Frobenius monoids, weak bimonoids, and corelations

Brandon Coya

November 5, 2017

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such that $I_1 = -I_2$. They also care about "voltage" V where $V = \phi_2 - \phi_1$.

FinCorel

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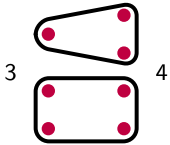

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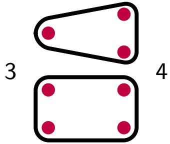

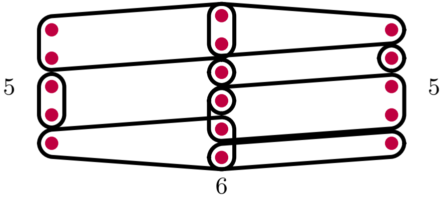

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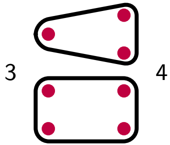
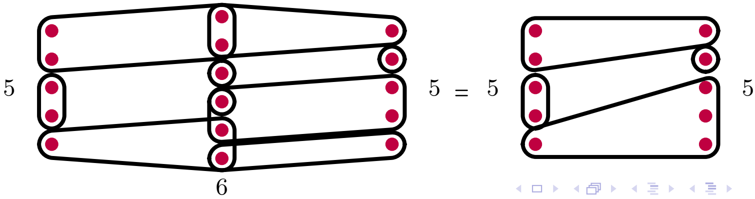
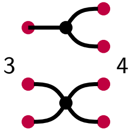




FinCorel

These are also called "corelations" and they determine the category $\mathrm{FinCorel}$. We can draw them as string diagrams:



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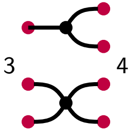

Then to study pairs of wires we study the objects $2n \in \mathrm{FinCorel}$.

Frobenius monoids

The object 2 can be equipped with two different *Frobenius monoid* structures.

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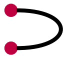
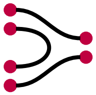
The first Frobenius monoid arises from using the unit and counit pair:

$i_2\colon 0 \to 2$

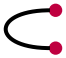


$e_2\colon 2 \to 0$

to build a multiplication and unit:



$m_2\colon 4 \to 2$



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Frobenius monoids

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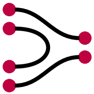
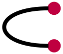


$m_2 \colon 4 \to 2$



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make 2 into a monoid:

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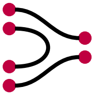
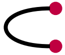


$m_2 : 4 \to 2$



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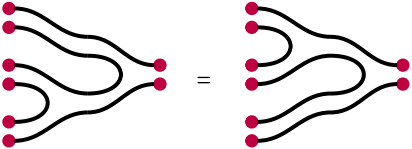

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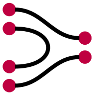

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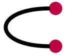

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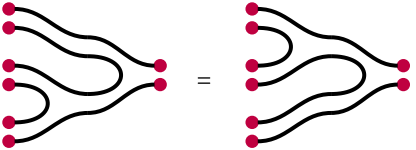

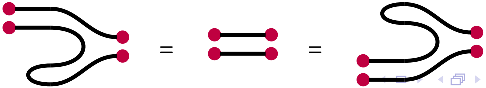

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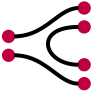

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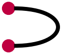

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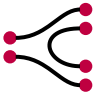

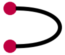

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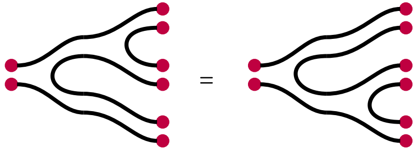

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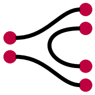

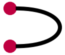

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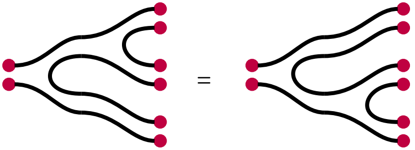

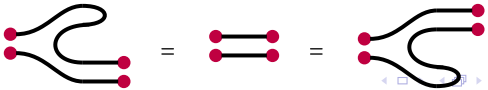

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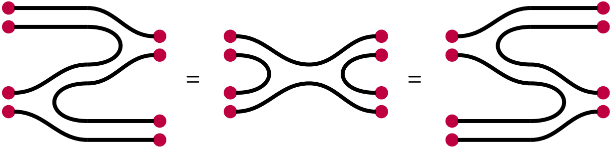

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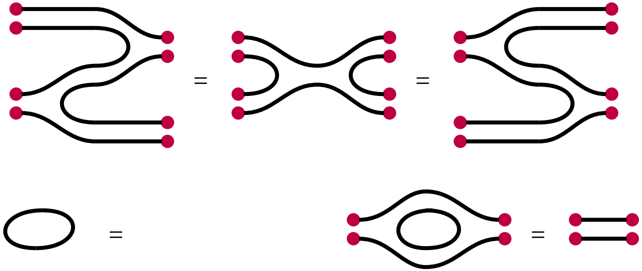

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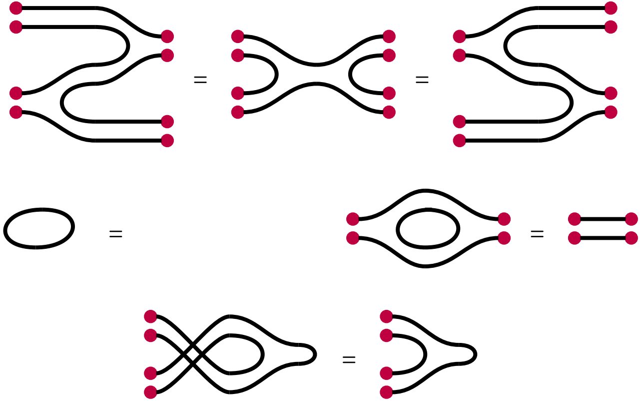

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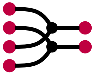
We can equip 2 with a different Frobenius monoid structure via another standard construction:



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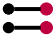

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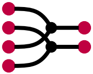

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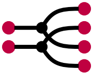
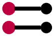


$\delta_2\colon 2 \to 4$



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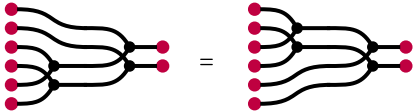
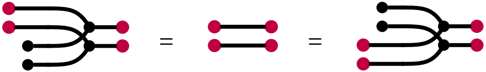
Frobenius monoids





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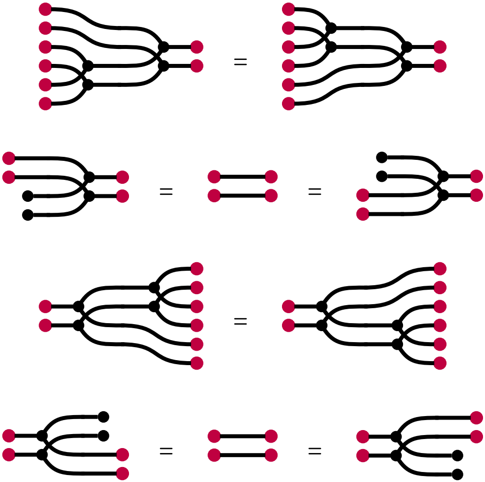

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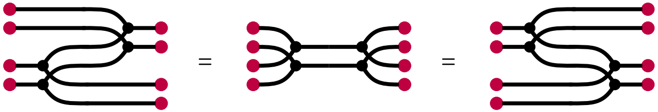

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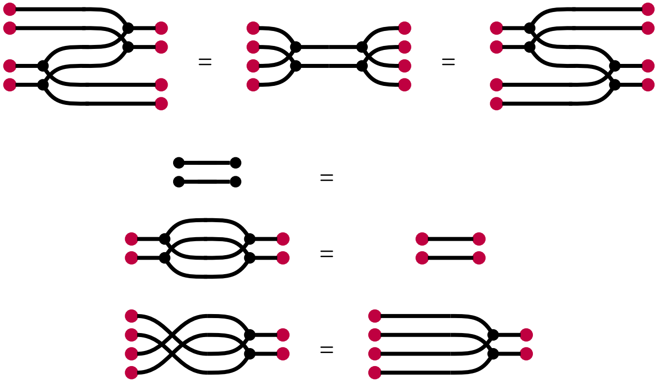
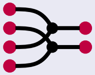


Weak bimonoids

From Pastro and Street [3] we get the following.

Theorem

The following morphisms make 2 *into a weak bimonoid:*



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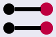

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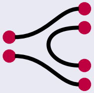
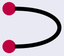


$d_2\colon 2 \to 4$



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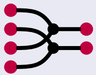

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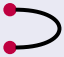

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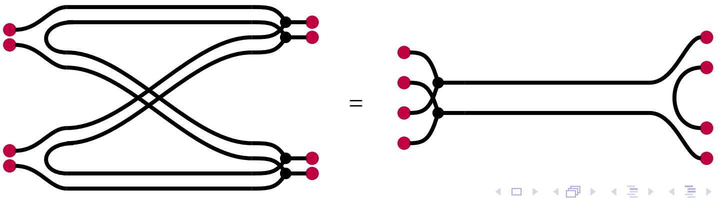

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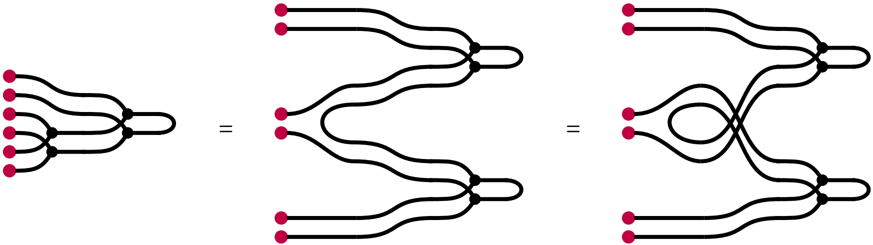

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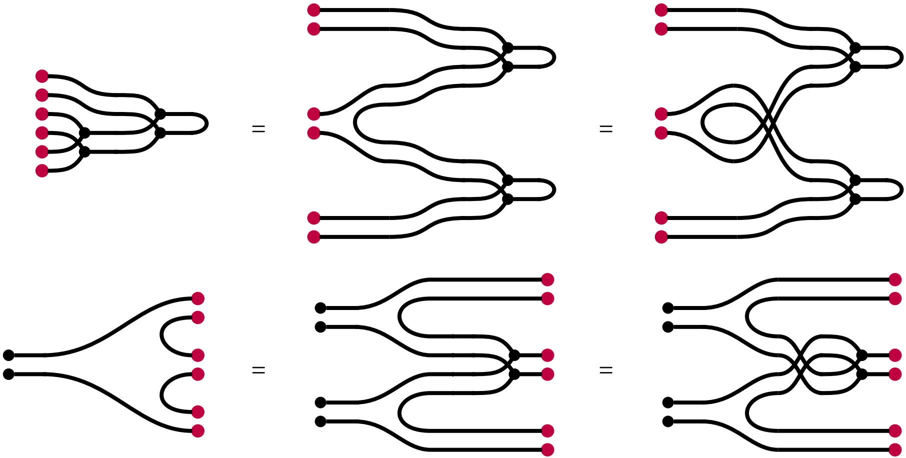

Black box functor

Now let's assign potentials and currents to our morphisms using the "black box" functor $\blacksquare\colon \mathrm{FinCorel} \to \mathrm{LagRel}_k$ given by Baez and Fong [2].

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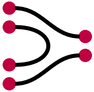

$$\overset{\blacksquare}{\longmapsto} \{(\phi_1, \ldots, I_6) : \; \phi_1 = \phi_5, I_1 = I_5, \phi_4 = \phi_6, \\ I_4 = I_6, \phi_2 = \phi_3, I_2 + I_3 = 0\}$$

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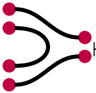

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$$I_4 = I_6, \phi_2 = \phi_3, I_2 + I_3 = 0\}$$

Then we impose that incoming current is opposite of outgoing current and write difference in potential as voltage.

$$I = I_1 = -I_2, I' = I_3 = -I_4, I'' = I_5 = -I_6$$

$$V = \phi_2 - \phi_1, V' = \phi_4 - \phi_3, V'' = \phi_6 - \phi_5$$

Series and parallel junctions

This results in the space $\{(V,\ldots,I''):\ V+V'=V'',I=I'=I''\}$ and we think of the morphism m_2 as summing voltages together while equalizing current.

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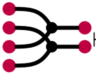
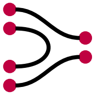
Doing this with the other multiplication gives us:

 $\longmapsto \{(V,\ldots,I''):\ I+I'=I'', V=V'=V''\}$

so that μ_2 equalizes voltage and sums voltage. Engineers call this a "parallel" junction.

FinCorel$^\circ$

Now we want to look at the subcategory FinCorel$^\circ$ of FinCorel generated by these 8 morphisms.



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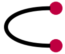
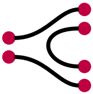
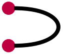


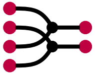
$i_2 \colon 0 \to 2$



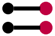
$d_2 \colon 2 \to 4$



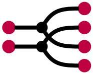
$e_2 \colon 2 \to 0$



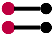
$\mu_2 \colon 4 \to 2$



$\iota_2 \colon 0 \to 2$



$\delta_2 \colon 2 \to 4$



$\epsilon_2 \colon 2 \to 0$

FinCorel°

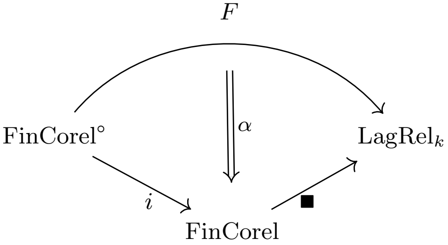
Now we want to look at the subcategory $\mathrm{FinCorel}^\circ$ of $\mathrm{FinCorel}$ generated by these 8 morphisms.

$m_2\colon 4 \to 2$ $i_2\colon 0 \to 2$ $d_2\colon 2 \to 4$ $e_2\colon 2 \to 0$

$\mu_2\colon 4 \to 2$ $\iota_2\colon 0 \to 2$ $\delta_2\colon 2 \to 4$ $\epsilon_2\colon 2 \to 0$

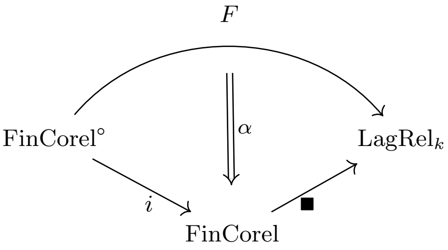
and we want to assign voltage and current with a functor $F\colon \mathrm{FinCorel}^\circ \to \mathrm{LagRel}_k$.

Then we want the following diagram:



where α comes from the relationships $V = \phi_2 - \phi_1$ and $I = I_1 = -I_2$.

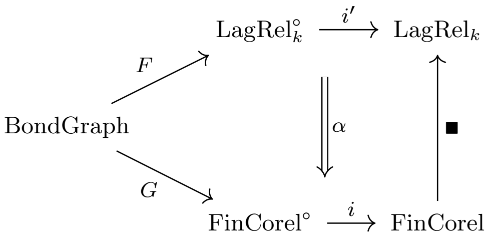
Then we want the following diagram:



where α comes from the relationships $V = \phi_2 - \phi_1$ and $I = I_1 = -I_2$. However, this cannot be done.

BondGraph

Instead this led to a lot more work where we define another category which maps into $\mathrm{FinCorel}^\circ$ and also a subcategory of LagRel_k. Then we get a nice diagram: [1]



[1] J. C. Baez, B. Coya, A compositional framework for bond graphs. Available at arXiv:1710.00098

[2] J. C. Baez, B. Fong, A compositional framework for passive linear circuits. Available at arXiv:1504.05625.

[3] C. Pastro, R. Street, Weak Hopf monoids in braided monoidal categories, *Algebra and Number Theory* **3**(2): 149–207, 2009. Available at
http://msp.org/ant/2009/3-2/ant-v3-n2-p02-s.pdf.