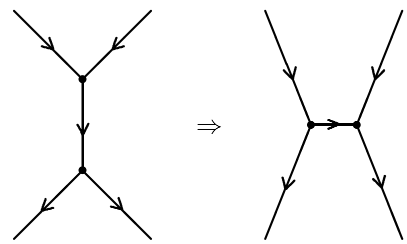
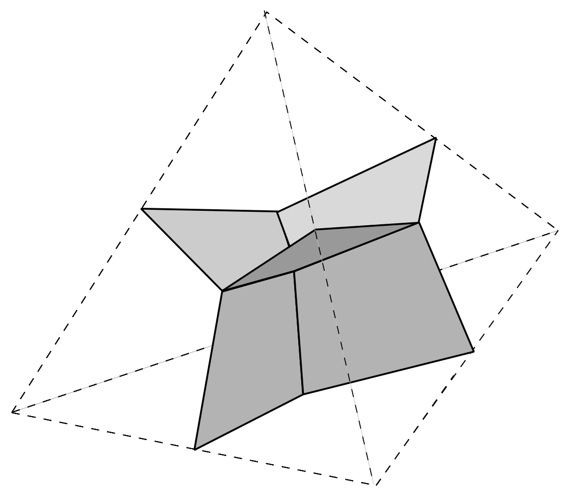
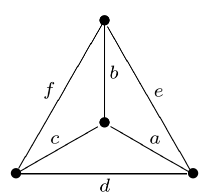
The associator:



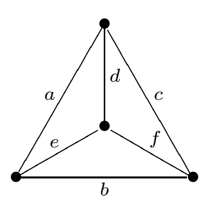
traces out the shaded surface Poincaré dual to a tetrahedron:



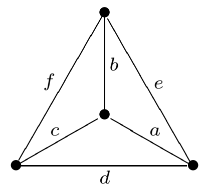
The boundary of this shaded surface is a graph with 4 vertices and 6 edges — again a tetrahedron! Each edge in the boundary of the shaded surface intersects one edge of the original tetrahedron. If we label the tetrahedron's edges like this:



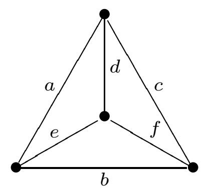
the corresponding edges of the boundary of the shaded surface get labelled like this:



In the Ponzano–Regge model of Riemannian $3d$ quantum gravity, we define the amplitude for a tetrahedron whose edges have lengths given by nonzero natural numbers:

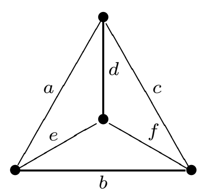


to be the result of evaluating the Poincaré dual spin network:



(the boundary of the shaded surface on the previous page). In this spin network, the numbers a, b, c, d, e, f stand for the dimensions of irreducible representations of SU(2).

Regge and Ponzano used a stationary phase approximation to argue that when we multiply all the numbers a, b, c, d, e, f by an ever larger natural number, we get an asymptotic formula:



$$\sim \quad \cos(S + \frac{\pi}{4}) \sqrt{\frac{2}{3\pi V}}$$

where V is the volume of the original tetrahedron and S is its so-called 'Ponzano-Regge action':

$$S = \sum_e \ell_e \theta_e .$$

Here the sum is taken over all 6 edges of the original tetrahedron, l_e is the length of the edge e, and θ_e is the dihedral angle of the edge e, that is, the angle between the outward normals of the two faces incident to this edge.

A rigorous proof of this asymptotic formula was given only in 1999, by Justin Roberts.

One can show that the Ponzano–Regge action is discretized version of the action for general relativity (namely the integral of the Ricci scalar curvature). Naively one might have hoped to get $\exp(iS)$ in the formula above. That one gets a cosine instead can be traced back to the fact that the lengths of the edges of a tetrahedron only determine its geometry modulo rotation *and reflection*. The phase $\frac{\pi}{4}$ shows up because calculating the asymptotics of the tetrahedral spin network involves a stationary phase approximation. The factor involving V is still mysterious to me.