi_1^{-1}(\chi_2(y))$.
- an assignment, for each $g \in G$ and all y, of a μ_y-class of Hilbert spaces $\phi_{y,x}$ and linear maps

  $$\Phi^g_{y,x}\colon \phi_{y,x} \to \phi_{(y,x)\lhd g^{-1}}$$

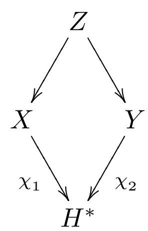
  satisfying the cocycle conditions

  $$\Phi^{g'g}_{y,x} = \Phi^{g'}_{(y,x)\lhd g^{-1}} \Phi^g_{y,x} \quad \text{and} \quad \Phi^1_{y,x} = 1_{\phi_{y,x}}$$

  μ-*a.e.* for each pair $g, g' \in G$, where $(y,x) \lhd g$ is short for $(y \lhd g, x \lhd g)$.

[1] Since we do not require *equality*, a more descriptive term would be 'quasi-equivariance'; we stick to 'equivariance' for simplicity.

There is a more geometric way to think of these intertwiners. First, the condition that each measure μ_y be supported on $\chi_1^{-1}(\chi_2(y))$ can be viewed as effectively restricting the field of Hilbert spaces $\phi_{y,x}$ on $X \times Y$ to the pullback of χ_1 and χ_2:



$$Z = \{(y, x) \in Y \times X : \chi_2(y) = \chi_1(x)\}$$

Indeed, the condition means precisely that, for each y, the measure $\delta_y \otimes \mu_y$ is supported on $Z \subseteq X \times Y$. Since the Hilbert spaces $\phi_{y,x}$ are really only defined up to $(\delta_y \otimes \mu_y)$-*a.e.* equivalence, we only care about the Hilbert spaces over Z.

Next, assume that, among the measure class of fields of linear operators

$$\Phi^g_{y,x} \colon \phi_{y,x} \to \phi_{(y,x)\lhd g^{-1}}$$

we may choose a representative such that the cocycle conditions hold everywhere in $Y \times X$ and for all $g \in G$. In fact, an intertwiner that does not satisfy some such condition seems quite ill-behaved, and we thus ultimately build such a condition into our definition of a *measurable* intertwiner (see Def. 60). Given this condition, we can think of the union of all the Hilbert spaces:

$$\phi = \coprod_{(y,x)} \phi_{y,x}$$

as a bundle of Hilbert spaces over the product space $Y \times X$, and then the group G acts on both the total space and the base space of this bundle. Indeed, the maps $\Phi^g_{y,x}$ give a map $\Phi^g \colon \phi \to \phi$; the cocycle conditions then become

$$\Phi^{g'g} = \Phi^{g'}\Phi^g \quad \text{and} \quad \Phi^1 = 1_\phi$$

which are simply the conditions that $\phi \mapsto \Phi^g\phi$ define a left action of G on ϕ. If we turn this into a right action by defining

$$\phi^g = \Phi^{g^{-1}}(\phi)$$

we find that the bundle map is equivariant with respect to this action of G on ϕ and the diagonal action of G on $Y \times X$, or rather on Z.

Putting all of this together, we conclude that a measurable intertwiner $\phi \colon \rho_1 \to \rho_2$ amounts to an equivariant family of measures μ_y and a μ-class of G-equivariant bundles of Hilbert spaces over the pullback $Z \subseteq Y \times X$.

As with representations, we introduce and discuss the notions of reducibility, retractability, and decomposability for intertwiners.

2-Intertwiners

Finally, the main results concerning the 2-intertwiners are as follows. Consider a pair of representations ρ_1 and ρ_2 of the skeletal 2-group $\mathcal{G}$ on the measurable categories H^X and H^Y, and two intertwiners $\phi, \psi \colon \rho_1 \Rightarrow \rho_2$. Suppose $\phi = (\mu, \phi, \Phi)$ and $\psi = (\nu, \psi, \Psi)$. For any y, we denote by $\sqrt{\mu_y \nu_y}$ the geometric mean of the measures μ_y and ν_y. A 2-intertwiner turns out to consist of:

- an assignment, for each y, of a $\sqrt{\mu_y \nu_y}$-class of linear maps $m_{y,x} \colon \phi_{y,x} \to \psi_{y,x}$, which satisfies the intertwining rule

$$\Psi^g_{y,x}\, m_{y,x} = m_{(y,x)g^{-1}}\, \Phi^g_{y,x}$$

$\sqrt{\mu\nu}$*a.e.*

In the geometric picture of intertwiners as equivariant bundles of Hilbert spaces, this characterization of a 2-intertwiner simply amounts to a **morphism of equivariant bundles**, up to almost-everywhere equality.

The intertwiners satisfy an analogue of *Schur's lemma*. Namely, in Prop. 105 we show that under some mild technical conditions, any 2-intertwiner between irreducible intertwiners is either null or an isomorphism.

4.2 Invertible morphisms and 2-morphisms in Meas

A 2-group representation ρ gives *invertible* morphisms $\rho(g)$ and *invertible* 2-morphisms $\rho(g,h)$ in the target 2-category. To understand 2-group representations in **Meas**, it is thus a useful preliminary step to characterize invertible measurable functors and invertible measurable natural transformations. We address these in this section, beginning with the 2-morphisms.

Consider two parallel measurable functors T and T'. A measurable natural transformation $\alpha \colon T \Rightarrow T'$ is **invertible** if it has a vertical inverse, namely a measurable natural transformation $\alpha' \colon T' \Rightarrow T$ such that $\alpha' \cdot \alpha = \mathbb{1}_T$ and $\alpha \cdot \alpha' = \mathbb{1}_{T'}$. We often call the invertible 2-morphism α in **Meas** a **2-isomorphism**, for short; we also say T and T' are **2-isomorphic**. The following theorem classifies 2-isomorphisms in the case where T and T' are matrix functors.

Theorem 43 *Let $(T,t),(T',t') \colon H^X \to H^Y$ be matrix functors. Then (T,t) and (T',t') are boundedly naturally isomorphic if and only if the measures t_y and t'_y are equivalent, for every y, and there is a measurable field of bounded linear operators $\alpha_{y,x} \colon T_{y,x} \to T'_{y,x}$ such that $\alpha_{y,x}$ is an isomorphism for each y and t_y-a.e. in x. In this case, there is one 2-isomorphism $T \Rightarrow T'$ for each t-class of fields $\alpha_{y,x}$.*

Proof: Suppose $\alpha \colon T \Rightarrow T'$ is a bounded natural isomorphism, with inverse $\alpha' \colon T' \Rightarrow T$. By Lemma 35, α and α' are both matrix natural transformations, hence defined by fields of bounded linear operators $\alpha_{y,x}$ and $\alpha'_{y,x}$ on $Y \times X$. By the composition formula (46), the composite $\alpha' \cdot \alpha = \mathbb{1}_T$ is given by

$$(\alpha' \cdot \alpha)_{y,x} = \sqrt{\frac{dt'_y}{dt_y}}\sqrt{\frac{dt_y}{dt'_y}}\; \alpha'_{y,x}\alpha_{y,x} = \mathbb{1}_{T_{y,x}} \qquad t_y\text{-}a.e.$$

We know by the chain rule (29) that the product of Radon-Nikodym derivatives in this formula equals one $\sqrt{t_y t'_y}$-*a.e.*, but not yet that it equals one t_y-*a.e.* However, by definition of the morphism (T,t), the Hilbert spaces $T_{y,x}$ are non-trivial t_y-*a.e.*; hence $\mathbb{1}_{T_{y,x}} \neq 0$. This shows that the product of Radon-Nikodym derivatives above is t_y-*a.e.* nonzero; in particular,

$$\frac{dt'^{t}_y}{dt_y}(x) \neq 0 \qquad t_y\text{-}a.e.$$

where ${t'}^t_y$ denotes the absolutely continuous part of t'_y in its Lebesgue decomposition $t'_y = t'^t_y + \overline{t'^t_y}$ with respect to t_y. But this property is equivalent to the statement that the measure t_y is absolutely continuous with respect to t'_y. To check this, pick a measurable set A and write

$$t'_y(A) = \int_A dt_y(x)\frac{dt'^t_y}{dt_y}(x) + \overline{t'^t_y}(A)$$

Now if $t'_y(A) = 0$, both terms of the right-hand-side of this equality vanish—in particular the integral term. But since the Radon-Nikodym derivative is a strictly positive function t_y-*a.e.*, this requires the t_y-measure of A to be zero. So we have shown that $t'_y(A) = 0$ implies $t_y(A) = 0$ for any measurable set A, i.e. $t_y \ll t'_y$. Starting with $\alpha \cdot \alpha' = \mathbb{1}_{T'}$, the same analysis leads to the conclusion $t_y \ll t'_y$. Hence the two measures are equivalent. From this it is immediate that

$$(\alpha' \cdot \alpha)_{y,x} = \alpha'_{y,x}\alpha_{y,x} = \mathbb{1}_{T_{y,x}} \qquad t_y\text{-}a.e.$$

and thus $\alpha'_{y,x} = \alpha^{-1}_{y,x}$. In particular, the operators $\alpha_{y,x}$ are invertible t_y-*a.e.*

Conversely, suppose the measures t_y and t'_y are equivalent and we are given a measurable field $\alpha\colon T \to T'$ such that for all y, the operators $\alpha_{y,x}$ are invertible for almost every x. It is easy to check, using the formula for vertical composition, that the matrix natural transformation defined by $\alpha_{y,x}$ has an inverse defined by $\alpha^{-1}_{y,x}$. ∎

A morphism $T\colon H^X \to H^Y$ is **strictly invertible** if it has a strict inverse, namely a 2-morphism $U\colon H^Y \to H^X$ such that $UT = \mathbb{1}_X$ and $TU = \mathbb{1}_Y$. In 2-category theory, however, it is more natural to weaken the notion of invertibility, so these equations hold only *up to 2-isomorphism.* In this case we say that T is **weakly invertible** or an **equivalence**.

We shall give two related characterizations of weakly invertible morphisms in **Meas**. For the first one, recall that if $f\colon Y \to X$ is a measurable function, then any measure μ on Y pushes forward to a measure $f_*\mu$ on X, by

$$f_*\mu(A) = \mu(f^{-1}A)$$

for each measurable set $A \subseteq X$. In the case where $\mu = \delta_y$, we have

$$f_*\delta_y = \delta_{f(y)}$$

Denoting by δ the Y-indexed family of measures $y \mapsto \delta_y$ on Y, the following theorem shows that every invertible matrix functor $T\colon H^X \to H^Y$ is essentially $(\mathbb{C}, f_*\delta)$ for some invertible measurable map $f\colon Y \to X$.

As shown by the following theorem, the condition for a morphism to be an equivalence is very restrictive [73]:

Theorem 44 *A matrix functor $(T,t)\colon H^X \to H^Y$ is a measurable equivalence if and only if there is an invertible measurable function $f\colon Y \to X$ between the underlying spaces such that, for all y, the measure t_y is equivalent to $\delta_{f(y)}$, and a measurable field of linear operators from $T_{y,x}$ to the constant field $\mathbb{C}$ that is t_y-a.e. invertible.*

Proof: If (T,t) is an equivalence, it has weak inverse that is also a matrix functor, say (U,u). The composite UT is 2-isomorphic to the identity morphism $\mathbb{1}_X$, and TU is 2-isomorphic to $\mathbb{1}_Y$. Since $\mathbb{1}_X\colon H^X \to H^X$ is 2-isomorphic to the matrix functor $(\mathbb{C}, \delta_x)$, and similarly for $\mathbb{1}_Y$, Thm. 43 implies that the composite measures ut and tu are equivalent to Dirac measures:

$$(ut)_x = \int_Y du_x(y)\, t_y \sim \delta_x \qquad (tu)_y = \int_X dt_y(x)\, u_x \sim \delta_y$$

An immediate consequence is that the measures u_x and t_y must be non-trivial, for all x and y. Also, for all x, the subset $X - \{x\}$ has zero $(ut)_x$-measure

$$\int_Y du_x(y)\, t_y(X - \{x\}) = 0$$

As a result the nonnegative function $y \mapsto t_y(X - \{x\})$ vanishes u_x-almost everywhere. This means that, for all x and u_x-almost all y, the measure t_y is equivalent to δ_x. Likewise, we find that, for all y and t_y-almost all x, the measure u_x is equivalent to δ_y.

Let us consider further the consequences of these two properties, by fixing a point $y_0 \in Y$. For t_{y_0}-almost every x, we know, on one hand, that $t_y \sim \delta_x$ for u_x-almost all y (since this actually holds for all x), and on the other hand, that $u_x \sim \delta_{y_0}$. It follows that for t_{y_0}-almost every x, we have $t_{y_0} \sim \delta_x$. The measure t_{y_0} being non-trivial, this requires $t_{y_0} \sim \delta_{f(y_0)}$ for at least one point $f(y_0) \in X$; moreover this point is unique, because two Dirac measures are equivalent only if they charge the same point. This defines a function $f : Y \to X$ such that t_y is equivalent to $\delta_{f(y)}$. Likewise, we can define a function $g : X \to Y$ such that u_x is equivalent to $\delta_{g(x)}$. Finally, by expressing the composite measures in terms of Dirac measures, we get $fg = \mathbb{1}_X$ and $gf = \mathbb{1}_Y$, establishing the invertibility of the function f.

The measurability of the function f can be shown as follows. Consider a measurable set $A \subseteq X$. Since the family of measures t_y is measurable, we know the function $y \mapsto t_y(A)$ is measurable. Since $t_y(A) = \delta_{f(y)}$, so this function is given by:

$$y \mapsto t_y(A) = \begin{cases} 1 & \text{if} \quad y \in f^{-1}(A) \\ 0 & \text{if not} \end{cases}$$

This coincides with the characteristic function of the set $f^{-1}(A) \subseteq Y$, which is measurable precisely when $f^{-1}(A)$ is measurable. Hence, f is measurable.

Finally, we can use (36) to compose the fields $U_{x,y}$ and $T_{y,x}$. Since $(tu)_y \sim \delta_y$, the only essential components of the composite field are the diagonal ones:

$$(TU)_{y,y} = \int_X^{\oplus} dk_{y,y}(x)\, T_{y,x} \otimes U_{x,y}.$$

Applying (37) in this case, we find that the measures $k_{y,y}$ are defined by the property

$$\int_X dk_{y,y}(x) F(x,y) = \int_X d\delta_{f(y)}(x) \int_Y \delta_{g(x)}(y) F(x,y)$$

for any measurable function F on $X \times Y$. From this we obtain $k_{y,y} = \delta_{f(y)}$ and $(TU)_{y,y} = T_{y,f(y)} \otimes U_{f(y),y}$ for all $y \in Y$. Since we know TU is 2-isomorphic to the matrix functor $(\mathbb{C}, \delta_y)$, we therefore obtain

$$(TU)_{y,y} = T_{y,f(y)} \otimes U_{f(y),y} \cong \mathbb{C} \quad \forall y \in Y.$$

where the isomorphism of fields is measurable. This can only happen if each factor in the tensor product is measurably isomorphic to the constant field $\mathbb{C}$.

Conversely, if the measures t_y are equivalent to $\delta_{f(y)}$ for an invertible measurable function f, and if $T_{y,f(y)} \cong \mathbb{C}$, construct a matrix functor $U \colon H^Y \to H^X$ from the family of measures $\delta_{f^{-1}(x)}$ and the constant field $U_{x,y} = \mathbb{C}$. One can immediately check that U is a weak inverse for T. ∎

Taken together, these theorems have the following corollary:

Corollary 45 *If $T \colon H^X \to H^Y$ is a weakly invertible measurable functor, there is a unique measurable isomorphism $f \colon Y \to X$ such that T is boundedly naturally isomorphic to the matrix functor $(\mathbb{C}, \delta_{f(y)})$.*

Proof: Any measurable functor is boundedly naturally isomorphic to a matrix functor, say $T \cong (T_{y,x}, t_y)$. By Thm. 44, we may in fact take $T_{y,x} = \mathbb{C}$ and $t_y = \delta_{f(y)}$ for some measurable isomorphism $f\colon Y \to X$. By Thm. 43, two such matrix functors, say $(\mathbb{C}, \delta_{f(y)})$ and $(\mathbb{C}, \delta_{f'(y)})$ are boundedly naturally isomorphic if and only if $f = f'$, so the choice of f is unique. ■

We have classified measurable equivalences by giving one representative—a specific *matrix* equivalence—of each 2-isomorphism class. These representatives are quite handy in calculations, but they do have one drawback: matrix functors are not strictly closed under composition. In particular, the composite of two of our representatives $(\mathbb{C}, f_*\delta)$ is isomorphic, but not equal, to another of this form. While in general this is the best we might expect, it is natural to wonder whether these 2-isomorphism classes have a set of representations that is closed under composition. They do.

If X and Y are measurable spaces, any measurable function

$$f\colon Y \to X$$

gives a functor H^f called the **pullback**

$$H^f\colon H^X \to H^Y$$

defined by pulling back measurable fields of Hilbert spaces and linear operators along f. Explicitly, given a measurable field of Hilbert spaces $\mathcal{H} \in H^X$, the field $H^f\mathcal{H}$ has components

$$(H^f\mathcal{H})_y = \mathcal{H}_{f(y)}$$

Similarly, for $\phi\colon \mathcal{H} \to \mathcal{H}'$ a measurable field of linear operators on X,

$$(H^f\phi)_y = \phi_{f(y)}.$$

It is easy to see that this is functorial; to check that $\mathcal{H}^f$ is a *measurable* functor, we note that it is boundedly naturally isomorphic to the matrix functor $(\mathbb{C}, \delta_{f(y)})$, which sends an object $\mathcal{H} \in H^X$ to

$$\int_X^{\oplus} d\delta_{f(y)}(x)\, \mathbb{C} \otimes \mathcal{H}_x \cong \mathcal{H}_{f(y)} = (H^f\mathcal{H})_y$$

and does the analogous thing to morphisms in H^X. The obvious isomorphism in this equation is natural, and has unit norm, so is bounded.

Proposition 46 *If $T\colon H^X \to H^Y$ is a weakly invertible measurable functor, there exists a unique measurable isomorphism $f\colon Y \to X$ such that T is boundedly naturally isomorphic to the pullback H^f.*

Proof: Any measurable functor from H^X to H^Y is equivalent to some matrix functor; by Cor. 45, this matrix functor may be taken to be $(\mathbb{C}, \delta_{f(x)})$ for a *unique* isomorphism of measurable spaces $f\colon Y \to X$. This matrix functor is 2-isomorphic to H^f. ■

While the pullbacks H^f are closely related to the matrix functors $(\mathbb{C}, f_*\delta)$, the former have several advantages, all stemming from the basic equations:

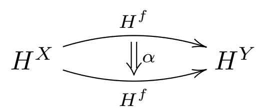
$$H^{1_X} = \mathbb{1}_{H^X} \quad \text{and} \quad H^f H^g = H^{gf} \tag{51}$$

In particular, composition of pullbacks is *strictly* associative, and each pullback H^f has *strict* inverse $H^{f^{-1}}$. In fact, there is a 2-category M with measurable spaces as objects, *invertible* measurable functions as morphisms, and only identity 2-morphisms. The assignments $X \mapsto H^X$ and $f \mapsto H^f$ give a contravariant 2-functor $M \to \mathbf{Meas}$. The forgoing analysis shows this 2-functor is faithful at the level of 1-morphisms.

If f, f' are distinct measurable isomorphisms, the measurable functors H^f and $H^{f'}$ are never 2-isomorphic. However, each H^f has many 2-automorphisms:

Theorem 47 *Let $f\colon Y \to X$ be an isomorphism of measurable spaces, and $H^f\colon H^X \to H^Y$ be its pullback. Then the group of 2-automorphisms of H^f is isomorphic to the group of measurable maps $Y \to \mathbb{C}^\times$, with pointwise multiplication.*

Proof: Let α be a 2-automorphism of H^f, where f is invertible.



Using the 2-isomorphism $\beta\colon H^f \Rightarrow (\mathbb{C}, f_*\delta)$, we can write α as a composite

$$\alpha = \beta^{-1} \cdot \tilde{\alpha} \cdot \beta$$

By Thm. 43, $\tilde{\alpha}\colon (\mathbb{C}, f_*\delta) \Rightarrow (\mathbb{C}, f_*\delta)$ is necessarily a matrix functor given by a measurable field of linear operators $\tilde{\alpha}_{y,x}\colon \mathbb{C} \to \mathbb{C}$, defined and invertible $\delta_{f(y)}$-*a.e.* for all y. Such a measurable field is just a measurable function $\tilde{\alpha}\colon Y \times X \to \mathbb{C}$, with $\tilde{\alpha}_{y,f(y)} \in \mathbb{C}^\times$. From the definition of matrix natural transformations, we can then compute for each object $\mathcal{H} \in H^X$, the morphism $\alpha_{\mathcal{H}}\colon H^f\mathcal{H} \to H^f\mathcal{H}$. Explicitly,

$$\begin{array}{ccccccc}
\mathcal{H}_{f(y)} & \xrightarrow{\beta_{\mathcal{H}}} & \int^{\oplus} d\delta_{f(y)}(x) H_x & \xrightarrow{\tilde{\alpha}_{\mathcal{H}}} & \int^{\oplus} d\delta_{f(y)}(x) H_x & \xrightarrow{\beta_{\mathcal{H}}^{-1}} & \mathcal{H}_{f(y)} \\
\psi_{f(y)} & \longmapsto & \int^{\oplus} d\delta_{f(y)}(x)\psi_x & \longmapsto & \int^{\oplus} d\delta_{f(y)}(x)\alpha_{y,x}\psi_x & \longmapsto & \tilde{\alpha}_{y,f(y)}\psi_{f(y)}
\end{array}$$

So, the natural transformation α acts via multiplication by

$$\alpha(y) := \tilde{\alpha}_{y,f(y)} \in \mathbb{C}^\times.$$

It is easy to show that $\alpha(y)\colon Y \to \mathbb{C}^\times$ is measurable, since $\tilde{\alpha}$ and f are both measurable.

Conversely, given a measurable map $\alpha(y)$, we get a 2-automorphism α of H^f by letting

$$\alpha_{\mathcal{H}}\colon H^f\mathcal{H} \to H^f\mathcal{H}$$

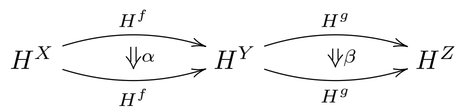
be given by

$$\begin{aligned}
(\alpha_{\mathcal{H}})_y\colon \mathcal{H}_{f(y)} &\to \mathcal{H}_{f(y)} \\
\psi_y &\mapsto \alpha(y)\psi_y
\end{aligned}$$

One can easily check that the procedures just described are inverses, so we get a one-to-one correspondence. Moreover, composition of 2-automorphisms α_1, α_2, corresponds to multiplication of the functions $\alpha_1(y)$, $\alpha_2(y)$, so this correspondence gives a group isomorphism. ∎

It will also be useful to know how to compose pullback 2-automorphisms horizontally:

Proposition 48 *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be measurable isomorphisms, and consider the following diagram in* **Meas***:*



where α and β are 2-automorphisms corresponding to measurable maps

$$\alpha\colon Y \to \mathbb{C}^\times \quad \textit{and} \quad \beta\colon Z \to \mathbb{C}^\times$$

as in the previous theorem. Then the horizontal composite $\beta \circ \alpha$ corresponds to the measurable map from Z to $\mathbb{C}^\times$ defined by

$$(\beta \circ \alpha)(z) = \beta(z)\alpha(g(z))$$

Proof: This is a straightforward computation from the definition of horizontal composition. ∎

4.3 Structure theorems

We now begin the precise description of the representation theory, as outlined in Section 4.1. We first give the detailed structure of representations, followed by that of intertwiners and 2-intertwiners.

4.3.1 Structure of representations

Given a generic 2-group $\mathcal{G} = (G, H, \rhd, \partial)$, we are interested in the structure of a representation ρ in the target 2-category **Meas**. Since any object of **Meas** is C^*-equivalent to one of the form H^X, we shall assume that

$$\rho(\star) = H^X$$

for some measurable space X. The representation ρ also gives, for each $g \in G$, a morphism $\rho(g)\colon H^X \to H^X$, and we assume from now on that all of these morphisms are pullbacks of measurable automorphisms of X.

Theorem 49 (Representations) *Let ρ be a representation of $\mathcal{G} = (G, H, \partial, \rhd)$ on H^X, and assume that each $\rho(g)$ is of the form H^{f_g} for some $f_g\colon X \to X$. Then ρ is determined uniquely by:*

- *a right action $\lhd$ of G as measurable transformations of X, and*
- *an assignment to each $x \in X$ of a group homomorphism $\chi(x)\colon H \to \mathbb{C}^\times$.*

satisfying the following properties:

(i) for each $h \in H$, the function $x \mapsto \chi(x)[h]$ is measurable

(ii) any element of the image of ∂ acts trivially on X via $\lhd$.

(iii) the field of homomorphisms is equivariant under the actions of G on H and X:

$$\chi(x)[g \rhd h] = \chi(x \lhd g)[h].$$

Proof: Consider a representation ρ on H^X and suppose that for each g,

$$\rho(g) = H^{f_g},$$

where $f_g \colon X \to X$ is a measurable isomorphism. Thanks to the strict composition laws (51) for such 2-morphisms, the conditions that ρ respects composition of morphisms and the identity morphism, namely

$$\rho(g'g) = \rho(g')\,\rho(g) \quad \text{and} \quad \rho(1) = \mathbb{1}_{H^X}$$

can be expressed as conditions on the functions f_g:

$$f_{g'g} = f_g f_{g'} \quad \text{and} \quad f_1 = 1_X. \tag{52}$$

Introducing the notation $x \lhd g = f_g(x)$, these equations can be rewritten

$$x \lhd g'g = (x \lhd g') \lhd g \quad \text{and} \quad x \lhd 1 = x$$

Thus, the mapping $(x, g) \mapsto x \lhd g$ is a right action of G on X.

Next, consider a 2-morphism $\rho(u)$, where $u = (g, h)$ is a 2-morphism in $\mathcal{G}$. Since u is invertible, so is $\rho(u)$. In particular, applying ρ to the 2-morphism $(1, h)$, we get a 2-isomorphism $\rho(1, h) \colon \mathbb{1}_{H^X} \Rightarrow H^{f_{\partial h}}$ for each $h \in H$. Such 2-isomorphisms exists only if $f_{\partial h} = 1_X$ for all h; that is,

$$x \lhd \partial(h) = x$$

for all $x \in X$ and $h \in H$. Thus, the image $\partial(H)$ of the homomorphism ∂ fixes every element $x \in X$ under the action $\lhd$.

For arbitrary, $u \in G \times H$, Thm. 47 implies $\rho(u)$ is given by a measurable function on X, which we also denote by $\rho(g, h)$:

$$\rho(g, h) \colon X \to \mathbb{C}.$$

We can derive conditions on the these functions from the requirement that ρ respect both kinds of composition of 2-morphisms.

First, by Thm. 47, vertical composition corresponds to pointwise multiplication of functions, so the condition (10) that ρ respect vertical composition becomes:

$$\rho(g, h'h)(x) = \rho(\partial h g, h')(x)\,\rho(g, h)(x). \tag{53}$$

Similarly, using the formula for horizontal composition provided by Prop. 48, we obtain

$$\rho(g'g, h'(g' \rhd h))(x) = \rho(g', h')(x)\,\rho(g, h)(x \lhd g'). \tag{54}$$

Applying this formula in the case $g' = 1$ and $h = 1$, we find that the functions $\rho(g, h)$ are independent of g:

$$\rho(g, h)(x) = \rho(1, h)(x)$$

This allows a drastic simplification of the formula for vertical composition (53). Indeed, if we define

$$\chi(x)[h] = \rho(1, h)(x), \tag{55}$$

then (53) is simply the statement that $h \mapsto \chi(x)[h]$ is a homomorphism for each x:

$$\chi(x)[h'h] = \chi(x)[h']\,\chi(x)[h].$$

To check that the field of homomorphisms $\chi(x)$ satisfies the equivariance property

$$\chi(x \lhd g)[h] = \chi(x)[g \rhd h], \tag{56}$$

one simply uses (54) again, this time with $g = h' = 1$.

To complete the proof, we show how to reconstruct the representation $\rho\colon \mathcal{G} \to \mathbf{Meas}$, given the measurable space X, right action of G on X, and field χ of homomorphisms from H to $\mathbb{C}^\times$. This is a straightforward task. To the unique object of our 2-group, we assign $H^X \in \mathbf{Meas}$. If $g \in G$ is a morphism in $\mathcal{G}$, we let $\rho(g) = H^{f_g}$, where $f_g(x) = x \lhd g$; if $u = (g, h) \in G \times H$ is a 2-morphism in $\mathcal{G}$, we let $\rho(u)$ be the automorphism of H^{f_g} defined by the measurable function $x \mapsto \chi(x)[h]$. ∎

This theorem suggests an interesting question: is every representation of $\mathcal{G}$ on H^X equivalent to one of the above type? As a weak piece of evidence that the answer might be 'yes', recall from Prop. 46 that any invertible morphism from H^X to itself is isomorphic to one of the form H^f. However, this fact alone is not enough.

The above theorem also suggests that we view representations of 2-groups in a more geometric way, as equivariant bundles. In a representation of a 2-group $\mathcal{G}$ on H^X, the assignment $x \mapsto \chi(x)$ can be viewed as promoting X to the total space of a kind of bundle over the set $\hom(H, \mathbb{C}^\times)$ of homomorphisms from H to $\mathbb{C}^\times$:

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \chi} \\ \hom(H, \mathbb{C}^\times) \end{array}$$

Here we are using 'bundle' in a very loose sense: no topology is involved. The group G acts on both the total space and the base of this bundle: the right action $\lhd$ of G on X comes from the representation, while its left action $\rhd$ on H induces a right action $(\chi, g) \mapsto \chi_g$ on $\hom(H, \mathbb{C}^\times)$, where

$$\chi_g[h] = \chi[g \rhd h].$$

The equivariance property in Thm. 49 means that the map χ satisfies

$$\chi(x \lhd g) = \chi(x)_g.$$

So, we say $\chi\colon X \to \hom(H, \mathbb{C}^\times)$ is a 'G-equivariant bundle'.

So far we have ignored any measurable structure on the groups G and H, treating them as discrete groups. In practice these groups will come with measurable structures of their own, and the maps involved in the 2-group will all be measurable. For such 2-groups the interesting representations will be the 'measurable' ones, meaning roughly that all the maps defining the above G-equivariant bundle are measurable.

To make this line of thought precise, we need a concept of 'measurable group':

Definition 50 *We define a* **measurable group** *to be a topological group whose topology is locally compact, Hausdorff, and second countable.*

Varadarajan calls these **lcsc groups**, and his book is an excellent source of information about them [71]. By Lemma 14, they are a special case of *Polish groups*: that is, topological groups G that are homeomorphic to complete separable metric spaces. For more information on Polish groups, see the book by Becker and Kechris [19].

It may seem odd to define a 'measurable group' to be a special sort of *topological* group. The first reason is that every measurable group has an underlying measurable space, by Lemma 14.

The second is that by Lemma 114, any measurable homomorphism between measurable groups is automatically continuous. This implies that the topology on a measurable group can be uniquely reconstructed from its group structure together with its σ-algebra of measurable subsets.

Next, instead of working with the set $\hom(H, \mathbb{C}^\times)$ of *all* homomorphisms from H to $\mathbb{C}^\times$, we restrict attention to the *measurable* ones:

Definition 51 *If H is a measurable group, let H^* denote the set of measurable (hence continuous) homomorphisms $\chi\colon H \to \mathbb{C}^\times$.*

We make H^* into a group with pointwise multiplication as the group operation:

$$(\chi\chi')[h] = \chi[h]\,\chi'[h].$$

H^* then becomes a topological group with the compact-open topology. This is the same as the topology where $\chi_\alpha \to \chi$ when $\chi_\alpha(h) \to \chi(h)$ uniformly for h in any fixed compact subset of H.

Unfortunately, H^* may not be a measurable group! An example is the free abelian group on countably many generators, for which H^* fails to be locally compact. However, H^* is measurable when H is a measurable group with finitely many connected components. For more details, including a necessary and sufficient condition for H^* to be measurable, see Appendix A.3.

In our definition of a 'measurable 2-group', we will demand that H and H^* be measurable groups. The left action of G on H gives a right action of G on H^*:

$$\begin{array}{rcl} \lhd\colon H^* \times G & \to & H^* \\ (\chi, g) & \mapsto & \chi_g \end{array}$$

where

$$\chi_g[h] = \chi[g \rhd h].$$

We will demand that both these actions be measurable. We do not know if these are independent conditions. However, in Lemma 119 we show that if the action of G on H is continuous, its action on H^* is continuous and thus measurable. This handles most of the examples we care about.

With these preliminaries out of the way, here are the main definitions:

Definition 52 *A* **measurable 2-group** *$\mathcal{G} = (G, H, \rhd, \partial)$ is a 2-group for which G, H and H^* are measurable groups and the maps*

$$\rhd\colon G \times H \to H, \qquad \lhd\colon H^* \times G \to H^*, \qquad \partial\colon H \to G$$

are measurable.

Definition 53 *Let $\mathcal{G} = (G, H, \rhd, \partial)$ be a measurable 2-group and suppose the representation ρ of $\mathcal{G}$ on H^X is specified by the maps*

$$\lhd\colon X \times G \to X, \qquad \chi\colon X \to H^*$$

as in Thm. 49. Then ρ is a **measurable representation** *if both these maps are measurable.*

From now on, we will always be interested in *measurable* representations of *measurable* 2-groups. For such a representation, Lemma 120 guarantees that we can choose a topology for X, compatible with its structure as a measurable space, such that the action of G on X is continuous. This may not make $\chi\colon X \to H^*$ continuous. However, Lemma 114 implies that each $\chi(x)\colon H \to \mathbb{C}^\times$ is continuous.

Before concluding this section, we point out a corollary of Thm. 49 that reveals an interesting feature of the representation theory in the 2-category **Meas**. This corollary involves a certain skeletal 2-group constructed from $\mathcal{G}$ (recall that a 2-group is 'skeletal' when its corresponding crossed module has $\partial = 0$). Let $\mathcal{G}$ be a 2-group, *not necessarily measurable*, with corresponding crossed module $(G, H, \partial, \rhd)$. Then, let

$$\bar{G} = G/\partial(H), \qquad \bar{H} = H/[H, H]$$

Note that the image $\partial(H)$ is a normal subgroup of G by (2), and the commutator subgroup $[H, H]$ is a normal subgroup of H. One can check that the action $\rhd$ naturally induces an action $\bar{\rhd}$ of $\bar{G}$ on $\bar{H}$. If we also define $\bar{\partial}\colon \bar{H} \to \bar{G}$ to be the trivial homomorphism, it is straightforward to check that these data define a new crossed module, from which we get a new 2-group:

Definition 54 *Let $\mathcal{G}$ be a 2-group with corresponding crossed module $(G, H, \partial, \rhd)$. Then the 2-group $\bar{\mathcal{G}}$ constructed from the crossed module $(\bar{G}, \bar{H}, \bar{\partial}, \bar{\rhd})$ is called the* **skeletization** *of $\mathcal{G}$.*

Now consider a representation ρ of the 2-group $\mathcal{G}$. First, by Thm. 49, $\partial(H)$ acts trivially on X, so $\bar{G}$ acts on X. Second, the group $\mathbb{C}^\times$ being abelian, $[H, H]$ is contained in the kernel of the homomorphisms $\chi(x)\colon H \to \mathbb{C}^\times$ for all x. In light of Thm. 49, these remarks lead to the following corollary:

Corollary 55 *For any 2-group, its representations of the form described in Thm. 49 are in natural one-to-one correspondence with representations of the same form of its skeletization.*

This corollary means measurable representations in **Meas** fail to detect the 'non-skeletal part' of a 2-group. However, the representation theory of $\mathcal{G}$ as a whole is generally richer than the representation theory of its skeletization $\bar{\mathcal{G}}$. One can indeed show that, while $\mathcal{G}$ and $\bar{\mathcal{G}}$ can not be distinguished by looking at their *representations*, they generally do not have the same *intertwiners*. In what follows, we will nevertheless restrict our study to the case of skeletal 2-groups.

Thus, from now on, we suppose the group homomorphism $\partial\colon H \to G$ to be trivial, and hence the group H to be abelian. Considering Thm. 49 in light of the preceding discussion, we easily obtain the following geometric characterization of measurable representations of skeletal 2-groups.

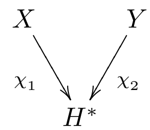
Theorem 56 *A measurable representation ρ of a measurable skeletal 2-group $\mathcal{G} = (G, H, \rhd)$ on H^X is determined uniquely by a measurable right G-action on X, together with a G-equivariant measurable map $\chi\colon X \to H^*$.*

Since we consider only skeletal 2-groups and measurable representations in the rest of the paper, this is the description of 2-group representations to keep in mind. It is helpful to think of this description as giving a 'measurable G-equivariant bundle'

$$\begin{array}{c} X \\ \big\downarrow{\scriptstyle \chi} \\ H^* \end{array}$$

4.3.2 Structure of intertwiners

In this section we study intertwiners between two fixed measurable representations ρ_1 and ρ_2 of a skeletal 2-group $\mathcal{G}$. Suppose ρ_1 and ρ_2 are specified, respectively, by the measurable G-equivariant bundles χ_1 and χ_2 , as in Thm. 56:



To state our main structure theorem for intertwiners, it is convenient to first define two properties that a Y-indexed measurable of measures μ_y on X might satisfy. First, we say the family μ_y is **fiberwise** if each μ_y is supported on the fiber, in X, over the point $\chi_2(y)$. That is, μ_y is fiberwise if

$$\mu_y(X) = \mu_y(\chi_1^{-1}(\chi_2(y)))$$

for all y. We also recall from Section 4.1 that we say a measurable family of measures is **equivariant** if for every $g \in G$ and $y \in Y$, $\mu_{y \lhd g}$ is *equivalent* to the transformed measure μ_y^g defined by:

$$\mu_y^g(A) := \mu_y(A \lhd g^{-1}). \tag{57}$$

Note that, to check that a given *equivariant* family of measures is fiberwise, it is enough to check that, for a set of representatives y_o of the G-orbits in Y, the measure μ_{y_o} concentrates on the fiber over $\chi_2(y_o)$.

We are now ready to give a concrete characterization of intertwiners $\phi\colon \rho_1 \to \rho_2$ between measurable representations. For notational simplicity we now omit the symbol '$\lhd$' for the right G-actions on X and Y defined by the representations, using simple concatenation instead.

Theorem 57 (Intertwiners) *Let ρ_1, ρ_2 be measurable representations of $\mathcal{G} = (G, H, \rhd)$, specified respectively by the G-equivariant bundles $\chi_1\colon X \to H^*$ and $\chi_2\colon Y \to H^*$, as in Thm. 56. Given an intertwiner $\phi\colon \rho_1 \to \rho_2$, we can extract the following data:*

(i) an equivariant and fiberwise Y-indexed measurable family of measures μ_y on X;

(ii) a μ-class of fields of Hilbert spaces $\phi_{y,x}$ on $Y \times X$;

(iii) for each $g \in G$, a μ-class of fields of invertible linear maps $\Phi^g_{y,x}\colon \phi_{y,x} \to \phi_{(y,x)g^{-1}}$ such that, for all $g, g' \in G$, the cocycle condition

$$\Phi^{g'g}_{y,x} = \Phi^{g'}_{(y,x)g^{-1}} \Phi^g_{y,x}$$

holds for all y and μ_y-almost every x.

Conversely, such data can be used to construct an intertwiner.

Before commencing with the proof, note what this theorem does *not* state. It does not state that the data extracted from an intertwiner are unique, nor that starting with these data and constructing an intertwiner gives 'the same' intertwiner. This does turn out to be essentially true, at least for an certain broad class of intertwiners. The sense in which this result classifies intertwiners will be clarified in Propositions 71 and 72.

Proof: Recall that an intertwiner provides a morphism $\phi\colon H^X \to H^Y$ in **Meas**, together with a family

$$\phi(g)\colon \rho_2(g)\phi \Rightarrow \phi\rho_1(g) \quad g \in G$$

of invertible 2-morphisms, subject to the compatibility conditions (14), (16) and (18), namely

$$\phi(1) = \mathbb{1}_\phi \tag{58}$$

and

$$\left[\phi(g') \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \left[\mathbb{1}_{\rho_2(g')} \circ \phi(g)\right] = \phi(g'g) \tag{59}$$

and

$$[\mathbb{1}_\phi \circ \rho_1(u)] \cdot \phi(g) = \phi(g) \cdot [\rho_2(u) \circ \mathbb{1}_\phi] \tag{60}$$

where $u = (g, h)$.

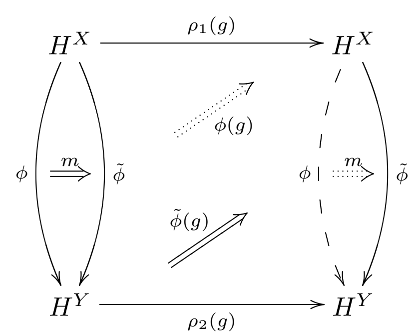
Let us show first that we may assume ϕ is a matrix functor. Since ϕ is a measurable functor, we can pick a bounded natural isomorphism

$$m\colon \phi \Rightarrow \tilde{\phi}$$

where $\tilde{\phi}$ is a matrix functor. We then define, for each $g \in G$, a measurable natural transformation

$$\tilde{\phi}(g) = \left[m \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \phi(g) \cdot \left[\mathbb{1}_{\rho_2(g)} \circ m^{-1}\right]$$

chosen to make the following diagram commute:



The matrix functor $\tilde{\phi}$, together with the family of measurable natural transformations $\tilde{\phi}(g)$, gives an intertwiner, which we also denote $\tilde{\phi}$. The natural isomorphism m gives an invertible 2-intertwiner $m\colon \phi \to \tilde{\phi}$. So, every intertwiner is equivalent to one for which $\phi\colon H^X \to H^Y$ is a matrix functor.

Hence, we now assume $\phi = (\phi, \mu)$ is a matrix functor, and work out what equations (59) and (60) amount to in this case. We use the following result, which simply collects in one place several useful composition formulas:

Lemma 58 *Let ρ_1 and ρ_2 be representations corresponding to G-equivariant bundles X and Y over H^*, as in the theorem.*

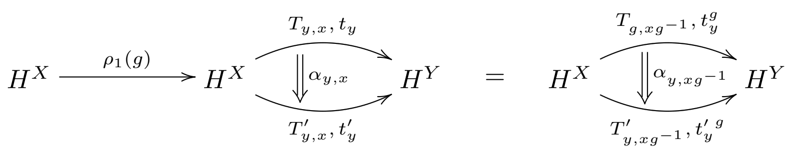
1. *Given any matrix functor $(T, t)\colon H^X \to H^Y$:*
    - *The composite $T\rho_1(g)$ is a matrix functor; in particular, it is defined by the field of Hilbert spaces $T_{y,xg^{-1}}$ and the family of measures t_y^g.*

- *The composite $\rho_2(g)T$ is a matrix functor; in particular, it is defined by the field of Hilbert spaces $T_{yg,x}$ and the family of measures t_{yg}.*

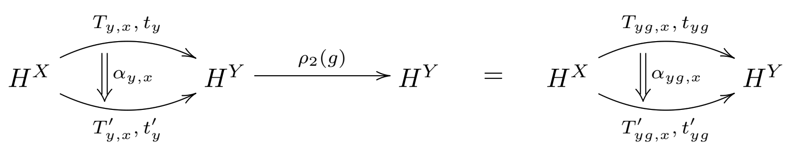
2. *Given a pair of such matrix functors (T,t), (T',t'), and any matrix natural transformation $\alpha\colon T \Rightarrow T'$:*

   - *Whiskering by $\rho_1(g)$ produces a matrix natural transformation whose field of linear operators is $\alpha_{yg,x}$.*
   - *Whiskering by $\rho_2(g)$ produces a matrix natural transformation whose field of linear operators is $\alpha_{yg,x}$.*

That is:



and



Proof: This is a direct computation from the definitions of composition for functors and natural transformations. □

We return to the proof of the theorem. Using this lemma, we immediately obtain explicit descriptions of the source and target of each $\phi(g)$: we find that composites $\rho_2(g)\phi$ and $\phi\rho_1(g)$ are the matrix functors whose families of measures are given by

$$\mu_{yg} \quad \text{and} \quad \mu_y^g$$

respectively, and whose fields of Hilbert spaces read

$$[\rho_2(g)\phi]_{y,x} = \phi_{yg,x} \quad \text{and} \quad [\phi\rho_1(g)]_{y,x} = \phi_{y,xg^{-1}}$$

An immediate consequence is that the family μ_y is equivariant. Indeed, since each $\phi(g)$ is a matrix natural isomorphism, Thm. 43 implies the source and target measures μ_{yg} and μ_y^g are equivalent for all g. Thus, for all g, the 2-morphism $\phi(g)$ defines a field of invertible operators

$$\phi(g)_{y,x} : \phi_{yg,x} \longrightarrow \phi_{y,xg^{-1}}, \tag{61}$$

determined for each y and $\sqrt{\mu_y^g \mu_{yg}}$-*a.e.* in x, or equivalently μ_{yg}-*a.e.* in x, by equivariance.

The lemma also helps make the compatibility condition (59) explicit. The composites $\phi(g') \circ \mathbb{1}_{\rho_1(g)}$ and $\mathbb{1}_{\rho_2(g')} \circ \phi(g)$ are matrix natural transformations whose fields of operators read

$$\left[\phi(g') \circ \mathbb{1}_{\rho_1(g)}\right]_{y,x} = \phi(g')_{y,xg^{-1}} \quad \text{and} \quad \left[\mathbb{1}_{\rho_2(g')} \circ \phi(g)\right]_{y,x} = \phi(g)_{yg',x}$$

Hence, (59) can be rewritten as

$$\phi(g')_{y,xg^{-1}}\, \phi(g)_{yg',x} = \phi(g'g)_{y,x}.$$

Defining a field of linear operators

$$\Phi^g_{y,x} \equiv \phi(g)_{yg^{-1},x}\,, \tag{62}$$

the condition (59) finally becomes:

$$\Phi^{g'g}_{y,x} = \Phi^{g'}_{(y,x)g^{-1}} \Phi^g_{y,x}. \tag{63}$$

We note that since $\phi(g)_{y,x}$ is defined and invertible μ_{yg}-*a.e.*, $\Phi^g_{y,x}$ is defined and invertible μ_y-*a.e.*

Finally, we must work out the consequences of the "pillow condition" (60). We start by evaluating the "whiskered" compositions $\rho_2(u) \circ \mathbb{1}_\phi$ and $\mathbb{1}_\phi \circ \rho_1(u)$, using the formula (48) for horizontal composisiton. By the lemma above, the composites $\rho_2(g)\phi$ and $\phi\rho_1(g)$ are matrix functors. Hence the 2-isomorphisms $[\rho_2(u) \circ \mathbb{1}_\phi]$ and $[\mathbb{1}_\phi \circ \rho_1(u)]$ are necessarily matrix natural transformations. We can work out their matrix components using the definition of horizontal composition,

$$[\rho_2(u) \circ \mathbb{1}_\phi]_{y,x} = \chi_2(y)[h]$$

and

$$[\mathbb{1}_\phi \circ \rho_1(u)]_{y,x} = \chi_1(xg^{-1})[h].$$

The vertical compositions with $\phi(g)$ can then be performed with (46); since all the measures involved are equivalent to each other, these compositions reduce to pointwise compositions of operators—here multiplication of complex numbers. Thus the condition (60) yields the equation

$$\left(\chi_2(y)[h] - \chi_1(xg^{-1})[h]\right) \phi(g)_{y,x} = 0$$

which holds for all h, all y and μ_{yg}-almost every x. Thanks to the covariance of the fields of characters, this equation can equivalently be written as

$$\left(\chi_2(y) - \chi_1(x)\right) \Phi^g_{y,x} = 0 \tag{64}$$

for all y and μ_y-almost every x.

This last equation actually expresses a condition for the family of measures μ_y. Indeed, it requires that, for every y, the subset of the $x \in X$ such that $\chi_1(x) \neq \chi_2(y)$ as well as $\Phi^g_{y,x} \neq 0$ is a null set for the measure μ_y. But we know that, for μ_y-almost every x, $\Phi^g_{y,x}$ is an invertible operator with a non-trivial source space $\phi_{y,x}$, so that it does not vanish. Therefore the condition expressed by (64) is that for each y, the measure μ_y is supported within the set $\{x \in X |\, \chi_1(x) = \chi_2(y)\} = \chi_1^{-1}(\chi_2(y))$. So, the family μ_y is fiberwise.

Conversely, given an equivariant and fiberwise Y-indexed measurable family μ_y of measures on X, a measurable field of Hilbert spaces $\phi_{y,x}$ on $Y \times X$, and a measurable field of invertible linear maps $\Phi^g_{y,x}$ satisfying the cocycle condition (63) μ-*a.e.* for each $g, g' \in G$, we can easily construct an intertwiner. The pair $(\mu_y, \phi_{y,x})$ gives a morphism $\phi\colon H^X \to H^Y$. For each $g \in G$, $y \in Y$, and $x \in X$, we let

$$\phi(g)_{y,x} = \Phi^g_{yg,x} \colon [\rho_2(g)\phi]_{y,x} \to [\phi\rho_1(g)]_{y,x}$$

This gives a 2-morphism in **Meas** for each morphism in $\mathcal{G}$. The cocycle condition, and the fiberwise property of μ_y, ensure that the equations (59) and (60) hold. ∎

Given that the maps $\Phi^g_{y,x}$ are invertible, the cocycle condition (63) implies that $g = 1$ gives the identity:

$$\Phi^1_{y,x} = \mathbb{1}_{\phi_{y,x}} \quad \mu\text{-}a.e. \tag{65}$$

In fact, given given the cocycle condition, this equation is clearly equivalent to the statement that the maps $\Phi^g_{y,x}$ are *a.e.*-invertible. We also easily get a useful formula for inverses:

$$\left(\Phi^g_{y,x}\right)^{-1} = \Phi^{g^{-1}}_{(y,x)g^{-1}} \quad \mu\text{-}a.e. \text{ for each } g. \tag{66}$$

Following our classification of representations, we noted that only some of them deserve to be called 'measurable representations' of a measurable 2-group. Similarly, here we introduce a notion of 'measurable intertwiner'.

First, in the theorem, there is no statement to the effect that the linear maps $\Phi^g_{y,x} \colon \phi_{y,x} \to \phi_{(y,x)g^{-1}}$ are 'measurably indexed' by $g \in G$. To correct this, for an intertwiner to be 'measurable' we will demand that $\Phi^g_{y,x}$ give a measurable field of linear operators on $G \times Y \times X$, where the field $\phi_{y,x}$ can be thought of as a measurable field of Hilbert spaces on $G \times Y \times X$ that is independent of its g-coordinate.

Second, in the theorem, for each pair of group elements $g, g' \in G$, we have a *separate* cocycle condition

$$\Phi^{g'g}_{y,x} = \Phi^{g'}_{(y,x)g^{-1}} \Phi^g_{y,x} \quad \mu\text{-}a.e.$$

In other words, for each choice of g, g', there is a set $U_{g,g'}$ with $\mu_y(X - U_{g,g'}) = 0$ for all y, such that the cocycle condition holds on $U_{g,g'}$. Unless the group G is countable, the union of the sets $X - U_{g,g'}$ may have positive measure. This seems to cause serious problems for characterization of such intertwiners, unless we impose further conditions. For an intertwiner to be 'measurable', we will thus demand that the cocycle condition holds outside some null set, independently of g, g'. Similarly, the theorem implies $\Phi^g_{y,x}$ is invertible μ-*a.e.*, but *separately* for each g; for measurable intertwiners we demand invertibility outside a fixed null set, independently of g.

Let us now formalize these concepts:

Definition 59 *Let $\Phi^g_{y,x} \colon \phi_{y,x} \to \phi_{(y,x)g^{-1}}$ be a measurable field of linear operators on $G \times Y \times X$, with Y, X measurable G-spaces. We say Φ is* **invertible** *at (y, x) if $\Phi^g_{y,x}$ is invertible for all $g \in G$; we say Φ is* **cocyclic** *at (y, x) if $\Phi^{g'g}_{y,x} = \Phi^{g'}_{(y,x)g^{-1}} \Phi^g_{y,x}$ for all $g, g' \in G$.*

Definition 60 *An intertwiner (ϕ, Φ, μ), of the form described in Thm. 57, is* **measurable** *if:*

- *The fields Φ^g are obtained by restriction of a measurable field of linear operators Φ on $G \times Y \times X$;*
- *Φ has a representative (from within its μ-class) that is invertible and cocyclic at all points in some fixed subset $U \subseteq Y \times X$ with $\bar{\mu}_y(Y \times X) = \bar{\mu}_y(U)$ for all y.*

More generally a **measurable intertwiner** *is an intertwiner that is isomorphic to one of this form.*

The generalization in the last sentence of this definition is needed for two composable measurable intertwiners to have measurable composite. From now on, we will always be interested in measurable intertwiners between measurable representations; we sometimes omit the word 'measurable' for brevity, but it is always implicit.

The measurable field $\Phi^g_{y,x}$ in an intertwiner is very similar to a kind of cocycle used in the theory of induced representations on locally compact groups (see, for example, the discussion in Varadarajan's book [71, Sec. V.5]). However, one major difference is that our cocycles here are much better behaved with respect to null sets. In particular, we easily find that if Φ is cocyclic and invertible at a point, it is satisfies the same properties everywhere on an G-orbit:

Lemma 61 *Let* $\Phi^g_{y,x}\colon \phi_{y,x} \to \phi_{(y,x)g^{-1}}$ *be a measurable field of linear operators on* $G\times Y\times X$*. If* Φ *is invertible and cocyclic at* (y_o,x_o)*, then it is invertible and cocyclic at every point on the* G*-orbit of* (y_o,x_o)*.*

Proof: If Φ is invertible and cocyclic at (y_o,x_o), then for any g,g' we have

$$\Phi^{g'}_{(y_o,x_o)g^{-1}} = \Phi^{g'g}_{y_o,x_o}\left(\Phi^g_{y_o,x_o}\right)^{-1}$$

so $\Phi^{g'}$ at $(y_o,x_o)g^{-1}$ is the composite of two invertible maps, hence is invertible. Since g,g' were arbitrary, this shows Φ is invertible everywhere on the orbit. Replacing g' in the previous equation with a product $g''g'$, and using only the cocycle condition at (y_o,x_o), we easily find that Φ is cocyclic at $(y_o,x_o)g^{-1}$ for arbitrary g. ∎

This lemma immediately implies, for any measurable intertwiner (ϕ,Φ,μ), that a representative of Φ may be chosen to be invertible and cocyclic not only on some set with null complement, but actually everywhere on any orbit that meets this set. This fact simplifies many calculations.

Definition 62 *The measurable field of linear operators* $\Phi^g_{y,x}\colon \phi_{y,x} \to \phi_{(y,x)g^{-1}}$ *on* $G\times Y\times X$ *is called a* **strict *G*-cocycle** *if the equations*

$$\Phi^1_{y,x} = 1_{\phi_{y,x}} \quad \textit{and} \quad \Phi^{g'g}_{y,x} = \Phi^{g'}_{(y,x)g^{-1}}\Phi^g_{y,x}$$

hold for all $(g,y,x)\in G\times Y\times X$*. An intertwiner* (Φ,ϕ,μ) *for which the measure-class of* $\Phi^g_{y,x}$ *has such a strict representative is a* **measurably strict** *intertwiner.*

An interesting question is which measurable intertwiners are measurably strict. This may be a difficult problem in general. However, there is one case in which it is completely obvious from Lemma 61: when the action of G on $Y\times X$ is *transitive*. In fact, it is enough for the G-action on $Y\times X$ to be 'essentially transitive', with respect to the family of measures μ. We introduce a special case of intertwiners for which this is true:

Definition 63 *A* Y*-indexed measurable family of measures* μ_y *on* X *is* **transitive** *if there is a single* G*-orbit* o *in* $Y\times X$ *such that, for every* $y\in Y$, $\bar\mu_y = \delta_y\otimes\mu_y$ *is supported on* o*. A* **transitive intertwiner** *is a measurable intertwiner* (Φ,ϕ,μ) *such that the family* μ *is transitive.*

It is often convenient to have a description of transitive families of measures using the measurable field μ_y of measures on X directly, rather than the associated fibered measure distribution $\bar\mu_y$. It is easy to check that μ_y is transitive if and only if there is a G-orbit $o\subseteq Y\times X$ such that whenever $(\{y\}\times A)\cap o = \emptyset$, we have $\mu_y(A)=0$. Transitive intertwiners will play an important role in our study of intertwiners.

Theorem 64 *A transitive intertwiner is measurably strict.*

Proof: The orbit $o\subseteq Y\times X$, on which the measures $\bar\mu_y$ are supported, is a measurable set (see Lemma 121). We may therefore take a representative of $\Phi^g_{y,x}\colon \phi_{y,x}\to\phi_{(y,x)g^{-1}}$ for which $\phi_{y,x}$ is trivial on the null set $(Y\times X)-o$. The cocycle condition then automatically holds not only on o, by Lemma 61, but also on its complement. ∎

In fact, it is clear that a transitive intertwiner has an essentially *unique* field representative. Indeed, any two representatives of Φ must be equal at almost every point on the supporting orbit, but then Lemma 61 implies that they must be equal *everywhere* on the orbit.

Let us turn to the geometric description of intertwiners. For simplicity, we restrict our attention to measurably strict intertwiners, for which the geometric correspondence is clearest. Following Mackey, we can view the measurable field ϕ as a measurable bundle of Hilbert spaces over $Y \times X$:

$$\begin{array}{c} \phi \\ \big\downarrow \\ Y \times X \end{array}$$

whose fiber over (y, x) is the Hilbert space $\phi_{y,x}$. As pointed out in Section 4.1, the strict cocycle $\Phi_{y,x}$ can be viewed as a left action of G on the 'total space' ϕ of this bundle since, by (63) and (65), $\Phi^g \colon \phi \to \phi$ satisfies

$$\Phi^{g'g} = \Phi^{g'}\Phi^g \quad \text{and} \quad \Phi^1 = 1_\phi.$$

The corresponding right action $g \mapsto \Phi^{g^{-1}}$ of G on ϕ is then an action of G over the diagonal action on $Y \times X$:

$$\begin{array}{ccc} \phi & \xrightarrow{\Phi^{g^{-1}}} & \phi \\ \big\downarrow & & \big\downarrow \\ Y \times X & \xrightarrow{\cdot \triangleleft g} & Y \times X \end{array}$$

So, loosely speaking, an intertwiner can be viewed as providing a 'measurable G-equivariant bundle of Hilbert spaces' over $Y \times X$. The associated equivariant family of measures μ serves to indicate, via μ-*a.e.* equivalence, when two such Hilbert space bundles actually describe the same intertwiner.

While these 'Hilbert space bundles' are determined only up to measure-equivalence, in general, they do share many of the essential features of their counterparts in the topological category. In particular, the 'fiber' $\phi_{y,x}$ is a linear representation of the stabilizer group $S_{y,x} \subseteq G$, since the cocycle condition reduces to:

$$\Phi^{s's}_{y,x} = \Phi^{s'}_{y,x}\Phi^{s}_{y,x} \colon \phi_{y,x} \to \phi_{y,x}$$

for $s, s' \in S_{y,x}$.

Definition 65 *Given any measurable intertwiner $\phi = (\phi, \Phi, \mu)$, we define the* **stabilizer representation** *at $(y, x) \in Y \times X$ to be the linear representation of $S_{y,x} = \{s \in G : (y, x)s = (y, x)\}$ on $\phi_{y,x}$ defined by*

$$\mathcal{R}^\phi_{y,x}(s) = \Phi^s_{y,x}.$$

These representations are defined μ_y-a.e. for each y.

Along a given G-orbit o in $Y \times X$, the stabilizer groups are all conjugate in G, so if we choose $(y_o, x_o) \in o$ with stabilizer $S_o = S_{y_o,x_o}$, then the stabilizer representations elsewhere on o can be viewed as representations of S_o. Explicitly,

$$s \mapsto \mathcal{R}^\phi_{y,x}(g^{-1}sg)$$

defines a linear representation of S_o on $\phi_{y,x}$, where $(y, x) = (y_o, x_o)g$. Moreover, the cocycle condition implies

$$\begin{array}{ccc} \phi_{y,x} & \xrightarrow{\mathcal{R}^{\phi}_{y,x}(g^{-1}sg)} & \phi_{y,x} \\ {\scriptstyle \Phi^g_{y,x}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi^g_{y,x}} \\ \phi_{y_o,x_o} & \xrightarrow[\mathcal{R}^{\phi}_{y_o,x_o}(s)]{} & \phi_{y_o,x_o} \end{array}$$

commutes for all $s \in S_o$, and all g such that $(y, x) = (y_o, x_o)g$. In other words, the maps Φ^g are intertwiners between stabilizer representations. We thus see that the assignment $\phi_{y,x}, \Phi^g_{y,x}$ defines, for each orbit in $Y \times X$, a representation of the stabilizer group as well as a consistent way to 'transport' it along the orbit with invertible intertwiners.

In the case of a transitive intertwiner, the only relevant Hilbert spaces are the ones over the special orbit o, so we may think of a transitive intertwiner as a Hilbert space bundle over a single orbit in $Y \times X$:

$$\begin{array}{c} \phi \\ \big\downarrow \\ o \end{array}$$

We have also observed that the Hilbert spaces on the orbit o are *uniquely* determined, so there is no need to mod out by μ-equivalence. We therefore obtain:

Theorem 66 *A transitive intertwiner is uniquely determined by:*

- *A transitive family of measures μ_y on X, with $\bar{\mu}_y$ supported on the G-orbit o,*
- *A measurable field of linear operators $\Phi^g_{y,x} \colon \phi_{y,x} \to \phi_{(y,x)g^{-1}}$ on $G \times o$ that is cocyclic and invertible at some (and hence every) point.*

4.3.3 Structure of 2-intertwiners

We now turn to the problem of classifying the all 2-intertwiners between a fixed pair of parallel intertwiners $\phi, \psi \colon \rho_1 \to \rho_2$. If ρ_1 and ρ_2 are representations of the type described in Thm. 49, then ϕ and ψ are, up to equivalence, of the type described in Thm. 57. Thus, we let ϕ and ψ be given respectively by the equivariant and fiberwise families of measures μ_y and ν_y, and the (classes of) fields of Hilbert spaces and invertible maps $\phi_{y,x}, \Phi^g_{y,x}$ and $\psi_{y,x}, \Psi^g_{y,x}$. A characterization of 2-intertwiners between such intertwiners is given by the following theorem:

Theorem 67 (2-Intertwiners) *Let ρ_1, ρ_2 be representations on H^X and H^Y, and let intertwiners $\phi, \psi \colon \rho_1 \to \rho_2$ be specified by the data (ϕ, Φ, μ) and (ψ, Φ, ν) as in Thm. 57. A 2-intertwiner $m \colon \phi \to \psi$ is specified uniquely by a $\sqrt{\mu\nu}$-class of fields of linear maps $m_{y,x} \colon \phi_{y,x} \to \psi_{y,x}$ satisfying*

$$\Psi^g_{y,x}\, m_{y,x} = m_{(y,x)g^{-1}}\, \Phi^g_{y,x}$$

$\sqrt{\mu\nu}$-a.e.

As usual, by $\sqrt{\mu\nu}$-class of fields we mean equivalence class of fields modulo identification of the fields which coincide for all y and $\sqrt{\mu_y \nu_y}$-almost every x.

Proof: By definition, a 2-intertwiner m between the given intertwiners defines a 2-morphism in **Meas** between the morphisms (ϕ, μ) and (ψ, ν), which satisfies the pillow condition (21), namely

$$\psi(g) \cdot \left[\mathbb{1}_{\rho_2(g)} \circ m\right] = \left[m \circ \mathbb{1}_{\rho_1(g)}\right] \cdot \phi(g) \tag{67}$$

By Thm. 34, since m is a measurable natural transformation between matrix functors, it is automatically a *matrix* natural transformation. We thus have merely to show that the conditions (67) imposes on its matrix components $m_{y,x}$ are precisely those stated in the theorem.

First, using Lemma 58, the two whiskered composites $\mathbb{1}_{\rho_2(g)} \circ m$ and $m \circ \mathbb{1}_{\rho_1(g)}$ in (67) are matrix natural transformations whose fields of operators read

$$[\mathbb{1}_{\rho_2(g)} \circ m]_{y,x} = m_{yg,x} \quad \text{and} \quad [m \circ \mathbb{1}_{\rho_1(g)}]_{y,x} = m_{y,xg^{-1}}$$

respectively. Next, we need to perform the vertical compositions on both sides of the equality (67). For this, we use the general formula (46) for vertical composition of matrix natural transformations, which involves the square root of a product of three Radon-Nykodym derivatives. These derivatives are, in the present context:

$$\frac{d\nu_y^g}{d\mu_{yg}} \frac{d\nu_{yg}}{d\nu_y^g} \frac{d\mu_{yg}}{d\nu_{yg}} \qquad \text{and} \qquad \frac{d\nu_y^g}{d\mu_{yg}} \frac{d\mu_y^g}{d\nu_y^g} \frac{d\mu_{yg}}{d\mu_y^g} \tag{68}$$

for the left and right sides of (67), respectively. Now the equivariance of the families μ_y, ν_y yields

$$\frac{d\mu_{yg}}{d\nu_{yg}} = \frac{d\mu_{yg}}{d\nu_y^g} \frac{d\nu_y^g}{d\nu_{yg}}, \qquad \frac{d\mu_y^g}{d\nu_y^g} = \frac{d\mu_{yg}}{d\nu_y^g} \frac{d\mu_y^g}{d\nu_y^g}$$

so that both products in (68) reduce to

$$\frac{d\nu_y^g}{d\mu_{yg}} \frac{d\mu_{yg}}{d\nu_y^g}$$

Thanks to the chain rule (29), namely

$$\frac{d\mu}{d\nu} \frac{d\nu}{d\mu} = 1 \qquad \sqrt{\mu\nu} - a.e.,$$

this last term equals 1 almost everywhere for the geometric mean of the source and target measures for the 2-morphism described by either side of (67). This shows that the vertical composition reduces to the pointwise composition of the fields of operators. Performing this composition and reindexing, (67) takes the form

$$\Psi_{y,x}^g \, m_{y,x} = m_{(y,x)g^{-1}} \, \Phi_{y,x}^g \tag{69}$$

as we wish to show. This equation holds for all y and $\sqrt{\mu_y \nu_y}$-almost every x. ∎

Thus, a 2-intertwiner $m \colon \phi \Rightarrow \psi$ essentially assigns linear maps $m_{y,x} : \phi_{y,x} \to \psi_{y,x}$ to elements $(y,x) \in Y \times X$, in such a way that (69) is satisfied. Diagrammatically, this equation can be written:

$$\begin{array}{ccc}
\phi_{y,x} & \xrightarrow{\Phi_{y,x}^g} & \phi_{(y,x)g^{-1}} \\
{\scriptstyle m_{y,x}} \downarrow & & \downarrow {\scriptstyle m_{(y,x)g^{-1}}} \\
\psi_{y,x} & \xrightarrow[\Psi_{y,x}^g]{} & \psi_{(y,x)g^{-1}}
\end{array}$$

which commutes $\sqrt{\mu\nu}$-*a.e.* for each g. It is helpful to think of this as a generalization of the equation for an intertwiner between ordinary group representations. Indeed, when restricted to elements of the stabilizer $S_{y,x} \subseteq G$ of (y,x) under the diagonal action on $Y \times X$, it becomes:

$$\mathcal{R}^{\psi}_{y,x}(s)\, m_{y,x} = m_{y,x}\, \mathcal{R}^{\phi}_{y,x}(s) \qquad s \in S_{y,x}$$

This states that $m_{y,x}$ is an intertwining operator, in the ordinary group-theoretic sense, between the stabilizer representations of ϕ and ψ.

If equation (69) is satisfied everywhere along some G-orbit o in $Y \times X$, the maps $m_{y,x}$ of such an assignment are determined by the one $m_o \colon \phi_o \to \psi_o$ assigned to a fixed point (y_o, x_o), since for $(y,x) = (y_o, x_o)g^{-1}$, we have

$$m_{y,x} = \Psi^g_o m_o \left(\Phi^g_o\right)^{-1}$$

If the measure class of $m_{y,x}$ has a representative for which equation (69) is satisfied everywhere, $m_{y,x}$ is determined by its values at one representative of each G-orbit.

In the previous two sections, we introduced 'measurable' versions of representations and intertwiners. For 2-intertwiners, there are no new data indexed by morphisms or 2-morphisms in our 2-group. Since a 2-group has a unique object, there are no new measurability conditions to impose. We thus make the following simple definition.

Definition 68 *A* **measurable 2-intertwiner** *is a 2-intertwiner between measurable intertwiners, as classified in Thm. 67.*

4.4 Equivalence of representations and of intertwiners

In the previous sections we have characterized representations of a 2-group $\mathcal{G}$ on measurable categories, as well as intertwiners and 2-intertwiners. In this section we would like to describe the *equivalence classes* of representations and intertwiners. The general notions of equivalence for representations and intertwiners was introduced, for a general target 2-category, in Section 2.2.3. Recall from that section that two representations are equivalent when there is a (weakly) invertible intertwiner between them. In the case of representations in **Meas**, it is natural to specialize to 'measurable equivalence' of representations:

Definition 69 *Two measurable representations of a 2-group are* **measurably equivalent** *if they are related by a pair of measurable intertwiners that are weak inverses of each other.*

In what follows, by 'equivalence' of representations we always mean measurable equivalence.

Similarly, recall that two parallel intertwiners are equivalent when there is an invertible 2-intertwiner between them. Since measurable 2-intertwiners are simply 2-intertwiners with measurable source and target, there are no extra conditions necessary for equivalent intertwiners to be 'measurably' equivalent.

Let ρ_1 and ρ_2 be measurable representations of $\mathcal{G} = (G, H, \rhd)$ on the measurable categories H^X and H^Y defined by G-equivariant bundles $\chi_1 \colon X \to H^*$ and $\chi_2 \colon Y \to H^*$. We use the same symbol "$\lhd$" for the action of G on both X and Y. The following theorem explains the geometric meaning of equivalence of representations.

Theorem 70 (Equivalent representations) *Two measurable representations ρ_1 and ρ_2 are equivalent if and only if the corresponding G-equivariant bundles $\chi_1 \colon X \to H^*$ and $\chi_2 \colon Y \to H^*$ are isomorphic. That is, $\rho_1 \sim \rho_2$ if and only if there is an invertible measurable function $f \colon Y \to X$ that is G-equivariant:*

$$f(y \lhd g) = f(y) \lhd g$$

and fiber-preserving:

$$\chi_1(f(y)) = \chi_2(y).$$

Proof: Suppose first the representations are equivalent, and let ϕ be an invertible intertwiner between them. Recall that each intertwiner defines a morphism in **Meas**; moreover, as shown by the law (23), the morphism defined by the composition of two intertwiners in the 2-category of representations $\mathbf{2Rep}(\mathcal{G})$ coincides with the composition of the two morphisms in **Meas**. As a consequence, the invertibility of ϕ yields the invertibility of its associated morphism (ϕ, μ). By Theorem 44, this means the measures μ_y are equivalent to Dirac measures $\delta_{f(y)}$ for some invertible (measurable) function $f\colon Y \to X$.

On the other hand, by definition of an intertwiner, the family μ_y is equivariant. This means here that the measure $\delta_{f(y \lhd g)}$ is equivalent to the measure $\delta^g_{f(y)} = \delta_{f(y)\lhd g}$. Thus, the two Dirac measures charge the same point, so $f(y \lhd g) = f(y) \lhd g$. We also know that the support of μ_y, that is, the singlet $\{f(y)\}$, is included in the set $\{x \in X | \, \chi_1(x) = \chi_2(y)\}$. This yields $\chi_1(f(y)) = \chi_2(y)$.

Conversely, suppose there is a function f which satisfies the conditions of the theorem. One can immediately construct from it an invertible intertwiner between the two representations, by considering the family of measures $\delta_{f(y)}$, the constant field of one-dimensional spaces $\mathbb{C}$ and the constant field of identity maps $\mathbb{1}$. ∎

We now consider two intertwiners ϕ and ψ between the same pair of representations ρ_1 and ρ_2, specified by equivariant and fiberwise families of measures μ_y and ν_y, and classes of fields $\phi_{y,x}, \Phi^g_{y,x}$ and $\psi_{y,x}, \Psi^g_{y,x}$. As we know, these carry standard linear representations $\mathcal{R}^\phi_{y,x}$ and $\mathcal{R}^\psi_{y,x}$ of the stabilizer $S_{y,x}$ of (y, x) under the diagonal action of G, respectively in the Hilbert spaces $\phi_{y,x}$ and $\psi_{y,x}$.

The following proposition gives necessary conditions for intertwiners to be equivalent:

Proposition 71 *If the intertwiners ϕ and ψ are equivalent, then for all $y \in Y$, μ_y and ν_y are in the same measure class and the stabilizer representations $\mathcal{R}^\phi_{y,x}$ and $\mathcal{R}^\psi_{y,x}$ are equivalent for μ_y-almost every $x \in X$.*

Proof: Assume $\phi \sim \psi$, and let $m\colon \phi \Rightarrow \psi$ be an invertible 2-intertwiner. Recall that any 2-intertwiner defines a 2-morphism in **Meas**; moreover, the morphism defined by the composition of two 2-intertwiners in the 2-category of representations $\mathbf{2Rep}(\mathcal{G})$ coincides with the composition of the two 2-morphisms in **Meas**. As a consequence, the invertibility of m yields the invertibility of its associated 2-morphism. By Thm. 43, this means that the measures of the source and the target of m are equivalent. Thus, for all y, μ_y and ν_y are in the same measure class.

We know that m defines a μ-class of fields of linear maps $m_{y,x}\colon \phi_{y,x} \to \psi_{y,x}$, such that for all y and μ_y-almost every x, $m_{y,x}$ intertwines the stabilizer representations $\mathcal{R}^\phi_{y,x}$ and $\mathcal{R}^\psi_{y,x}$. Moreover, since m is invertible as a 2-morphism in **Meas**, we know by Thm. 43 that the maps $m_{y,x}$ are invertible. Thus, for all y and almost every x, the two group representations $\mathcal{R}^\phi_{y,x}$ and $\mathcal{R}^\psi_{y,x}$ are equivalent. ∎

This proposition admits a partial converse, if one restricts to transitive intertwiners:

Proposition 72 (Equivalent transitive intertwiners) *Suppose the intertwiners ϕ and ψ are transitive. If for all y, μ_y and ν_y are in the same measure class and the stabilizer representations $\mathcal{R}^\phi_{y,x}$ and $\mathcal{R}^\psi_{y,x}$ are equivalent for μ_y-almost every $x \in X$, then ϕ and ψ are equivalent.*

Proof: Let o be an orbit of $Y \times X$ such that $\mu_y(A) = 0$ for each $\{y\} \times A$ in $(Y \times X) - o$. First of all, if the family μ_y is trivial, so is ν_y; and in that case the intertwiners are obviously equivalent.

Otherwise, there is a point $u_o = (y_o, x_o)$ in o at which the representations $\mathcal{R}^\phi_o$ and $\mathcal{R}^\psi_o$ of the stabilizer S_o are equivalent. Now, assume the two intertwiners are specified by the assignments of Hilbert spaces ϕ_u, ψ_u and invertible maps $\Phi^g_u \colon \phi_u \to \phi_{ug^{-1}}$ and $\Psi^g_u \colon \psi_u \to \psi_{ug^{-1}}$ to the points of the orbit, satisfying cocycle conditions. These yield, for $u = u_o k^{-1}$,

$$\Phi^g_u = \Phi^{gk}_o \left(\Phi^k_o\right)^{-1}, \qquad \Psi^g_u = \Psi^{gk}_o \left(\Psi^k_o\right)^{-1} \tag{70}$$

where ϕ_o, Φ^g_o denote the value of the fields at the point u_o. Now, let $m_o \colon \phi_o \to \psi_o$ be an invertible intertwiner between the representations $\mathcal{R}^\phi_o$ and $\mathcal{R}^\psi_o$. Then for $u = u_o k^{-1}$, the formula

$$m_u = \Psi^k_o m_o \left(\Phi^k_o\right)^{-1}$$

defines invertible maps $m_u \colon \phi_u \to \psi_u$. It is then straightforward to show that (70) yields the intertwining equation $\Psi^g_u \, m_u = m_{ug^{-1}} \, \Phi^g_u$. Thus, the maps m_u define a 2-intertwiner $m \colon \phi \Rightarrow \psi$. We furthermore deduce from the Thm. 43 that m is invertible. Thus, the intertwiners ϕ and ψ are equivalent. ■

In fact, any transitive intertwiner is equivalent to one for which the field of Hilbert spaces $\phi_{y,x}$ is *constant*, $\phi_{y,x} \equiv \phi_o$. More generally, this is true, for any intertwiner, on any single G-orbit $o \subseteq Y \times X$ on which the cocycle is strict. To see this, pick $u_o = (y_o, x_o)$ in o and let $S_o = S_{y_o,x_o}$ be its stabilizer. Since $o \cong G/S_o$ is a homogeneous space of G, there is a measurable section (see Lemma 123)

$$\sigma \colon o \to G$$

defined by the properties

$$\sigma(y_o, x_o) = 1 \in G \quad \text{and} \quad (y_o, x_o)\sigma(y, x) = (y, x)$$

If we define $\phi_o = \phi_{y_o,x_o}$, then for each $(y, x) \in o$, we get a specific isomorphism of $\phi_{y,x}$ with ϕ_o:

$$\alpha_{y,x} = \Phi^{\sigma(y,x)}_{y,x} \colon \phi_{y,x} \to \phi_o$$

If we then define

$$P^g_{y,x} = \alpha_{(y,x)g^{-1}} \Phi^g_{y,x} \left(\alpha_{y,x}\right)^{-1}$$

a straightforward calculation shows that P is cocyclic:

$$\begin{aligned} P^{g'g}_{y,x} &= \alpha_{(y,x)(g'g)^{-1}} \Phi^{g'g}_{y,x} \left(\alpha_{y,x}\right)^{-1} \\ &= \alpha_{(y,x)g^{-1}g'^{-1}} \Phi^{g'}_{(y,x)g^{-1}} \left(\alpha_{(y,x)g^{-1}}\right)^{-1} \alpha_{(y,x)g^{-1}} \Phi^g_{y,x} \left(\alpha_{y,x}\right)^{-1} \\ &= P^{g'}_{(y,x)g^{-1}} P^g_{y,x} \end{aligned}$$

We thus get a new measurable intertwiner (ϕ_o, P, μ), which is equivalent to the original intertwiner (ϕ, Φ, μ) via an invertible 2-intertwiner defined by $\alpha_{y,x}$.

In geometric language, this shows that any 'measurable G-equivariant bundle' can be trivialized by a 'measurable bundle isomorphism', while maintaining G-equivariance. So there are no global 'twists' in such 'bundles', as there are in topological or smooth categories.

4.5 Operations on representations

Some of the most interesting features of ordinary group representation theory arise because there are natural notions of 'direct sum' and 'tensor product', which we can use to build new representations from old. The same is true of 2-group representation theory. In the group case, these sums and products of representations are built from the corresponding operations in Vect. Likewise, for sums and products in our representation theory, we first need to develop such notions in the 2-category **Meas**.

Thus, in this section, we first consider direct sums and tensor products of measurable categories and measurable functors. We then use these to describe direct sums and tensor products of measurable representations, and measurable intertwiners.

4.5.1 Direct sums and tensor products in Meas

We now introduce important operations on 'higher vector spaces', analogous to taking 'tensor products' and 'direct sums' of ordinary vector spaces. These operations are well understood in the case of **2Vect** [18, 44]; here we discuss their generalization to **Meas**.

We begin with 'direct sums'. As emphasized by Barrett and Mackaay [18] in the case of **2Vect**, there are several levels of 'linear structure' in a 2-category of higher vector spaces. In ordinary linear algebra, the *set* $\mathrm{Vect}(V, V')$ of all linear maps between fixed vector spaces V, V' is itself a *vector space*. But the *category* Vect has a similar structure: we can take direct sums of both vector spaces and linear maps, making Vect into a (symmetric) *monoidal category*.

In *categorified* linear algebra, this 'microcosm' of linearity goes one layer deeper. Here we can add *2-maps* between fixed maps, so the top-dimensional hom sets form vector spaces. But there are now two distinct ways of taking 'direct sums' of *maps*. Namely, since we can think of a map between 2-vector spaces as a 'matrix of vector spaces', we can either take the 'matrix of direct sums', when the matrices have the same size, or, more generally, we can take the 'direct sum of matrices'. These ideas lead to two distinct operations which we call the 'direct sum' and the '2-sum'. The direct sum leads to the idea that the hom *categories*, consisting of all maps between fixed 2-vector spaces, as well as 2-maps between those, should be *monoidal* categories; the second leads to the idea that a 2-category of 2-vector spaces should itself be a 'monoidal 2-category'.

Let us make these ideas more precise, in the case of **Meas**. The most obvious level of linear structure in **Meas** applies only at 2-morphism level. Since sums and constant multiples of bounded natural transformations are bounded, the set of measurable natural transformations between fixed measurable functors is a complex vector space.

Next, fixing two measurable spaces X and Y, let $\mathrm{Mat}(X, Y)$ be the category with:

- matrix functors $(T, t)\colon H^X \to H^Y$ as objects
- matrix natural transformations as morphisms

$\mathrm{Mat}(X, Y)$ is clearly a linear category, since composition is bilinear with respect to the vector space structure on each hom set.

Next, there is a notion of *direct sum* in $\mathrm{Mat}(X, Y)$, which corresponds to the intuitive idea of a 'matrix of direct sums'. Intuitively, given two matrix functors $(T, t), (T', t') \in \mathrm{Mat}(X, Y)$, we would like to form a new matrix functor with matrix components $T_{y,x} \oplus T'_{y,x}$. This makes sense as long as the families of measures t_y and t'_y are equivalent, but in general we must be a bit more careful. We first define a y-indexed measurable family of measures $t \oplus t'$ on X by

$$(t \oplus t')_y = t_y + t'_y \tag{71}$$

This will be the family of measures for a matrix functor we will call the direct sum of T and T'. To obtain the corresponding field of Hilbert spaces, we use the Lebesgue decompositions of the measures with respect to each other:

$$t = t^{t'} + \overline{t^{t'}}, \qquad t' = t'^{t} + \overline{t'^{t}}$$

with $t^{t'} \ll t'$ and $\overline{t^{t'}} \perp t'$, and similarly $t'^{t} \ll t$ and $\overline{t'^{t}} \perp t$. The subscript y indexing the measures has been dropped for simplicity. The measures $t^{t'}$ and t'^{t} are equivalent, and these are singular with respect to both $\overline{t^{t'}}$ and $\overline{t'^{t}}$; moreover, these latter two measures are mutually singular. For each $y \in Y$, we can thus write X as a disjoint union

$$X = A_y \amalg B_y \amalg C_y$$

with $\overline{t^{t'}}$ supported on A_y, $\overline{t'^{t}}$ supported on B_y, and $t^{t'}, t'^{t}$ supported on C_y. (In particular, t_y is supported on $A_y \amalg C_y$ and t'_y is supported on $B_y \amalg C_y$.) We then define a new $(t \oplus t')$-class of fields of Hilbert spaces $T \oplus T'$ by setting

$$[T \oplus T']_{y,x} = \begin{cases} T_{y,x} & x \in A_y \\ T'_{y,x} & x \in B_y \\ T_{y,x} \oplus T'_{y,x} & x \in C_y \end{cases} \tag{72}$$

The $(t \oplus t')$-class does not depend on the choice of sets A_y, B_y, C_y, so the data $(T \oplus T', t \oplus t')$ give a well defined matrix functor $H^X \to H^Y$, an object of $\mathrm{Mat}(X, Y)$. We call this the **direct sum** of (T, t) and (T', t'), and denote it by $(T, t) \oplus (T', t')$, or simply $T \oplus T'$ for short. Note that this direct sum is boundedly naturally isomorphic to the functor mapping $\mathcal{H} \in H^X$ to the H^Y-object with components $(T\mathcal{H})_y \oplus (T'\mathcal{H})_y$.

There is an obvious unit object $0 \in \mathrm{Mat}(X, Y)$ for the tensor product, defined by the trivial Y-indexed family of measures on X, $\mu_y \equiv 0$. In fact, this is a *strict* unit object, meaning that we have the equations:

$$(T, t) \oplus 0 = (T, t) = 0 \oplus (T, t)$$

for any object $(T, t) \in \mathrm{Mat}(X, Y)$. We might expect these to hold only up to isomorphism, but since $t + 0 = t$, and T is defined up to measure-class, the equations hold strictly.

Also, given any pair of 2-morphisms in $\mathrm{Mat}(X, Y)$, say matrix natural transformations α and α':

$$H^X \underset{U,u}{\overset{T,t}{\underset{\Downarrow\alpha}{\rightrightarrows}}} H^Y \qquad \text{and} \qquad H^X \underset{U',u'}{\overset{T',t'}{\underset{\Downarrow\alpha'}{\rightrightarrows}}} H^Y$$

we can construct their direct sum, a matrix natural transformation

$$H^X \underset{U\oplus U',u\oplus u'}{\overset{T\oplus T',t\oplus t'}{\underset{\Downarrow\alpha\oplus\alpha'}{\rightrightarrows}}} H^Y$$

as follows. Again, dealing with measure-classes is the tricky part. This time, let us decompose X in two ways, for each y:

$$X = A_y \amalg B_y \amalg C_y = A'_y \amalg B'_y \amalg C'_y$$

with t_y supported on $A_y \amalg C_y$, u_y on $B_y \amalg C_y$, and t_y and u_y equivalent on C_y, and similarly, t'_y supported on $A'_y \amalg C'_y$, u'_y on $B'_y \amalg C'_y$, and t'_y and u'_y equivalent on C'_y. We then define

$$[\alpha \oplus \alpha']_{y,x} = \begin{cases} \alpha_{y,x} \oplus \alpha'_{y,x} & x \in C_y \cap C'_y \\ \alpha_{y,x} & x \in C_y - C'_y \\ \alpha'_{y,x} & x \in C'_y - C_y \\ 0 & \text{otherwise} \end{cases} \tag{73}$$

For this to determine a matrix natural transformation between the indicated matrix functors, we must show that our formula determines the field of linear operators for each y and μ_y-almost every x, where

$$\mu_y = \sqrt{(t_y + t'_y)(u_y + u'_y)}$$

On the set $C_y \cap C'_y$, the measures t_y, u_y, t'_y, u'_y are all equivalent, hence are also equivalent to μ, so α is clearly determined on this set. On the set $C_y - C'_y$, we have $t_y \sim u_y$, while $t'_y \perp u'_y$. Using these facts, we show that

$$\mu_y \sim \sqrt{(t_y + t'_y)(t_y + u'_y)} \sim t_y + \sqrt{t'_y u'_y} = t_y \sim \sqrt{t_y u_y} \qquad \text{on } C_y - C'_y.$$

But the matrix components of $\alpha \oplus \alpha$ given in (73) are determined precisely $\sqrt{t_y u_y}$-*a.e.*, hence μ_y-*a.e.* on $C_y - C'_y$. By an identical argument with primed and un-primed symbols reversing roles, we find

$$\mu_y \sim \sqrt{t'_y u'_y} \qquad \text{on } C'_y - C_y.$$

So the components of $\alpha \oplus \alpha'$ are determined μ_y-*a.e.* for each y, hence give a matrix natural transformation.

We have defined the 'direct sum' in $\mathrm{Mat}(X, Y)$ as a binary operation on objects (matrix functors) and a binary operation on morphisms (matrix natural transformations). One can check that the direct sum is functorial, i.e. it respects composition and identities:

$$(\beta \cdot \alpha) \oplus (\beta' \cdot \alpha') = (\beta \oplus \beta') \cdot (\alpha \oplus \alpha')$$

and

$$\mathbb{1}_T \oplus \mathbb{1}_{T'} = \mathbb{1}_{T \oplus T'}.$$

Definition 73 *The* **direct sum** *in* $\mathrm{Mat}(X, Y)$ *is the functor:*

$$\oplus \colon \mathrm{Mat}(X, Y) \times \mathrm{Mat}(X, Y) \to \mathrm{Mat}(X, Y).$$

defined by

- *The direct sum of objects* $T, T' \in \mathrm{Mat}(X, Y)$ *is the object* $T \oplus T'$ *specified by the family of measures* $t \oplus t'$ *given in (71), and by the* $t \oplus t'$*-class of fields* $[T \oplus T']_{y,x}$ *given in (72);*
- *The direct sum morphisms* $\alpha \colon T \to U$ *and* $\alpha' \colon T' \to U'$ *is the morphism* $\alpha \oplus \alpha' \colon T \oplus T' \to U \oplus U'$ *specified by the* $\sqrt{(t + u)(t' + u')}$*-class of fields of linear maps given in (73).*

The direct sum can be used to promote $\mathrm{Mat}(X,Y)$ to a monoidal category. There is an obvious 'associator' natural transformation; namely, given objects $T, T', T'' \in \mathrm{Mat}(X,Y)$, we get a morphism

$$A_{T,T',T''} \colon (T \oplus T') \oplus T'' \to T \oplus (T' \oplus T'')$$

obtained by using the usual associator for direct sums of Hilbert spaces, on the common support of the respective measures t, t', and t''. The left and right 'unit laws', as mentioned already, are identity morphisms. A straightforward exercise shows that that $\mathrm{Mat}(X,Y)$ becomes a monoidal category under direct sum.

There is also an obvious 'symmetry' natural transformation in $\mathrm{Mat}(X,Y)$,

$$S_{T,T'} \colon T \oplus T' \to T' \oplus T$$

making $\mathrm{Mat}(X,Y)$ into a *symmetric* monoidal category.

We can go one step further. Given *any* measurable categories H and H', the 'hom-category' $\mathbf{Meas}(\mathsf{H},\mathsf{H}')$ has

- measurable functors $T \colon \mathsf{H} \to \mathsf{H}$ as objects
- measurable natural transformations as morphisms

An important corollary of Thm. 34 is that this category is equivalent to some $\mathrm{Mat}(X,Y)$. Picking an adjoint pair of equivalences:

$$\mathbf{Meas}(\mathsf{H},\mathsf{H}') \underset{\overline{F}}{\overset{F}{\rightleftarrows}} \mathrm{Mat}(X,Y)$$

we can transport the (symmetric) monoidal structure on $\mathrm{Mat}(X,Y)$ to one on $\mathbf{Meas}(\mathsf{H},\mathsf{H}')$ by a standard procedure. For example, we define a tensor product of $T, T' \in \mathbf{Meas}(\mathsf{H},\mathsf{H}')$ by

$$T \oplus T' = \overline{F}(F(T) \oplus F(T')).$$

This provides a way to take direct sums of arbitrary parallel measurable functors, and arbitrary measurable natural transformations between them.

We now explain the notion of '2-sum', which is a kind of sum that applies not only to measurable functors and natural transformations, like the direct sum defined above, but also to measurable categories themselves.

First, we define to 2-sum of measurable categories of the form H^X by the formula

$$H^X \boxplus H^{X'} = H^{X \amalg X'}$$

where $\amalg$ denotes disjoint union. Thus, an object of $H^X \boxplus H^{X'}$ consists of a measurable field of Hilbert spaces on X, and one on X'.

Next, for arbitrary matrix functors $(T,t) \colon H^X \to H^Y$ and $(T',t') \colon H^{X'} \to H^{Y'}$, we will define a matrix functor $(T \boxplus T', t \boxplus t')$ called the 2-sum of T and T'. Intuitively, whereas the 'direct sum' was like a 'matrix of direct sums', the '2-sum' should be like a 'direct sum of matrices'. Thus, we use the fields of Hilbert spaces T on $Y \times X$ and T' on $Y' \times X'$ to define a field $T \boxplus T'$ on $Y \amalg Y' \times X \amalg X'$, given by

$$[T \boxplus T']_{y,x} = \begin{cases} T_{y,x} & (y,x) \in Y \times X \\ T'_{y,x} & (y,x) \in Y' \times X' \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

This is well defined on measure-equivalence classes, almost everywhere with respect to the $Y \amalg Y'$-indexed family $t \boxplus t'$ of measures on $X \amalg X'$, defined by:

$$[t \boxplus t']_y = \begin{cases} t_y & y \in Y \\ t'_y & y \in Y' \end{cases} \tag{75}$$

In this definition we have identified t_y with its obvious extension to a measure on $X \amalg X'$.

Finally, suppose we have two arbitrary matrix natural transformations, defined by the fields of linear maps $\alpha_{y,x}\colon T_{y,x} \to U_{y,x}$ and $\alpha'_{y',x'}\colon T'_{y',x'} \to U'_{y',x'}$. From these, we construct a new field of maps from $[T \boxplus T']_{y,x}$ to $[U \boxplus U']_{y,x}$, given by

$$[\alpha \boxplus \alpha']_{y,x} = \begin{cases} \alpha_{y,x} & (y,x) \in Y \times X \\ \alpha'_{y,x} & (y,x) \in Y' \times X' \\ 0 & \text{otherwise} \end{cases} \tag{76}$$

This is determined $\sqrt{(t \boxplus t')(u \boxplus u')}$-*a.e.*, and hence defines a matrix natural transformation $\alpha \boxplus \alpha'\colon T \boxplus T' \Rightarrow U \boxplus U'$.

Definition 74 *The term* **2-sum** *refers to any of the following binary operations, defined on certain objects, morphisms, and 2-morphisms in* **Meas***:*

- *The* **2-sum** *of measurable categories H^X and $H^{X'}$ is the measurable category $H^X \boxplus H^{X'} = H^{X \amalg X'}$;*
- *The* **2-sum** *of matrix functors $(T,t)\colon H^X \to H^Y$ and $(T',t')\colon H^{X'} \to H^{Y'}$ is the matrix functor $(T \boxplus T', t \boxplus t')\colon H^{X \amalg X'} \to H^{Y \amalg Y'}$ specified by the family of measures $t \boxplus t'$ given in (75) and the class of fields $T \boxplus T'$ given in (74);*
- *The* **2-sum** *of matrix natural transformations $\alpha\colon (T,t) \Rightarrow (U,u)$ and $\alpha'\colon (T',t') \Rightarrow (U,u')$ is the matrix natural transformation $\alpha \boxplus \beta\colon (T \boxplus T', t \boxplus t') \Rightarrow (U \boxplus U', u \boxplus u')$ specified by the class of fields of linear operators given in (76).*

It should be possible to extend the notion of 2-sum to apply to arbitrary objects, morphisms, or 2-morphisms in **Meas**, and define additional structure so that **Meas** becomes a 'monoidal 2-category'. While we believe our limited definition of '2-sum' is a good starting point for a more thorough treatment, we make no such attempts here. For our immediate purposes, it suffices to know how to take 2-sums of objects, morphisms, and 2-morphisms of the special types described.

There is an important relationship between the direct sum $\oplus$ and the 2-sum $\boxplus$. Given arbitrary—not necessarily parallel—matrix functors $(T,t)\colon H^X \to H^Y$ and $(T',t')\colon H^{X'} \to H^{Y'}$, their 2-sum can be written as a direct sum:

$$T \boxplus T' \cong [T \boxplus 0'] \oplus [0 \boxplus T'] \tag{77}$$

Here 0 and $0'$ denote the unit objects in the monoidal categories $\mathrm{Mat}(X,Y)$ and $\mathrm{Mat}(X',Y')$. A similar relation holds for matrix natural transformations.

We now briefly discuss 'tensor products'. As with the additive structures discussed above, there may be multiple layers of related multiplicative structures. In particular, we can presumably use the ordinary tensor product of Hilbert spaces and linear maps to turn each $\mathrm{Mat}(X,Y)$, and ultimately each $\mathbf{Meas}(\mathsf{H},\mathsf{H}')$, into a (symmetric) monoidal category. But, we should also be able to turn **Meas** itself into a monoidal 2-category, using a 'tensor 2-product' analogous to the 'direct 2-sum'.

We shall not develop these ideas in detail here, but it is perhaps worthwhile outlining the general structure we expect. First, the tensor product in $\mathrm{Mat}(X,Y)$ should be given as follows:

- Given objects (T,t), (T',t'), define their tensor product $(T\otimes T', t\otimes t')$ by the family of measures

$$(t\otimes t')_y = \sqrt{t_y t'_y}$$

and the field of Hilbert spaces

$$[T\otimes T']_{y,x} = T_{y,x}\otimes T_{y,x}$$

- Given morphisms $\alpha\colon T\to U$ and $\alpha'\colon T'\to U'$, define their tensor product $\alpha\otimes\alpha'\colon T\otimes T'\to U\otimes U'$ by the class of fields defined by

$$(\alpha\otimes\alpha')_{y,x} = \alpha_{y,x}\otimes\alpha'_{y,x}$$

These are simpler than the corresponding formulae for the direct sum, as null sets turn out to be easier to handle. As with the direct sum, we expect the tensor product to give $\mathrm{Mat}(X,Y)$ the structure of a symmetric monoidal category, allowing us to transport this structure to any hom-category $\mathbf{Meas}(\mathsf{H},\mathsf{H}')$ in **Meas**.

Next, let us describe the 'tensor 2-product'.

- Given two measurable categories of the form H^X and $H^{X'}$, we define their tensor 2-product to be
$$H^X\boxtimes H^{X'} := H^{X\times X'}$$

- Given matrix functors $(T,t)\colon H^X\to H^Y$ and $(T',t')\colon H^{X'}\to H^{Y'}$, define their tensor 2-product to be the matrix functor $(T\boxtimes T', t\boxtimes t')$ defined by the $Y\times Y'$-indexed family of measures on $X\times X'$
$$[t\boxtimes t']_{y,y'} = t_y\otimes t'_{y'},$$
where $\otimes$ on the right denotes the ordinary tensor product of measures, and the field of Hilbert spaces
$$[T\boxtimes T']_{(y,y'),(x,x')} = T_{y,x}\otimes T_{y',x'}.$$

- Given matrix natural transformations $\alpha\colon (T,t)\Rightarrow(U,u)$ and $\alpha'\colon(T',t')\Rightarrow(U,u')$, define their tensor 2-product to be the matrix natural transformation $\alpha\boxtimes\beta\colon(T\boxtimes T', t\boxtimes t')\Rightarrow(U\boxtimes U', u\boxtimes u')$ specified by:
$$[\alpha\boxtimes\alpha']_{(y,y'),(x,x')} = \alpha_{y,x}\otimes\alpha_{y',x'}$$
determined almost everywhere with respect to the family of geometric mean measures:
$$\sqrt{(t\boxtimes t')(u\boxtimes u')} = \sqrt{tu}\boxtimes\sqrt{t'u'}$$

As with the 2-sum, it should be possible to use this tensor 2-product to make **Meas** into a monoidal 2-category. We leave this to further work.

4.5.2 Direct sums and tensor products in $\mathbf{2Rep}(\mathcal{G})$

Now let $\mathcal{G}$ be a skeletal measurable 2-group, and consider the representation 2-category $\mathbf{2Rep}(\mathcal{G})$ of (measurable) representations of $\mathcal{G}$ in **Meas**. Monoidal structures in **Meas** give rise to monoidal structures in this representation category in a natural way.

Let us consider the various notions of 'sum' that **2Rep**($\mathcal{G}$) inherits from **Meas**. First, and most obvious, since the 2-morphisms in **Meas** between a fixed pair of morphisms form a vector space, so do the 2-intertwiners between fixed intertwiners.

Next, fix two representations ρ_1 and ρ_2, on the measurable categories H^X and H^Y, respectively. An intertwiner $\phi\colon \rho_1 \to \rho_2$ gives an object of $\phi \in \mathbf{Meas}(H^X, H^Y)$ and for each $g \in G$ a morphism in $\mathbf{Meas}(H^X, H^Y)$. Since $\mathbf{Meas}(H^X, H^Y)$ is equivalent to $\mathrm{Mat}(X, Y)$, the former becomes a symmetric monoidal category with direct sum, and this in turn induces a direct sum of intertwiners between ρ_1 and ρ_2. We get a direct sum of 2-intertwiners in an analogous way.

Definition 75 *Let ρ_1, ρ_2 be representations on H^X and H^Y. The* **direct sum of intertwiners** *$\phi, \phi'\colon \rho_1 \to \rho_2$ is the intertwiner $\phi \oplus \phi'\colon \rho_1 \to \rho_2$ given by the morphism $\phi \oplus \phi'$ in* **Meas***, together with the 2-morphisms $\phi(g) \oplus \phi'(g)$ in* **Meas***. The* **direct sum of 2-intertwiners** *$m\colon \phi \to \psi$ and $m'\colon \phi' \to \psi'$ is the 2-intertwiner given by the measurable natural transformation $m \oplus m'\colon \phi \oplus \phi' \to \psi \oplus \psi'$.*

The intertwiners define families of measures μ_y and μ'_y, and classes of fields of Hilbert spaces $\phi_{y,x}$ and $\phi'_{y,x}$ and invertible maps $\Phi^g_{y,x}$ and $\Phi'^g_{y,x}$ that are invertible and cocyclic. It is straightforward to deduce the structure of the direct sum of intertwiners in terms of these data:

Proposition 76 *Let $\phi = (\phi, \Phi, \mu)$, $\phi' = (\phi', \Phi', \mu')$ be measurable intertwiners with the same source and target representations. Then the intertwiner $\phi \oplus \phi'$ specified by the family of measures $\mu + \mu'$, and the classes of fields $\phi_{y,x} \oplus \phi'_{y,x}$ and $\Phi^g_{y,x} \oplus \Phi'^g_{y,x}$, is a direct sum for ϕ and ϕ'.*

The intertwiner specified by the family of trivial measures, $\mu_y \equiv 0$, plays the role of unit for the direct sum. This unit is the **null intertwiner** between ρ_1 and ρ_2.

Finally, **2Rep**($\mathcal{G}$) inherits a notion of '2-sum'. We begin with the representations.

Definition 77 *The* **2-sum of representations** *$\rho \boxplus \rho'$ is the representation defined by*

$$(\rho \boxplus \rho')(\varsigma) = \rho(\varsigma) \boxplus \rho'(\varsigma)$$

where ς denotes the object $\star$, or any morphism or 2-morphism in $\mathcal{G}$.

We immediately deduce, from the definition of the 2-sum in **Meas**, the structure of the 2-sum of representations:

Proposition 78 *Let ρ, ρ' be measurable representations of $\mathcal{G} = (G, H, \rhd)$, with corresponding equivariant maps $\chi\colon X \to H^*$, $\chi'\colon X \to H^*$. The* **2-sum of representations** *$\rho \boxplus \rho'$ is the representation on the measurable category $H^{X \amalg X'}$, specified by the action of G induced by the actions on X and X', and the obvious equivariant map $\chi \amalg \chi'\colon X \amalg X' \to H^*$.*

The empty space $X = \emptyset$ defines a representation[1] which plays the role of unit element for the direct sum. This unit element is the **null representation**.

There is a notion of 2-sum for intertwiners, which allows one to define the sum of intertwiners that are not necessarily parallel. This notion can essentially be deduced from that of the direct sum, using (77). Indeed, if $\phi = (\phi, \Phi, \mu)$ is a measurable intertwiner, a 2-sum of the form $\phi \boxplus 0$ is simply given by the trivial extensions of the fields ϕ, Φ, μ to a disjoint union, and likewise for $0 \boxplus \phi'$; we then simply write $\phi \boxplus \phi'$ as a direct sum via (77) and the analogous equation for 2-morphisms.

There should also be notions of 'tensor product' and 'tensor 2-product' in the representation 2-category **2Rep**($\mathcal{G}$). Since we have not constructed these products in detail in **Meas**, we shall not give the details here; the constructions should be analogous to the 'direct sum' and '2-sum' just described.

[1] Note that the measurable category $H^{\emptyset}$ is the category with just one object and one morphism.

4.6 Reduction, retraction, and decomposition

In this section, we introduce notions of reducibility and decomposability, in analogy with group representation theory, as well as an *a priori* intermediate notion, 'retractability'. These notions make sense not only for representations, but also for intertwiners. We classify the indecomposable, irretractable and irreducible measurable representations, and intertwiners between these, up to equivalence.

4.6.1 Representations

Let us start with the basic definitions.

Definition 79 *A representation ρ' is a* **subrepresentation** *of a given representation ρ if there exists a weakly monic intertwiner $\rho' \to \rho$.*

We remind the reader that an intertwiner $\phi\colon \rho' \to \rho$ is (strictly) **monic** if whenever $\xi, \xi'\colon \tau \to \rho$ are intertwiners such that $\phi \cdot \xi = \phi \cdot \xi'$, we have $\xi = \xi'$; we say it is **weakly monic** if this holds up to invertible 2-intertwiners, i.e. $\phi \cdot \xi \cong \phi \cdot \xi'$ implies $\xi \cong \xi'$.

Definition 80 *A representation ρ' is a* **retract** *of ρ if there exist intertwiners $\phi\colon \rho' \to \rho$ and $\psi\colon \rho \to \rho'$ whose composite $\psi\phi$ is equivalent to the identity intertwiner of ρ'*

$$\rho' \xrightarrow{\phi} \rho \xrightarrow{\psi} \rho' \quad \simeq \quad \rho' \xrightarrow{\mathbb{1}_{\rho'}} \rho'$$

Definition 81 *A representation ρ' is a* **2-summand** *of ρ if $\rho \simeq \rho' \boxplus \rho''$ for some representation ρ''.*

It is straightforward to show that any 2-summand is automatically a retract, since the diagram

$$\rho' \to \rho' \boxplus \rho'' \to \rho',$$

built from the obvious 'injection' and 'projection' intertwiners, is equivalent to the identity. On the other hand, we shall see that a representation ρ generally has retracts that are not 2-summands; this is in stark contrast to linear representations of ordinary groups, where summands and retracts coincide.

Similarly, any retract is automatically a subrepresentation, since $\psi\phi \simeq 1$ easily implies ϕ is weakly monic.

Any representation ρ has both itself and the null representation as subrepresentations, as retracts, and as summands. This leads us to the following definitions:

Definition 82 *A representation ρ is* **irreducible** *if it has exactly two subrepresentations, up to equivalence, namely ρ itself and the null representation.*

Definition 83 *A representation ρ is* **irretractable** *if it has exactly two retracts, up to equivalence, namely ρ itself and the null representation.*

Definition 84 *A representation ρ is* **indecomposable** *if it has exactly two 2-summands, up to equivalence, namely ρ itself and the null representation.*

Note that according to these definitions, the null representation is neither irreducible, nor indecomposable, nor irretractable. An irreducible representation is automatically irretractable, and an irretractable representation is automatically indecomposable. A priori, neither of these implications is reversible.

Indecomposable representations are characterized by the following theorem:

Theorem 85 (Indecomposable representations) *Let ρ be a measurable representation on H^X, making X into a measurable G-space. Then ρ is indecomposable if and only if X is nonempty and G acts transitively on X.*

Proof: Observe first that, since the null representation is not indecomposable, the theorem is obvious for the case $X = \emptyset$. We may thus assume ρ is not the null representation.

Assume first ρ indecomposable, and let U and V be two disjoint G-invariant subsets such that $X = U \amalg V$. ρ naturally induces representations ρ_U in H^U and ρ_V in H^V, and furthermore $\rho = \rho_U \boxplus \rho_V$. Since by hypothesis ρ is indecomposable, at least one of these representations is the null representation. Consequently $U = \emptyset$ or $V = \emptyset$. This shows that the G-action is transitive.

Conversely, assume G acts transitively on X, and suppose $\rho \sim \rho_1 \boxplus \rho_2$ for some representations ρ_i in H^{X_i}. There is then a splitting $X = X_1' \amalg X_2'$, where X_i' is measurably identified with X_i and G-invariant. Since by hypothesis G acts transitively on X, we deduce that $X_i' = \emptyset = X_i$ for at least one i. Thus, ρ_i is the null representation for at least one i; hence ρ is indecomposable. ∎

Let o be any G-orbit in H^*; pick a point x_o^*, and let S_o^* denote its stabilizer group. The orbit can be identified with the homogeneous space G/S_o^*. Let also $S \subset S_o^*$ be any closed subgroup of S. Then $X := G/S$ is a measurable G-space (see Lemma 122 in the Appendix). The canonical projection onto G/S_o^* defines a G-equivariant map $\chi \colon X \to H^*$. This map is measurable: to see this, write $\chi = \pi s$, where s is a **measurable section** of G/S as in Lemma 123, and $\pi \colon G \to G/S_o^*$ is the measurable projection. Hence, the pair (o, S) defines a measurable representation; this representations is clearly indecomposable.

Next, consider the representations given by two pairs (o, S) and (o', S'). When are they equivalent? Equivalence means that there is an isomorphism $f \colon G/S \to G/S'$ of measurable G-equivariant bundles over H^*. Such an isomorphism exists if and only if the orbits are the same $o = o'$ and the subgroups S, S' are conjugate in S_o^*. Hence, there is class of inequivalent indecomposable representations labelled by an orbit o in H^* and a conjugacy class of subgroups $S \subset S_o^*$.

Now, let ρ be any indecomposable representation on H^X. Thm. 85 says X is a transitive measurable G-space. Transitivity forces the G-equivariant map $\chi \colon X \to H^*$ to map onto a single orbit $o \simeq G/S_o^*$ in H^*. Moreover, it implies that X is isomorphic as a G-equivariant bundle to G/S for some closed subgroup $S \subset S_o^*$. Hence, ρ is equivalent to the representation defined by the orbit o and the subgroup S.

These remarks yield the following:

Corollary 86 *Indecomposable representations are classified, up to equivalence, by a choice of G-orbit o in the character group H^*, along with a conjugacy class of closed subgroups $S \subset S_o^*$ of the stabilizer of one of its points.*

Irretractable representations are characterized by the following theorem:

Theorem 87 (Irretractable representations) *Let ρ be a measurable representation, given by a measurable G-equivariant map $\chi \colon X \to H^*$, as in Thm. 56. Then ρ is irretractable if and only if χ induces a G-space isomorphism between X and a single G-orbit in H^*.*

Proof: First observe that, since a G-orbit in H^* is always nonempty, and the null representation is not irretractable, the theorem is obvious for the case $X = \emptyset$. We may thus assume ρ is not the null representation.

Now suppose ρ is irretractable, and consider a single G-orbit X^* contained in the image $\chi(X) \subset H^*$. X^* is a measurable subset (see Lemma. 121 in the Appendix), so it naturally becomes a measurable G-space, with G-action induced by the action on H^*. The canonical injection $X^* \to H^*$ makes X^* a measurable equivariant bundle over the character group. These data give a non-null representation ρ^* of the 2-group on the measurable category H^{X^*}.

We want to show that ρ^* is a retract of ρ. To do so, we first construct an X-indexed family of measures μ_x on X^* as follows: if $\chi(x) \in X^*$, we choose μ_x to be the Dirac measure $\delta_{\chi(x)}$ which charges the point $\chi(x)$; otherwise we choose μ_x to be the trivial measure. This family is fiberwise by construction; the covariance of the field of characters ensures that it is also equivariant:

$$\delta^g_{\chi(x)} = \delta_{\chi(x)g} = \delta_{\chi(xg)}.$$

To check that the family is measurable, pick a measurable subset $A^* \subset X^*$. The function $x \mapsto \mu_x(A^*)$ coincides with the characteristic function of the set $A = \chi^{-1}(A^*)$, whose value at x is 1 if $x \in A$ and 0 otherwise; this function is measurable if the set A is. Now, since we are working with measurable representations, the map χ is measurable: therefore A is measurable, as the pre-image of the measurable A^*. Thus, the family of measures μ_x is measurable. So, together with the μ-classes of one-dimensional fields of Hilbert spaces and identity linear maps, it defines an intertwiner $\phi\colon \rho^* \to \rho$.

Next, we want to construct an X^*-indexed equivariant and fiberwise family of measures ν_{x^*} on X. To do so, pick an element $x^*_o \in X^*$, denote by $S^*_o \subset G$ its stabilizer group. We require some results from topology and measure theory (see Appendix A.4). First, S^*_o is a closed subgroup, and the orbit X^* can be measurably identified with the homogenous space G/S^*_o; second, there exists a measurable section for G/S^*_o, namely a measurable map $n\colon G/S^*_o \to G$ such that πn =1, where $\pi\colon G \to G/S^*_o$ is the canonical projection, and $n\pi(e) = e$. Also, the action of G on X induces a measurable S^*_o-action on the fiber over x^*_o; any orbit of this fiber can thus be measurably identified with a homogeneous space S^*_o/S, on which nonzero quasi-invariant measures are known to exist.

So let $\nu_{x^*_o}$ be (the extension to X of) a S^*_o-quasi-invariant measure on the fiber over x^*_o. Using a measurable section $n\colon G/S^*_o \to G$, each $x^* \in X^*$ can then be written unambiguously as $x^*_o n(k)$ for some coset $k \in G/S^*_o$. Define

$$\nu_{x^*} := \nu^{n(k)}_{x^*_o}$$

where by definition $\nu^g(A) = \nu(Ag^{-1})$. We obtain by this procedure a measurable fiberwise and equivariant family of measures on X. Together with the (ν-classes of) constant one-dimensional field(s) of Hilbert spaces $\mathbb{C}$ and constant field of identity linear maps, this defines an intertwiner $\psi\colon \rho \to \rho^*$.

We can immediately check that the composition $\psi\phi$ of these two intertwiners defined above is equivalent to the identity intertwiner $\mathbb{1}_{\rho^*}$, since the composite measure at x^*,

$$\int_X d\nu_{x^*}(x)\mu_x = \nu_{x^*}(\chi^{-1}(x^*))\,\delta_{x^*}$$

is equivalent to the delta function δ_{x^*}. This shows that ρ^* is a retract of ρ.

Now, by hypothesis ρ is irretractable; since the retract ρ^* is not null, it must therefore be equivalent to ρ. We know by Thm. 70 that this equivalence gives a measurable isomorphism f : $X^* \to X$, as G-equivariant bundles over H^*. In our case, f being a bundle map means

$$\chi(f(x^*)) = x^*.$$

Together with the invertibility of f, this relation shows that the image of the map χ is X^*, and furthermore that $\chi = f^{-1}$. We have thus proved that $\chi\colon X \to X^*$ is an invertible map of X onto the orbit $X^* \subseteq H^*$.

Conversely, suppose χ is invertible and maps X to a single orbit X^* in H^* and consider a non-null retract ρ' of ρ. We denote by X' the underlying space and by χ' the field of characters associated to ρ'. Pick two intertwiners $\phi\colon \rho' \to \rho$ and $\psi\colon \rho \to \rho'$ such that $\psi\phi \simeq \mathbb{1}_{\rho'}$. These two intertwiners provide an X-indexed family of measures μ_x on X' and a X'-indexed family of measures $\nu_{x'}$ on X which satisfy the property that, for each x', the composite measure at x' is equivalent to a Dirac measure:

$$\int_X d\nu_{x'}(x)\mu_x \sim \delta_{x'} \tag{78}$$

An obvious consequence of this property is that the measures $\nu_{x'}$ are all non-trivial. Since $\nu_{x'}$ concentrates on the fiber over $\chi'(x')$ in X, this fiber is therefore not empty. This shows that $\chi'(X')$ is included in the G-orbit $\chi(X) = X^*$. The G-invariance of the subset $\mathrm{im}\chi'$ shows furthermore that this inclusion is an equality, so $\chi(X) = \chi'(X')$. Consequently the map $f = \chi^{-1}\chi'$ is a well defined measurable function from X' to X; it is surjective, commutes with the action of g and obviously satisfies $\chi f = \chi'$. Now, by hypothesis, the fiber over $\chi'(x')$ in X, on which $\nu_{x'}$ concentrates, consists of the singlet $\{f(x')\}$: we deduce that $\nu_{x'} \sim \delta_{f(x')}$. The property (78) thus reduces to $\mu_{f(x')} \sim \delta_{x'}$ for all x', which requires f to be injective. Thus, we have found an invertible measurable map $f\colon X' \to X$ that is G-equivariant and preserves fibers of $\chi\colon X \to H^*$. By Thm. 70, the representations ρ and ρ' are equivalent; hence ρ is irretractable. ∎

Any irretractable representation is indecomposable; up to equivalence, it thus takes the form (o, S), where o is a G-orbit in H^* and S is a subgroup of S^*_o. However, the converse is not true: there are in general many indecomposable representations (o, S) that are retractable. Indeed, (o, S) defines an invertible map $\chi\colon G/S \to G/S^*_o$ only when $S = S^*_o$. The existence of retractable but indecomposable representations has been already noted by Barrett and Mackaaay [18] in the context of the representation theory of 2-groups on finite dimensional 2-vector spaces. We see here that this is also true for representations on more general measurable categories.

Corollary 88 *Irretractable measurable representations are classified, up to equivalence, by G-orbits in the character group H^*.*

4.6.2 Intertwiners

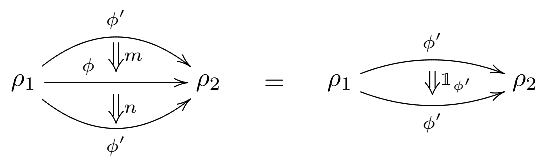
Because 2-group representation theory involves not only intertwiners between representations, but also 2-intertwiners between intertwiners, there are obvious analogs for intertwiners of the concepts discussed in the previous section for representations. We define sub-intertwiners, retracts and 2-summands of intertwiners in a precisely analogous way, obtaining notions of irreducibility, irretractability, and indecomposability for intertwiners, as for representations.

Definition 89 *An intertwiner $\phi'\colon \rho_1 \to \rho_2$ is a* **sub-intertwiner** *of $\phi\colon \rho_1 \to \rho_2$ if there exists a monic 2-intertwiner $m\colon \phi' \Rightarrow \phi$.*

We remind the reader that a 2-intertwiner $m\colon \phi' \Rightarrow \phi$ is **monic** if whenever $n, n'\colon \psi \Rightarrow \phi'$ are 2-intertwiners such that $m \cdot n = m \cdot n'$, we have $n = n'$.

Definition 90 *An intertwiner $\phi'\colon \rho_1 \to \rho_2$ is a* **retract** *of $\phi\colon \rho_1 \to \rho_2$ if there exist 2-intertwiners*

$m\colon \phi' \Rightarrow \phi$ and $n\colon \phi \Rightarrow \phi'$ such that the vertical product $n \cdot m$ equals the identity 2-intertwiner of ϕ'



Definition 91 *An intertwiner $\phi'\colon \rho_1 \to \rho_2$ is a* **summand** *of $\phi\colon \rho_1 \to \rho_2$ if $\phi \cong \phi' \boxplus \phi''$ for some intertwiner ϕ''.*

Any summand is a retract, and any retract is a sub-intertwiner. Recall from Section 4.5.1 that the **null intertwiner** between measurable representations on H^X and H^Y is defined by the trivial family of measures, $\mu_y = 0$ for all y. It is easy to see that the null intertwiner is a summand (hence also a retract, and a sub-intertwiner) of any intertwiner.

Definition 92 *An intertwiner ϕ is* **irreducible** *if it has exactly two sub-intertwiners, up to 2-isomorphism, namely ϕ itself and the null intertwiner.*

Definition 93 *An intertwiner ϕ is* **irretractable** *if it has exactly two retracts, up to 2-isomorphism, namely ϕ itself and the null intertwiner.*

Definition 94 *An intertwiner ϕ is* **indecomposable** *if it has exactly two summands, up to 2-isomorphism, namely ϕ itself and the null intertwiner.*

According to these definitions, the null representation is neither irreducible, nor indecomposable, nor irretractable. An irreducible intertwiner is automatically irretractable, and an irretractable intertwiner is automatically indecomposable. A priori, neither of these implications is reversible.

To dig deeper into these notions, we need some concepts from ergodic theory: ergodic measures, and their generalization to measurable families of measures. In what follows, we denote by $\triangle$ the symmetric difference operation on sets:

$$U \triangle V = (U \cup V) - (U \cap V)$$

When U is a subset of a G-set X, we use the notation $Ug = \{ug \,|\, u \in U\}$.

Definition 95 *A measure μ on X is* **ergodic** *under a G-action if for any measurable subset $U \subset X$ such that $\mu(U \triangle Ug) = 0$ for all g, we have either $\mu(U) = 0$ or $\mu(X - U) = 0$.*

In the case of quasi-invariant measures, there is a useful alternative criterion for ergodicity. Roughly speaking, an ergodic quasi-invariant measure has as many null sets as possible without vanishing entirely. More precisely, we have the following lemma:

Lemma 96 *Let μ be a quasi invariant measure with respect to a G-action. Then μ is ergodic if and only if any quasi-invariant measure ν that is absolutely continuous with respect to μ is either zero or equivalent to μ.*

Proof: Assume first μ is ergodic. Let ν be a quasi-invariant measure with $\nu \ll \mu$. Consider the Lebesgue decomposition $\mu = \mu^\nu + \overline{\mu^\nu}$. As shown in Prop. 107, the two measures are mutually singular, so there is a measurable set U such that $\mu^\nu(A) = \mu^\nu(A \cap U)$ for every measurable set A, and $\overline{\mu^\nu}(U) = 0$. Hence, for all $g \in G$, $\mu^\nu(Ug - U) = 0$. Now, $\nu \ll \mu$ implies $\mu^\nu \sim \nu$, so we know μ^ν is also quasi-invariant. This implies $\mu^\nu(U - Ug) = \mu((Ug^{-1} - U)g) = 0$ for all g. We then have

$$\mu(U \triangle Ug) = \mu(Ug - U) + \mu(U - Ug) = 0$$

for all $g \in G$. Since μ is ergodic, we conclude that either $\mu(U) = 0$, in which case $\mu^\nu = 0$ and therefore $\nu = 0$, or $\mu(X - U) = 0$, in which case $\mu \sim \mu^\nu$, and hence $\mu \sim \nu$.

Conversely, suppose every quasi-invariant measure subordinate to μ is either zero or equivalent to μ. Let U be a measurable set such that $\mu(U \triangle Ug) = 0$ for all $g \in G$. Define a measure ν by setting $\nu(A) = \mu(A \cap U)$ for each measurable set A. Obviously $\nu \ll \mu$. Since $U \triangle Ug$ is μ-null,

$$\nu(A) = \mu(A \cap U) = \mu(A \cap Ug)$$

for all g and every measurable set A. In particular, applying this to Ag,

$$\nu(Ag) = \mu(Ag \cap U) = \mu((A \cap U)g),$$

so quasi-invariance of ν follows from that of μ. Thus, ν is a quasi-invariant measure such that $\nu \ll \mu$; this, by hypothesis, yields either $\nu = 0$, hence $\mu(U) = 0$, or $\nu \sim \mu$, hence $\mu(X - U) = 0$. We conclude that μ is ergodic. ∎

The notion of ergodic measure has an important generalization to the case of measurable families of measures:

Definition 97 *Let X and Y be measurable G-spaces. A Y-indexed equivariant family of measures μ_y on X is* **minimal** *if:*

(i) there exists a G-orbit Y_o in Y such that $\mu_y = 0$ for all $y \in Y - Y_o$, and

(ii) for all y, μ_y is ergodic under the action of the stabilizer $S_y \subset G$ of y.

Notice that an ergodic measure is simply a minimal family whose index space is the one-point G-space. The criterion given in the previous lemma extends to the case of minimal equivariant families of measures:

Lemma 98 *Let μ_y be an equivariant family of measures. The family is minimal if and only if, for any equivariant family ν_y such that $\nu_y \ll \mu_y$ for all y, ν_y is either trivial or satisfies $\nu_y \sim \mu_y$ for all y.*

Proof: The 'only if' part of the statement is a direct application of Lemma 96; let us prove the 'if' part.

Suppose every equivariant family subordinate to μ_y is either zero or equivalent to μ_y. We first show that μ_y satisfies property (i) in Def. 97. Assuming the family μ_y is non-trivial, let Y_o be a G-orbit in Y on which $\mu_y \neq 0$. Define an equivariant family ν_y by setting $\nu_y = \mu_y$ if $y \in Y_o$ and 0 otherwise. This family is non-trivial and obviously satisfies $\nu_y \ll \mu_y$; this by hypothesis yields $\mu_y \sim \nu_y$. Therefore $\mu_y = 0$ for all $y \in Y - Y_o$.

We now turn to property (i) in Def. 97. Fix $y_o \in Y_o$, and let $S_o \subseteq G$ be its stabilizer. To show that μ_{y_o} is ergodic under the ation of S_o pick a measurable subset U such that $\mu_{y_o}(U \triangle Us) = 0$ for all $s \in S_o$. By equivariance of the family μ_y, this implies

$$\mu_{y_o g}(Ug \triangle Usg) \tag{79}$$

for all $s \in S_o$ and all $g \in G$. Then, for every $y = y_o g$ in Y_o, define a measure ν_y by setting $\nu_y(A) = \mu_y(A \cap Ug)$. This is well defined, since any g' such that $y = y_o g'$ is given by $g' = sg$ for some $s \in S_o$, and by (79) we have $\mu_y(A \cap Ug) = \mu_y(A \cap Usg)$.

The family ν_y is equivariant; indeed, for any $g \in G$, and $y = y_o g' \in Y_o$:

$$\nu_{yg}(Ag) = \mu_{yg}((A \cap Ug')g),$$

so equivariance of ν_y follows from that of μ_y. Since we also obviously have $\nu_y \ll \mu_y$ for all y, by hypothesis the family ν_y is either trivial, or satisfies $\nu_y \sim \mu_y$ for all y. In the former case, $\mu_{y_o}(U) = 0$; in the latter, $\mu_{y_o}(X - U) = 0$. Thus, μ_{y_o} is ergodic under the action of S_o. Since y_o was arbitrary, (ii) is proved, and the family μ_y is minimal. ∎

Transitive families of measures, for which there exists a G-orbit o in $Y \times X$ such that $\mu_y(A) = 0$ for every measurable $\{y\} \times A$ in the complement $Y \times X - o$, are particular examples of minimal families. Indeed, the obvious projection $Y \times X \to Y$ maps the orbit o into an orbit Y_o such that $\mu_y = 0$ unless $y \in Y_o$; furthermore for all $y \in Y_o$, μ_y is quasi-invariant under the action of the stabilizer S_y of y and concentrates on a single orbit, so it is clearly ergodic.

It is useful to investigate the converse: Is a minimal family of measures necessarily transitive?

This is not the case, in general. To understand this, we need to dwell further on the notion of quasi-invariant ergodic measure. First note that each orbit in X naturally defines a measure class of such measures: we indeed know that an orbit defines a measure class of quasi-invariant measures; now the uniqueness of such a class yields the minimality property stated in Lemma. 96, hence the ergodicity of the measures.

However, not every quasi-invariant and ergodic measure need belong to one of the classes defined by the orbits. In fact, given a measure μ on X, quasi-invariant and ergodic under a measurable G-action, there should be *at most* one orbit with positive measure, and its complement in X should be a null set. If there is an orbit with positive measure, μ belongs to the class that the orbit defines. But it may also very well be that *all* G-orbits are null sets. Consider for example the group $G = \mathbb{Z}$, acting on the unit circle $X = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane as $e^{i\theta} \mapsto e^{i\theta + \alpha\pi}$, where $\alpha \in \mathbb{R} - \mathbb{Q}$ is some fixed irrational number. It can be shown that the linear measure $d\theta$ on X is ergodic, whereas the orbits, which are all countable, are null sets.

This makes the classification of the equivalence classes of ergodic quasi-invariant measures quite difficult in general. Luckily, there is a simple criterion, stated in the following lemma, that precludes the kind of behaviour illustrated in the above example. For X a measurable G-space, we call a measurable subset $N \subset X$ a **measurable cross-section** if it intersects each G-orbit in exactly one point.

Lemma 99 **[71, Lemma 6.14]** *Let X be a measurable G-space. If X has a measurable cross-section, then any ergodic measure on X is supported on a single G-orbit.*

Roughly speaking, the existence of a measurable cross-section ensures that the orbit space is "nice enough". Thus, for example, making such assumption is equivalent to requiring that the orbit space is countably separated as a Borel space; or, in the case of a continuous group action, that it is a T_0 space [39].

Having introduced these concepts, we now begin our study of indecomposable, irretractable and irreducible intertwiners. Consider a pair of representations ρ_1 and ρ_2 on the measurable categories H^X and H^Y; denote by χ_1 and χ_2 the corresponding fields of characters. Let $\phi\colon \rho_1 \to \rho_2$ be an intertwiner; denote by μ_y the corresponding equivariant and fiberwise family μ_y of measures on X.

The following proposition gives a necessary condition for the intertwiner to be indecomposable (hence to be irretractable or irreducible):

Proposition 100 *If the intertwiner $\phi = (\phi, \Phi, \mu)$ is indecomposable, its family of measures μ_y is minimal.*

Proof: Assume ϕ is indecomposable, and consider an equivariant family of measures ν_y such that $\nu_y \ll \mu_y$ for all y. The Lebesgue decompositions:

$$\mu_y = \mu_y^{\nu_y} + \overline{\mu_y^{\nu_y}}$$

define two new fiberwise and equivariant families measures. Together with the μ^ν-classes of fields and the $\overline{\mu^\nu}$-classes of fields induced by the μ-classes of ϕ, these specify two intertwiners $\psi, \overline{\psi}$.

The measures $\mu_y^{\nu_y}$ and $\overline{\mu_y^{\nu_y}}$ are mutually singular for all y. Using the definition of the direct sum of intertwiners, we find

$$\phi = \psi \oplus \overline{\psi}.$$

Now, by hypothesis ϕ is indecomposable, so that either ψ or $\overline{\psi}$ is the null intertwiner. In the former case, the family μ^ν is trivial. This means that $\nu_y \perp \mu_y$ for all y; since, furthermore, $\nu_y \ll \mu_y$, it implies that ν is trivial. In the latter case, the family $\overline{\mu^\nu}$ is trivial. This mean that $\nu_y \sim \mu_y$ for all y. We conclude with Lemma 98 that the family μ_y is minimal. ∎

We can be more precise by focusing on the *transitive* intertwiners, as defined in Def. 63. Suppose the intertwiner $\phi\colon \rho_1 \to \rho_2$ is transitive, and specified by the assignments $\phi_{y,x}, \Phi^g_{y,x}$ of Hilbert spaces and invertible maps to the points of an G-orbit o in $Y \times X$. These define ordinary linear representations $\mathcal{R}^\phi_{y,x}$ of the stabilizer $S_{y,x}$ of (y,x) under the diagonal action of G.

The following propositions give a criterion for ϕ to be indecomposable, irretractable, or irreducible:

Proposition 101 (Indecomposable and irretractable transitive interwiners) *Let $\phi = (\phi, \Phi, \mu)$ be a transitive intertwiner. Then the following are equivalent:*

- *ϕ is indecomposable*
- *ϕ is irretractable*
- *the stabilizer representations $\mathcal{R}^\phi_{y,x}$ are indecomposable.*

Proposition 102 (Irreducible transitive interwiners) *Let $\phi = (\phi, \Phi, \mu)$ be a transitive intertwiner. Then ϕ is irreducible if and only if the stabilizer representations $\mathcal{R}^\phi_{y,x}$ are irreducible.*

Let us prove these two propositions together:

Proof: Fix a point $y_o \in Y$ such that $\mu_{y_o} \neq 0$, and let S_{y_o} be its stabilizer. The action of G on X induces an action of S_{y_o} on the fiber over $\chi_2(y_o)$ in X. Since by hypothesis ϕ is transitive, μ_{y_o} concentrates on a single S_{y_o}-orbit $\imath_o \subseteq X$. Next, fix x_o in $\imath_o$, and let $S_o = S_{y_o,x_o}$ denote stabilizer of (y_o, x_o) under the diagonal action. Let also $\phi_o = \phi_{y_o,x_o}, \Phi^g_o = \Phi^g_{y_o,x_o}$ be the space and maps assigned to the point (y_o, x_o), and let $\mathcal{R}^\phi_o$ be the corresponding linear representation $s \mapsto \Phi^s_o$ of S_o. Note that the representations $\mathcal{R}^\phi_{y,x}$ are all indecomposable (or irreducible) if $\mathcal{R}^\phi_o$ is.

We begin with Prop. 101. Suppose first that ϕ is indecomposable. Consider a Hilbert space decomposition $\phi_o = \phi'_o \oplus \phi''_o$ that is invariant under $\mathcal{R}^\phi_o$; assume that ϕ'_o is non-trivial. Given a point $(y,x) = (y_o, x_o)g^{-1}$ in the orbit, the isomorphism $\Phi^g_o\colon \phi_o \to \phi_{y,x}$ gives a splitting

$$\phi_{y,x} = \phi'_{y,x} \oplus \phi''_{y,x} \quad \text{where} \quad \phi'_{y,x} = \Phi^g_o(\phi'_o),\ \phi''_{y,x} = \Phi^g_o(\phi''_o)$$

This decomposition is independent of the representative of gS_o chosen, hence depends only on the point (y,x); indeed, for every $s \in S_o$, we have

$$\Phi_o^{gs}(\phi'_o) = \Phi_o^g \Phi_o^s(\phi'_o) = \Phi_o^g(\phi'_o) = \phi'_{y,x}$$

by invariance of ϕ'_o, and likewise for ϕ''. The decomposition of $\phi_{y,x}$ is also invariant under the representation $\mathcal{R}^{\phi}_{y,x}$ of $S_{y,x}$:

$$\Phi_{y,x}^s(\phi'_{y,x}) = \Phi_{y,x}^{sg}(\phi'_o) = \Phi^{s(g^{-1}sg)}(\phi'_o) = \Phi_o^g(\phi'_o) = \phi'_{y,x}$$

since for any $s \in S_{y,x}$, we have $(g^{-1}sg) \in S_o$, and likewise for $\phi''_{y,x}$. These data give us a transitive intertwiner $\phi' = (\phi'_{y,x}, \Phi'^g_{y,x}, \mu_y)$, where $\Phi'^g_{y,x}$ simply denotes the restriction of $\Phi^g_{y,x}$ to $\phi'_{y,x}$.

By construction, ϕ' is a summand of ϕ, distinct from the null intertwiner. Now, we have assumed ϕ is indecomposable; so we have that $\phi' \simeq \phi$. We then deduce from Prop. 71 that the representation $\mathcal{R}^{\phi}_o$ is equivalent to its restriction to ϕ'_o. Thus, $\mathcal{R}^{\phi}_o$ is indecomposable.

Next, suppose that the linear representations $\mathcal{R}^{\phi}_{y,x}$ are indecomposable. We will show that ϕ is irretractable; since an irretractable is automatically indecomposable, this will complete the proof of Prop. 101.

Let ϕ' be a retract of ϕ, specified by the family of measures μ'_y and the assignments of Hilbert spaces $\phi'_{y,x}$ and invertible maps $\Phi'^g_{y,x}$. By definition one can find 2-intertwiners $m\colon \phi \Rightarrow \phi'$ and $n\colon \phi' \Rightarrow \phi$ such that $n \cdot m = \mathbb{1}_{\phi'}$. This last equality requires that the geometric mean measures $\sqrt{\mu_y \mu'_y}$ be equivalent to μ'_y, or equivalently that $\mu'_y \ll \mu_y$. Hence, the S_o-quasi-invariant measure μ'_{y_o} concentrates on the orbit $\imath_o$. Non-trivial S_o-quasi-invariant measures on $\imath_o$ are unique up to equivalence, so we conclude that μ'_{y_o} is either trivial or equivalent to μ_{y_o}. In the first case, ϕ' is trivial, so we are done.

In the second case, where $\mu'_y \sim \mu_y$ for all y, the linear maps $m_{y,x}\colon \phi'_{y,x} \to \phi_{y,x}$ and $n_{y,x}\colon \phi_{y,x} \to \phi'_{y,x}$ are intertwining operators between the representations $\mathcal{R}^{\phi'}_{y,x}$ and $\mathcal{R}^{\phi}_{y,x}$; they satisfy $n_{y,x} m_{y,x} = \mathbb{1}_{\phi'_{y,x}}$. Thus, $\mathcal{R}^{\phi'}_{y,x}$ is a retract of $\mathcal{R}^{\phi}_{y,x}$, hence a direct summand. But $\mathcal{R}^{\phi}_{y,x}$ is indecomposable: so the two representations must be equivalent. Hence the map $m_{y,x}$ is invertible. The 2-intertwiner $m\colon \phi' \Rightarrow \phi$ is thus invertible, which shows ϕ' and ϕ are equivalent. We conclude that ϕ is irretractable.

We now prove Prop. 102. Suppose first that ϕ is irreducible. Consider a non-trivial subspace $\phi'_o \subset \phi_o$ that is invariant under $\mathcal{R}^{\phi}_o$. Given a point $(y,x) = (y_o, x_o)g^{-1}$ in the orbit, the isomorphism $\Phi^g_o\colon \phi_o \to \phi_{y,x}$ gives a subspace $\phi'_{y,x} := \Phi^g_{y,x}(\phi'_o)$ of $\phi_{y,x}$ that is invariant under $\mathcal{R}^{\phi}_{y,x}$. These data give us a transitive intertwiner $\phi' = (\phi'_{y,x}, \Phi'^g_{y,x}, \mu_y)$, where $\Phi'^g_{y,x}$ simply denotes the restriction of $\Phi^g_{y,x}$ to $\phi'_{y,x}$.

The canonical injections $\imath_{y,x}\colon \phi'_{y,x} \to \phi_{y,x}$ define a monic 2-intertwiner $\imath\colon \phi' \to \phi$; this shows that ϕ' is a sub-intertwiner of ϕ. But ϕ is irreducible: we therefore have that $\phi' \simeq \phi$. This means that $\mathcal{R}^{\phi}_o$ is equivalent to its restriction to ϕ'_o. Thus, $\mathcal{R}^{\phi}_o$ is irreducible.

Conversely, suppose that the representations $\mathcal{R}^{\phi}_{y,x}$ are irreducible. Consider a sub-intertwiner ϕ' of ϕ, giving a family of measures μ'_y. First of all, note that the existence of a monic 2-intertwiner $m\colon \phi' \to \phi$ forces μ' to be transitive, with $\bar{\mu}_y$ supported on the orbit o. In particular, we have that $\mu'_y \sim \mu_y$ for all y.

Next, fix a monic 2-intertwiner m. It gives injective linear maps $m_{y,x}\colon \phi'_{y,x} \to \phi_{y,x}$; these define subspaces $m_{y,x}(\phi'_{y,x})$ in $\phi_{y,x}$ that are invariant under $\mathcal{R}^{\phi}_{y,x}$. Since the representations are by hypothesis irreducible, this means that the maps $m_{y,x}$, and hence m, are invertible. We obtain that $\phi' \simeq \phi$, and conclude that ϕ is irreducible. ∎

These results allow us to classify, up to equivalence, the indecomposable and irreducible intertwiners between fixed measurable representations ρ_1, ρ_2. We may assume that these representations are indecomposable, and given by the pairs (o, S_1) and (o, S_2). They are thus specified by the G-equivariant bundles $X = G/S_1$ and $Y = G/S_2$ over the same G-orbit $o \simeq G/S_o^*$ in H^*; S_1 and S_2 are some closed subgroups of S_o^*. In the following, we denote $y_o = S_2 e$, and fix a (not necessarily measurable) cross-section of the S_2-space S_o^*/S_1 – namely, a subset that intersects each S_2-orbit $\imath_o$ in exactly one point $x_o := S_1 k_o$.

Let $\phi\colon \rho_1 \to \rho_2$ an indecomposable (resp. irreducible) intertwiner. We will assume that ϕ is *transitive*, keeping in mind the following consequence of Prop. 100 and Lemma 99:

Lemma 103 *Suppose that the S_2-space S_o^*/S_1 has a measurable cross-section. Then every indecomposable intertwiner $\phi\colon (o, S_1) \to (o, S_2)$ is transitive.*

The intertwiner ϕ gives a non-trivial S_2-quasi-invariant measure μ_{y_o} in the fiber $S_o^*/S_1 \subset X$ over $S_o^* e$. Moreover, the transitivity of ϕ implies that this measure is supported on a single S_2-orbit $\imath_o^\phi$ in S_o^*/S_1. Note that any two such measures are equivalent. ϕ also gives an indecomposable (resp. irreducible) linear representation $\mathcal{R}_o^\phi$ of the group

$$S_o = k_o^{-1} S_1 k_o \cap S_2.$$

So, ϕ gives a pair $(\imath_o^\phi, \mathcal{R}_o^\phi)$, where $\imath_o^\phi$ is a S_2-orbit in S_o^*/S_1 and $\mathcal{R}_o^\phi$ is an indecomposable (resp. irreducible) representation of S_o. We easily deduce from Prop. 72 that two equivalent transitive intertwiners give two pairs with the same orbit and equivalent linear representations.

Conversely, given any orbit $\imath_o$ and any linear representation $\mathcal{R}_o$ of S_o on some Hilbert space ϕ_o, there is an intertwiner $\phi = (\phi, \Phi, \mu)$ such that $\imath_o^\phi = \imath_o$ and $\mathcal{R}_o^\phi = \mathcal{R}_o$. Indeed, a measurable equivariant and fiberwise family of measures is obtained by choosing a S_2-quasi-invariant measure μ_o supported on $\imath_o$ and a measurable section $n\colon G/S_2 \to G$, and by setting, for each $y = y_o n(k)$:

$$\mu_y := \mu_o^{n(k)}$$

To construct the measurable fields of spaces and linear maps, fix a measurable section $\bar{n}\colon G/S_o \to G$, denote by $\pi\colon G \to G/S_o$ the canonical projection, and consider the function $\alpha\colon G \to S_o$ given by:

$$\alpha(g) = (\bar{n}\pi)(g^{-1})g.$$

This function satisfies the property that $\alpha(gs) = \alpha(g)s$ for all $s \in S_o$. Using this, we define a family Φ_o^g of isomorphisms of ϕ_o as:

$$\Phi_o^g = \mathcal{R}_o(\alpha(g))$$

and construct a measurable field $\Phi_{y,x}^g$ by setting, for each $(y, x) = (y_o, x_o)k^{-1}$:

$$\Phi_{y,x}^g = \Phi_o^g (\Phi_o^k)^{-1}.$$

These data specify a transitive intertwiner ϕ; this intertwiner is indecomposable (resp. irreducible) if $\mathcal{R}_o$ is.

These remarks yield the following:

Corollary 104 *Indecomposable (resp. irreducible) transitive intertwiners $\phi\colon (o, S_1) \to (o, S_2)$ are classified, up to equivalence, by a choice of a S_2-orbit $\imath_o$ in S_o^*/S_1, along with an equivalence class of indecomposable (resp. irreducible) linear representations $\mathcal{R}_o$ of the group $k_o^{-1} S_1 k_o \cap S_2$.*

We close this section with a version of Schur's lemma for irreducible intertwiners:

Proposition 105 (Schur's Lemma for Intertwiners) *Let $\phi, \psi\colon (o, S_1), \to (o, S_2)$ be two irreducible transitive intertwiners. Then any 2-intertwiner $m\colon \phi \Rightarrow \psi$ is either null or an isomorphism. In the latter case, m is unique, up to a normalization factor.*

Proof: We may assume ϕ and ψ are given by the pairs $(\imath_o^\phi, \mathcal{R}_o^\phi)$ and $(\imath_o^\psi, \mathcal{R}_o^\psi)$ of S_2-orbits in S_o^*/S_1 and irreducible linear representations. Let μ_y and ν_y denote the two families of measures. If the orbits are distinct $\imath_o^\phi \neq \imath_o^\psi$, the measures μ_y and ν_y have disjoint support, so that their geometric mean is trivial. In this case, any 2-intertwiner $m\colon \phi \Rightarrow \psi$ is trivial.

Suppose now $\imath_o^\phi = \imath_o^\psi$. In this case, we have that $\mu_y \sim \nu_y$ for all y. Let $m\colon \phi \Rightarrow \psi$ be a 2-intertwiner, given by the assignment of linear maps $m_{y,x}\colon \phi_{y,x} \to \psi_{y,x}$. Because of the intertwining rule (69), the assignment is entirely specified by the data $m_o := m_{y_o,x_o}$.

Now, m_o defines a standard intertwiner between the irreducible linear representations $\mathcal{R}_o^\phi$ and $\mathcal{R}_o^\psi$. Therefore m_o, and hence m, is either trivial or invertible; in the latter case, it is unique, up to a normalization factor. ∎

5 Conclusion

We conclude with some possible avenues for future investigation. First, it will be interesting to study examples of the general theory described here. As explained in the Introduction, representations of the Poincaré 2-group have already been studied by Crane and Sheppeard [26], in view of obtaining a 4-dimensional state sum model with possible relations to quantum gravity. Representations of the Euclidean 2-group (with $G = \mathrm{SO}(4)$ acting on $H = \mathbb{R}^4$ in the usual way) are somewhat more tractable. Copying the ideas of Crane and Sheppeard, this 2-group gives a state sum model [10, 11] with interesting relations to the more familiar Ooguri model.

There are also many other 2-groups whose representations are worth studying. For example, Bartlett has studied representations of finite groups G, regarded as 2-groups with trivial H [17]. He considers *weak* representations of these 2-groups, where composition of 1-morphisms is preserved only up to 2-isomorphism. More precisely, he considers *unitary* weak representations on finite-dimensional 2-Hibert spaces. These choices lead him to a beautiful geometrical picture of representations, intertwiners and 2-intertwiners — strikingly similar to our work here, but with U(1) gerbes playing a major role. So, it will be very interesting to generalize our work to weak representations, and specialize it to unitary ones.

To define *unitary* representations of measurable 2-groups, we need them to act on something with more structure than a measurable category: namely, some sort of infinite-dimensional 2-Hilbert space. This notion has not yet been defined. However, we may hazard a guess on how the definition should go.

In Section 3.3, we argued that the measurable category H^X should be a categorified analogue of $L^2(X)$, with direct integrals replacing ordinary integrals. However, we never discussed the inner product in H^X. We can define this only after choosing a measure μ on X. This measure appears in the formula for the inner product of vectors $\psi, \phi \in L^2(X)$:

$$\langle \psi, \phi \rangle = \int \overline{\psi}(x)\phi(x)\, d\mu(x) \in \mathbb{C}.$$

Similarly, we can use it to define the **inner product** of fields of Hilbert spaces $\mathcal{H}, \mathcal{K} \in H^X$:

$$\langle \mathcal{H}, \mathcal{K} \rangle = \int^{\oplus} \overline{\mathcal{H}}(x) \otimes \mathcal{K}(x)\, d\mu(x) \in \mathrm{Hilb}.$$

Here $\overline{\mathcal{H}}(x)$ is the complex conjugate of the Hilbert space $\mathcal{H}(x)$, where multiplication by i has been redefined to be multiplication by $-i$. This is naturally isomorphic to the Hilbert space dual $\mathcal{H}(x)^*$, so we can also write

$$\langle \mathcal{H}, \mathcal{K} \rangle \cong \int^{\oplus} \mathcal{H}(x)^* \otimes \mathcal{K}(x)\, d\mu(x).$$

Recall that throughout this paper we are assuming our measures are σ-finite; this guarantees that the Hilbert space $\langle \mathcal{H}, \mathcal{K} \rangle$ is *separable*. So, we may give a preliminary definition of a 'separable 2-Hilbert space' as a category of the form H^X where X is a measurable space equipped with a measure μ.

As a sign that this definition is on the right track, note that when X is a finite set equipped with a measure, H^X is a finite-dimensional 2-Hilbert space as previously defined [3]. Moreover, every finite-dimensional 2-Hilbert space is equivalent to one of this form [17, Sec. 2.1.2].

The main thing we lack in the infinite-dimensional case, which we possess in the finite-dimensional case, is an intrinsic definition of a 2-Hilbert space. An intrinsic definition should not refer to the measurable space X, since this space merely serves as a 'choice of basis'. The problem is that it seems tricky to define direct integrals of objects without mentioning this space X.

The same problem afflicted our treatment of measurable categories. Instead of giving an intrinsic definition of measurable categories, we defined a measurable category to be a C^*-category that is C^*-equivalent to H^X for some measurable space X. This made the construction of **Meas** rather roundabout. We could try a similar approach to defining a 2-category of separable 2-Hilbert spaces, but it would be equally roundabout.

Luckily there is another approach, essentially equivalent to the one just presented, that does not mention measure spaces or measurable categories! In this approach, we think of a 2-Hilbert space as a category of representations of a commutative von Neumann algebra.

The key step is to notice that when μ is a measure on a measurable space X, the algebra $L^\infty(X, \mu)$ acts as multiplication operators on $L^2(X, \mu)$. Using this one can think of $L^\infty(X, \mu)$ as a commutative von Neumann algebra of operators on a separable Hilbert space. Conversely, any commutative von Neumann algebra of operators on a separable Hilbert space is isomorphic—as a C^*-algebra—to one of this form [29, Part I, Chap. 7, Thm. 1]. The technical conditions built into our definition of 'measurable space' and 'measure' are precisely what is required to make this work (see Defs. 15 and 16).

This viewpoint gives a new outlook on fields of Hilbert spaces. Suppose A is commutative von Neumann algebra of operators on a separable Hilbert space. As a C^*-algebra, we may identify A with $L^\infty(X, \mu)$ for some measure μ on a measurable space X. Define a **separable representation** of A to be a representation of A on a separable Hilbert space. It can then be shown that every separable representation of A is equivalent to the representation of $L^\infty(X, \mu)$ as multiplication operators on $\int^\oplus \mathcal{H}(x) d\mu(x)$ for some field of Hilbert spaces $\mathcal{H}$ on X. Moreover, this field $\mathcal{H}$ is essentially unique [29, Part I, Chap. 6, Thms. 2 and 3].

This suggests that we define a **separable 2-Hilbert space** to be a category of separable representations of some commutative von Neumann algebra of operators on a separable Hilbert space. More generally, we could drop the separability condition and define a **2-Hilbert space** to be a category of representations of a commutative von Neumann algebra.

While elegant, this definition is not quite right. Any category 'equivalent' to the category of representations of a commutative von Neumann algebra—in a suitable sense of 'equivalent', probably stronger than C^*-equivalence—should also count as a 2-Hilbert space. A better approach would give an intrinsic characterization of categories of this form. Then it would become a *theorem* that every 2-Hilbert space is equivalent to the category of representations of a commutative von Neumann algebra.

Luckily, there is yet another simplification to be made. After all, a commutative von Neumann algebra can be recovered, up to isomorphism, from its category of representations. So, we can forget the category of representations and focus on the von Neumann algebra itself!

The problem is then to redescribe morphisms between 2-Hilbert spaces, and 2-morphisms between these, in the language of von Neumann algebras. There is a natural guess as to how this should work, due to Urs Schreiber. Namely, we can define a bicategory **2Hilb** for which:

- objects are commutative von Neumann algebras $A, B, \ldots$,
- a morphism $\mathcal{H}: A \to B$ is a Hilbert space $\mathcal{H}$ equipped with the structure of a (B, A)-bimodule,
- a 2-morphism $f: \mathcal{H} \to \mathcal{K}$ is a homomorphism of (B, A)-bimodules.

Composition of morphisms corresponds to tensoring bimodules. Note also that given an (B, A)-bimodule and a representation of A, we can tensor the two and get a representation of B. This is how an (B, A)-bimodule gives a functor from the category of representations of A to the category of representations of B. Similarly, a homomorphism of (B, A)-bimodules gives a natural transformation between such functors.

Let us briefly sketch the relation between this version of **2Hilb** and the 2-category **Meas** described in this paper. First, given separable commutative von Neumann algebras A and B, we can write $A \cong L^\infty(X, \mu)$ and $B \cong L^\infty(Y, \nu)$ where X, Y are measurable spaces and μ, ν are measures. Then, given an (B, A)-bimodule, we can think of it as a representation of $B \otimes A \cong L^\infty(Y \times X, \nu \otimes \mu)$. By the remarks above, this representation comes from a field of Hilbert spaces on $Y \times X$. Then, given a 2-morphism $f\colon \mathcal{H} \to \mathcal{K}$, we can represent it as a measurable field of bounded operators between the corresponding fields of Hilbert spaces.

While the details still need to be worked out, all this suggests that a theory of 2-Hilbert spaces based on commutative von Neumann algebras should be closely linked to the theory of measurable categories described here.

Even better, the bicategory 2Hilb just described sits inside a larger bicategory where we drop the condition that the von Neumann algebras be commutative. Representations of 2-groups in this larger bicategory should also be interesting. The reason is that Schreiber has convincing evidence that the work of Stolz and Teichner [70] provides a representation of the so-called 'string 2-group' [6] inside this larger bicategory. For details, see the last section of Schreiber's recent paper on two approaches to quantum field theory [69]. This is yet another hint that infinite-dimensional representations of 2-groups may someday be useful in physics.

A Tools from measure theory

This Appendix summarizes some tools of measure theory used in the paper. The first section recalls basic terminology and states the well-known Lebesgue decomposition and Radon-Nikodym theorems. The second section defines the geometric mean of two measures and derives of some of its key properties. The third section studies measurable abelian groups and their duals. Finally, the fourth section presents a few standard results about measure theory on G-spaces.

Recall that for us, a **measurable space** is shorthand for a **standard Borel space**: that is, a set X with a σ-algebra $\mathcal{B}$ of subsets generated by the open sets for some second countable locally compact Hausdorff topology on X. We gave two other equivalent definitions of this concept in Prop. 14.

Also recall that for us, all measures are σ-finite. So, a **measure** on X is a function $\mu\colon \mathcal{B} \to [0, +\infty]$ such that

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$$

for any sequence $(A_n)_{n\in\mathbb{N}}$ of mutually disjoint measurable sets, such that X is a countable union of $S_i \in \mathcal{B}$ with $\mu(S_i) < \infty$.

A.1 Lebesgue decomposition and Radon-Nikodym derivatives

In a fixed measurable space X, a measure t is **absolutely continuous** with respect to a measure u, written $t \ll u$, if every u-null set is also t-null. The measures are **equivalent**, written $t \sim u$, if they are absolutely continuous with respect to each other: in other words, they have the same null sets. The two measures are **mutually singular**, written $t \perp u$, if we can find a measurable set $A \subseteq X$ such that

$$t(A) = u(X - A) = 0.$$

If $A \subseteq X$ is a measurable set with $u(X - A) = 0$ we say the measure u is **supported** on A.

Theorem 106 (Lebesgue decomposition) *Let t and u be (σ-finite) measures on X. Then there is a unique pair of measures t^u and $\overline{t^u}$ such that*

$$t = t^u + \overline{t^u} \qquad \textit{with} \quad t^u \ll u \quad \textit{and} \quad \overline{t^u} \perp u.$$

The notation chosen here is particularly useful when we have more than two measures around and need to distinguish between Lebesgue decompositions with respect to different measures.

This result is completed by the following useful propositions. Fix two measures t and u on X.

Proposition 107 *In the Lebesgue decomposition $t = t^u + \overline{t^u}$, we have $t^u \perp \overline{t^u}$.*

Proof: Given that $\overline{t^u} \perp u$, there is a measurable set A such that u is supported on A and $\overline{t^u}$ is supported on $X - A$:

$$u(S) = u(S \cap A) \qquad \overline{t^u}(S) = \overline{t^u}(S - A)$$

for all measurable sets S. But then absolute continuity of t^u with respect to u implies $t^u(X - A) = 0$, and therefore $t^u(S) = t^u(S \cap A)$. That is, t^u is supported on A, so $t^u \perp \overline{t^u}$. ∎

Proposition 108 *Consider the Lebesgue decompositions $t = t^u + \overline{t^u}$ and $u = u^t + \overline{u^t}$. Then $\overline{t^u} \perp \overline{u^t}$ and $t^u \sim u^t$.*

Proof: Given that $\overline{t^u} \perp u$, there is a measurable set A such that u is supported on A and $\overline{t^u}$ is supported on $X - A$. Note first that $\overline{u^t}$ is supported on A, as u is. This shows that $\overline{t^u} \perp \overline{u^t}$.

Next, fix a t^u-null set S; we thus have that $t(S) = \overline{t^u}(S)$. Since $\overline{t^u}$ is supported on $X - A$, it follows that $t(S \cap A) = 0$. Using the fact that $u^t \ll t$, we obtain $u^t(S \cap A) = 0$. But u^t is supported on A, as u is; therefore $u^t(S) = u^t(S \cap A) = 0$. Thus, we have shown that $u^t \ll t^u$. We show similarly $t^u \ll u^t$, and conclude that $t^u \sim u^t$. ∎

The Lebesgue decomposition theorem is refined by the Radon–Nikodym theorem, which provides a classification of absolutely continuous measures:

Theorem 109 (Radon-Nikodym) *Let t and u be two σ-finite measures on X. Then $t \ll u$ if and only if t can be written as u times a function $\frac{dt}{du}$, the* **Radon–Nikodym derivative***: that is,*

$$t(A) = \int_A du \, \frac{dt}{du}$$

A.2 Geometric mean measure

Suppose X is a measurable space on which are defined two measures, u and t. If each measure is absolutely continuous with respect to the other, then we have the equality

$$\sqrt{\frac{dt}{du}} du = \sqrt{\frac{du}{dt}} dt$$

so we can define the 'geometric mean' $\sqrt{dudt}$ of the two measures to be given by either side of this equality. In the more general case, where u and t are not necessarily mutually absolutely continuous, we may still define $\sqrt{dudt}$, as we shall see.

Using the notation of the first section we have the following key fact. Recall once more that all our measures are assumed σ-finite.

Proposition 110 *If u and t are measures on the same measurable space X then*

$$\sqrt{\frac{dt^u}{du}} du = \sqrt{\frac{du^t}{dt}} dt$$

Proof: Our notation for the Lebesgue decomposition means

$$u = u^t + \overline{u^t} \qquad u^t \ll t \quad \overline{u^t} \perp t$$

and likewise,

$$t = t^u + \overline{t^u} \qquad t^u \ll u \quad \overline{t^u} \perp u.$$

Prop. 107 shows that u^t and $\overline{u^t}$ are mutually singular. So there is a measurable set A with t and u^t are supported on A and, and $\overline{u^t}$ supported on $X - A$. Similarly, there is a measurable set B with u and t^u supported on B, and $\overline{t^u}$ supported on $X - B$. These sets divide X into four subsets: $A \cap B$, $A - B$, $B - A$, and $X - (B \cup A)$. The uniqueness of the Lebesgue decomposition implies the decomposition of the restriction of a measure is given by the restriction of the decomposition. On $A \cap B$, u and t restrict to u^t and t^u, which are mutually absolutely continuous. Hence, on this subset, we have

$$\sqrt{\frac{dt^u}{du}} du = \sqrt{\frac{du^t}{dt}} dt$$

On the other three subsets of X, we have, respectively $u = 0$, $t = 0$, and $u = t = 0$. In each case, both sides of the previous equation are zero. ∎

Given this proposition, we define the **geometric mean** of the measures u and t to be:

$$\sqrt{dt du} := \sqrt{\frac{dt^u}{du}} du = \sqrt{\frac{du^t}{dt}} dt$$

Outside of this appendix, to reduce notational clutter, we generally drop the superscripts in Radon–Nikodym derivatives and simply write, for example:

$$\frac{dt}{du} := \frac{dt^u}{du}.$$

Proposition 111 *Let t, u be measures on X. Then a set is $\sqrt{tu}$-null if and only if it is the union of a t-null set and a u-null set. Equivalently, expressed in terms of almost-everywhere equivalence, the relation '$\sqrt{tu}$-a.e.' is the transitive closure of the union of the relations 't-a.e.' and 'u-a.e.'.*

Proof: First, $\sqrt{tu} \ll t$ and $\sqrt{tu} \ll u$; indeed $\sqrt{tu}$ is equivalent to both t^u and u^t. So clearly the union of a t-null set and a u-null set is also $\sqrt{tu}$-null.

Conversely, suppose $D \subseteq X$ has $\sqrt{tu}(D) = 0$. Then $u(D) = u^{\bar{t}}(D)$, and $t(D) = t^{\bar{u}}(D)$. But $u^{\bar{t}} \perp t^{\bar{u}}$, so we can pick a set $P \subseteq X$ on which $u^{\bar{t}}$ is supported and $t^{\bar{u}}$ vanishes. Then $t(D \cap P) = 0$ and $u(D - P) = 0$, so D is the union of a t-null set and a u-null set.

Expressing this in terms of equivalence relations, suppose $f_1(x) = f_2(x)$ $\sqrt{tu}$-*a.e.* in the variable x; we will construct $g(x)$ such that $g(x) = f_1(x)$ t-*a.e.* and $g(x) = f_2(x)$ u-*a.e.*. Let D be the set on which f_1 and f_2 differ, and let P be the set defined in the previous paragraph. Set $g(x) := f_1(x) = f_2(x)$ on $X - D$, $g(x) := f_1(x)$ on $D - P$, and $g(x) := f_2(x)$ on $D \cap P$. This defines g on all of x. Now f_1 and g differ only on $D \cap P$, which has t-measure 0; f_2 and g differ only on $D - P$, which has u-measure 0. ∎

Now suppose we have three measures t, u, and v on the same space. How are the geometric means $\sqrt{dtdu}$ and $\sqrt{dtdv}$ related? An answer to this question is given by the following lemma, which is useful for rewriting an integral with respect to one of these geometric means as an integral with respect the other.

Lemma 112 *Let t, u, and v be measures on X. Then we have an equality of measures*

$$\sqrt{dtdu}\sqrt{\frac{dv^u}{du}} = \sqrt{dtdv}\sqrt{\frac{dv^t}{dt}}\sqrt{\frac{du^v}{dv}}\sqrt{\frac{dt^u}{du}}$$

Proof: Let us first define a measure μ by the left side of the desired equality:

$$d\mu = \sqrt{dtdu}\sqrt{\frac{dv^u}{du}}$$

We then have, using the definition of geometric mean measure,

$$\begin{aligned} d\mu &= dt\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dt^u}{du}} \\ &= (dt^v + dt^{\bar{v}})\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dt^u}{du}} \end{aligned}$$

where the latter expression gives the Lebesgue decomposition of μ with respect to v. However, as we show momentarily, the singular part of this decomposition is identically zero. Assuming this result for the moment, we then have

$$\begin{aligned} d\mu &= dt^v \sqrt{\frac{du^t}{dt}}\sqrt{\frac{dt^u}{du}} \\ &= dv\frac{dt^v}{dv}\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dt^u}{du}} \\ &= \sqrt{dtdv}\sqrt{\frac{dv^t}{dt}}\sqrt{\frac{du^v}{dv}}\sqrt{\frac{dt^u}{du}} \end{aligned}$$

as we wished to show. To complete the proof, we thus need only see that the $\overline{t^v}$ part of μ vanishes:

$$d\overline{t^v}\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dv^u}{du}} = 0$$

That is, we must show that

$$\overline{\mu^v}(X) = \int_X d\overline{t^v}\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dv^u}{du}} = 0.$$

Let $Y \subseteq X$ be a measurable set such that $\overline{t^v}$ is supported on Y, while v and t^v are supported on its complement:

$$v = v|_{X-Y} \qquad t^v = t^v|_{X-Y} \qquad \overline{t^v} = \overline{t^v}|_Y$$

Similarly, let $A \subseteq X$ be such that

$$t = t|_A \qquad u^t = ut|_A \qquad \overline{u^t} = \overline{u^t}|_{X-A}$$

Note that

$$\frac{du^t}{dt}$$

vanishes t–*a.e.*, and hence $\overline{t^v}$–*a.e.* on $X - A$. Thus the measure

$$d\overline{t^v}\sqrt{\frac{du^t}{dt}}$$

is zero on $X - A$. Since we also have $\overline{t^v}$ vanishing on $X - Y$, we have

$$\overline{\mu^v}(X) = \int_{Y\cap A} d\overline{t^v}\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dv^u}{du}}.$$

Now by construction of Y, we have $v(Y \cap A) = 0$, and hence $v^u(Y \cap A) = 0$. So

$$\frac{dv^u}{du}$$

vanishes u–*a.e.*, and hence u^t–*a.e.*, on $Y \cap A$. If $C \subseteq Y \cap A$ is the set of points where the latter Radon–Nikodym derivative does not vanish, then $u^t(C) = 0$ implies that

$$\sqrt{\frac{du^t}{dt}}$$

vanishes t–*a.e.*, hence $\overline{t^v}$–*a.e.* on C. Thus

$$\overline{\mu^v}(X) = \int_C d\overline{t^v} \sqrt{\frac{du^t}{dt}} \sqrt{\frac{dv^u}{du}} = 0,$$

so μ is absolutely continuous with respect to v. ∎

Proposition 113 *Let t, u be measures on X, and consider the Lebesgue decompositions $t = t^u + \overline{t^u}$ and $u = u^t + \overline{u^t}$. Then:*

$$\frac{du^t}{dt}\frac{dt^u}{du} = 1 \qquad \sqrt{tu} - a.e.$$

Proof: Applying Lemma 112 with $v = u$ we get

$$\sqrt{dtdu} = \sqrt{dtdu}\sqrt{\frac{du^t}{dt}}\sqrt{\frac{dt^u}{du}}$$

Thus the function $\frac{du^t}{dt}\frac{dt^u}{du}$ differs from 1 at most on a set of $\sqrt{tu}$-measure zero. ∎

A.3 Measurable groups

Given a measurable group H, it is natural to ask whether H^* is again a measurable group. The main goal of this section is to present necessary and sufficient conditions for this to be so. These conditions are due to Yves de Cornulier and Todd Trimble. We also show that when H and H^* are measurable, a continuous action of a measurable group G on H gives a continuous action of G on H^*.

Recall that for us, a **measurable group** is a locally compact Hausdorff second countable topological group. Any measurable group becomes a measurable space with its σ-algebra of Borel subsets. The multiplication and inverse maps for the group are then measurable. However, not every measurable space that is a group with measurable multiplication and inverse maps can be promoted to a measurable group in our sense! There may be no second countable locally compact Hausdorff topology making these maps continuous. Luckily, all the counterexamples are fairly exotic [19, Sec. 1.6].

Lemma 114 *A measurable homomorphism between measurable groups is continuous.*

Proof: Various proofs can be found in the literature. For example, Kleppner showed that a measurable homomorphism between locally compact groups is automatically continuous [46]. ∎

Given a measurable group H, we let H^* be the set of measurable — or equivalently, by Lemma 114, continuous — homomorphisms from H to $\mathbb{C}^\times$. We make H^* into a topological space with the compact-open topology. H^* then becomes a topological group under pointwise multiplication.

The first step in analyzing H^* is noting that every continuous homomorphism $\chi\colon H \to \mathbb{C}^\times$ is trivial on the commutator subgroup $[H, H]$ and thus also on its closure $\overline{[H, H]}$. This lets us reduce the problem from H to

$$\mathrm{Ab}(H) = H/\overline{[H, H]},$$

which becomes a topological group with the quotient topology. Let $\pi\colon H \to \mathrm{Ab}(H)$ be the quotient map. Then we have:

Lemma 115 *Suppose H is a measurable group. Then* $\mathrm{Ab}(H)$ *is a measurable group.* $\mathrm{Ab}(H)^*$ *is a measurable group if and only if H^* is, and in this case the map*

$$\begin{array}{rccc} \pi^*\colon & \mathrm{Ab}(H)^* & \to & H^* \\ & \chi & \mapsto & \chi\pi \end{array}$$

is an isomorphism of measurable groups.

Proof: Suppose H is a measurable group: that is, a second countable locally compact Hausdorff group. By Lemma 122, the quotient $\mathrm{Ab}(H)$ is a second countable locally compact Hausdorff space because the subgroup $\overline{[H,H]}$ is closed. So, $\mathrm{Ab}(H)$ is a measurable group.

The map π^* is a bijection because every continuous homomorphism $\phi\colon H \to \mathbb{C}^\times$ equals the identity on $\overline{[H,H]}$ and thus can be written as $\chi\pi$ for a unique continuous homomorphism $\phi\colon \mathrm{Ab}(H) \to \mathbb{C}^\times$. We can also see that π^* is continuous. Suppose a net $\chi_\alpha \in \mathrm{Ab}(H)^*$ converges uniformly to $\chi \in \mathrm{Ab}(H)^*$ on compact subsets of $\mathrm{Ab}(H)$. Then if $K \subseteq H$ is compact, $\chi_\alpha\pi$ converges uniformly to $\chi\pi$ on K because χ_a converges uniformly to χ on the compact set $\pi(K)$.

It follows that $\pi^*\colon \mathrm{Ab}(H)^* \to H^*$ is a continuous bijection between second countable locally compact Hausdorff spaces. This induces a measurable bijection between measurable spaces. Such a map always has a measurable inverse [59, Chap. I, Cor. 3.3]. (This reference describes measurable spaces in terms of separable metric spaces, but we have seen in Lemma 14 that this characterization is equivalent to the one we are using here.) So, π^* is an isomorphism of measurable spaces. Since it is a group homomorphism, it is also an isomorphism of measurable groups. ∎

Thanks to the above result, we henceforth assume H is an abelian measurable group. Since

$$\mathbb{C}^\times \cong \mathrm{U}(1) \times \mathbb{R}$$

as topological groups, we have

$$H^* \cong \hom(H, \mathrm{U}(1)) \times \hom(H, \mathbb{R})$$

as topological groups, where hom denotes the space of continuous homomorphisms equipped with its compact-open topology and made into a topological group using pointwise multiplication. The topological group

$$\hat{H} = \hom(H, \mathrm{U}(1))$$

is the subject of Pontrjagin duality so this part of H^* is well-understood [1, 56, 60]. In particular:

Lemma 116 *If H is an abelian measurable group, so is its Pontrjagin dual $\hat{H}$.*

Proof: It is well-known that whenever H is an abelian locally compact Hausdorff group, so is $\hat{H}$ [56, Thm. 10]. So, let us assume in addition that H is second countable, and show the same for $\hat{H}$.

For this, first note by Lemma 14 that H is metrizable. A locally compact second-countable space is clearly σ-compact, so H is also σ-compact. Second, note that a locally compact Hausdorff abelian group H is metrizable if and only $\hat{H}$ is σ-compact [56, Thm. 29].

It follows that $\hat{H}$ is also σ-compact and metrizable. Since a compact metric space is second countable (for each n it admits a finite covering by balls of radius $1/n$), so is a σ-compact metric space. It follows that $\hat{H}$ is second countable. ∎

The issue thus boils down to: if H is an abelian measurable group, is $\hom(H, \mathbb{R})$ also measurable? Sadly, the answer is "no". Suppose H is the free abelian group on countably many generators. Then

$\hom(H,\mathbb{R})$ is a countable product of copies of $\mathbb{R}$, with its product topology. This space is not locally compact.

Luckily, there is a sense in which this counterexample is the only problem:

Lemma 117 *Suppose that H is an abelian measurable group. Then* $\hom(H,\mathbb{R})$ *is measurable if and only if the free abelian group on countably many generators is not a discrete subgroup of H.*

Proof: First suppose H is an abelian locally compact Hausdorff group. Then H has a compact subgroup K such that H/K is a Lie group, perhaps with infinitely many connected components [43, Cor. 7.54]. Since any connected abelian Lie group is the product of $\mathbb{R}^n$ and a torus, we can enlarge K while keeping it compact to ensure that the identity component of H/K is $\mathbb{R}^n$.

Any continuous homomorphism from a compact group to $\mathbb{R}$ must have compact range, and thus be trivial. It follows that K lies in the kernel of any $\chi \in \hom(H,\mathbb{R})$, so

$$\hom(H,\mathbb{R}) \cong \hom(H/K,\mathbb{R}).$$

So, without loss of generality we can replace H by H/K. In other words, we may assume that H is an abelian Lie group with $\mathbb{R}^n$ as its identity component. The only subtlety is that H may have infinitely many components.

Since $\mathbb{R}^n$ is a divisible abelian group, the inclusion $j\colon \mathbb{R}^n \to H$ comes with a homomorphism $p\colon H \to \mathbb{R}^n$ with $pj = 1$, so we actually have $H \cong \mathbb{R}^n \times A$ as abstract groups, where A is the range of p. Since $A \cap \mathbb{R}^n$ is trivial, A is actually a discrete subgroup of H. So, as a topological group H must be the product of $\mathbb{R}^n$ and a discrete abelian group A. It follows that

$$\hom(H,\mathbb{R}) \cong \mathbb{R}^n \times \hom(A,\mathbb{R}),$$

so without loss of generality we may replace H by the discrete abelian group A, and ask if $\hom(A,\mathbb{R})$ is measurable.

Since homomorphisms $\chi\colon A \to \mathbb{R}$ vanish on the torsion of A, we may assume A is torsion-free. There are two alternatives now:

1. A has finite rank: i.e., it is a subgroup of the discrete group $\mathbb{Q}^k$ for some finite k. If we choose the smallest such k, then A contains a subgroup isomorphic to $\mathbb{Z}^k$ such that the natural restriction map
$$\hom(A,\mathbb{R}) \to \hom(\mathbb{Z}^k,\mathbb{R})$$
is an isomorphism (actually of topological groups). Since $\hom(\mathbb{Z}^k,\mathbb{R})$ is locally compact, Hausdorff, and second countable, so is $\hom(A,\mathbb{R})$. So, in this case our original topological group $\hom(H,\mathbb{R})$ is measurable.

2. A has infinite rank. This happens precisely when our original group H contains the free abelian group on a countable infinite set of generators as a discrete subgroup. In this case we can show that $\hom(A,\mathbb{R})$ and thus our original topological group $\hom(H,\mathbb{R})$ is not locally compact.

   To see this, let U be any neighborhood of 0 in $\hom(A,\mathbb{R})$. By the definition of the compact-open topology, there is a compact (and thus finite) subset $K \subseteq A$ and a number $r > 0$ such that U contains the set V consisting of $\chi \in \hom(A,\mathbb{R})$ with $|\chi(a)| \le r$ for all $a \in K$. It suffices to show that V is not relatively compact.

   To do this, we shall find a sequence $\chi_n \in V$ with no cluster point. Since A has infinite rank, we can find $a \in A$ such that the subgroup generated by a has trivial intersection with the finite set K. For each $n \in \mathbb{N}$, there is a unique homomorphism ϕ_n from the subgroup generated by

a and K to $\mathbb{R}$ with $\phi_n(a) = n$ and $\phi_n(K) = 0$. Since $\mathbb{R}$ is a divisible abelian group, we can extend ϕ_n to a homomorphism $\chi_n : A \to \mathbb{R}$. Since χ_n vanishes on K, it lies in V. But since $\chi_n(a) = n$, there can be no cluster point in the sequence χ_n.

■

Combining all these lemmas, we easily conclude:

Theorem 118 *Suppose H is a measurable group. Then H^* is a measurable group if and only if the free abelian group on countably many generators is not a discrete subgroup of* $\mathrm{Ab}(H)$. *This is true, for example, if H has finitely many connected components.*

We also have:

Lemma 119 *Let G and H be measurable groups with a left action $\rhd$ of G as automorphisms of H such that the map*

$$\rhd : G \times H \to H$$

is continuous. Then the right action of G on H^* given by

$$\chi_g[h] = \chi[g \rhd h]$$

is also continuous.

Proof: Recall that H^* has the induced topology coming from the fact that it is a subset of the space of continuous maps $C(H, \mathbb{C}^\times)$ with its compact-open topology. So, it suffices to show that the following map is is continuous:

$$\begin{array}{ccc} C(H, \mathbb{C}^\times) \times G & \to & C(H, \mathbb{C}^\times) \\ (f, g) & \mapsto & f_g \end{array}$$

where

$$f_g[h] = f[g \rhd h].$$

This map is the composite of two maps:

$$\begin{array}{ccccc} C(H, \mathbb{C}^\times) \times G & \stackrel{1\times\alpha}{\longrightarrow} & C(H, \mathbb{C}^\times) \times C(H, H) & \stackrel{\circ}{\longrightarrow} & C(H, \mathbb{C}^\times) \\ (f, g) & \mapsto & (f, \alpha(g)) & \mapsto & f \circ \alpha(g) = f_g. \end{array}$$

where

$$\alpha(g)h = g \rhd h.$$

The first map in this composite is continuous because α is: in fact, any continuous map

$$\rhd : X \times Y \to X$$

determines a continuous map

$$\alpha : Y \to C(X, X)$$

by the above formula, as long as X and Y are locally compact Hausdorff spaces. The second map

$$C(H, \mathbb{C}^\times) \times C(H, H) \stackrel{\circ}{\to} C(H, \mathbb{C}^\times)$$

is also continuous, since composition

$$C(Y, Z) \times C(X, Y) \stackrel{\circ}{\to} C(X, Z)$$

is continuous in the compact-open topology whenever X, Y and Z are locally compact Hausdorff spaces. ■

A.4 Measurable G-spaces

Suppose G is a measurable group. A (right) action of G on a measurable space X is a **measurable** if the map $(g, x) \mapsto xg$ of $G \times X$ into X is measurable. A measurable space X on which G acts measurably is called a **measurable G-space**.

In fact, we can always equip a measurable G-space with a topology for which the action of G is continuous:

Lemma 120 [**19, Thm. 5.2.1**] *Suppose G is a measurable group and X is a measurable space with σ-algebra $\mathcal{B}$. Then there is a way to equip X with a topology such that:*

- *X is a Polish space—i.e., homeomorphic to separable complete metric space,*
- *$\mathcal{B}$ consists precisely of the Borel sets for this topology, and*
- *the action of G on X is continuous.*

Moreover:

Lemma 121 [**71, Cor. 5.8**] *Let G be a measurable group and let X be a measurable G-space. Then for every $x \in X$, the orbit $xG = \{xg : g \in G\}$ is a measurable subset of X; moreover the stabilizer $S_x = \{g \in G \,|\, xg = x\}$ is a closed subgroup of G.*

This result is important for the following reason. Given a point $x_o \in X$, the measurable map

$$g \mapsto x_o g$$

from G into X allows us to measurably identify the orbit $x_o G$ with the homogeneous space G/S_{x_o} of right cosets $S_{x_o} g$, on which G acts in the obvious way. Now, such spaces enjoy some nice properties, some of which are listed below.

Fix a measurable group G and a closed subgroup S of G.

Lemma 122 [**50, Thm. 7.2**] *The homogeneous space $X = G/S$, equipped with the quotient topology, is a Polish space. Since the action of G on X is continuous, it follows that X becomes a measurable G-space when endowed with its σ-algebra of Borel sets.*

Let $\pi \colon G \to G/S$ denote the canonical projection. A **measurable section** for G/S is a measurable map $s \colon G/S \to G$ such that πs is the identity on G/S and $s(\pi(1)) = 1$, where 1 is the identity in G.

Lemma 123 [**49, Lemma 1.1**] *There exist measurable sections for G/S.*

Next we present a classic result concerning quasi-invariant measures on homogeneous spaces. Let X be a measurable G-space, and μ a measure on X. For each $g \in G$, define a new measure μ^g by setting $\mu^g(A) = \mu(Ag^{-1})$. We say the measure is **invariant** if $\mu^g = \mu$ for each $g \in G$; we say it is **quasi-invariant** if $\mu^g \sim \mu$ for each $g \in G$.

Lemma 124 [**49, Thm. 1.1**] *Let G be a measurable group and S a closed subgroup of G. Then there exist non-trivial quasi-invariant measures on the homogeneous space G/S. Moreover, such measures are all equivalent.*

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