Open Systems in Classical Mechanics

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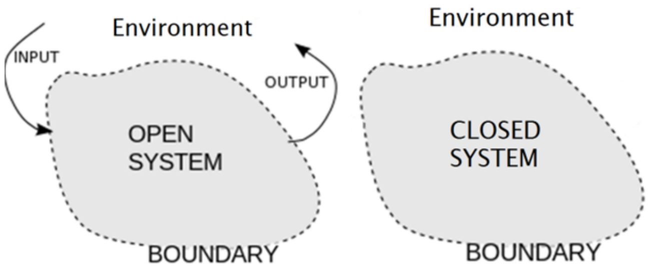
Background



Open systems are systems that have external interactions whereas a closed system does not have such interactions.

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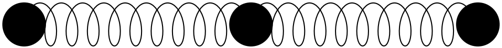
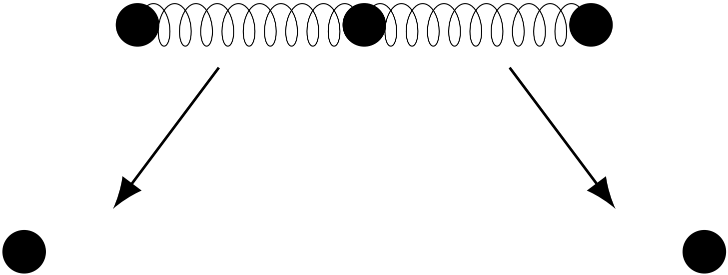


We can study open systems where the “outside world” decides the location of the left and right rocks, which affects the position of the middle rock.

Lower left and right rocks represent the outside world which decides the location of the upper left and right rock.

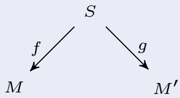


Lower left and right rocks represent the outside world which decides the location of the upper left and right rock.

Spans in Classical Mechanics

Definition

*A **span** from M to M' in a category $\mathcal{C}$ is an object S in $\mathcal{C}$ with a pair of morphisms $f\colon S \to M$ and $g\colon S \to M'$. M and M' are known as **feet** and S is known as the **apex** of the span.*



Remark

The advantage of spans is that we can build bigger systems by by “gluing” together smaller systems.

The composition of spans is done using a pullback. Spans are composable if the right foot **of one** is the same as the left foot **of the other**.

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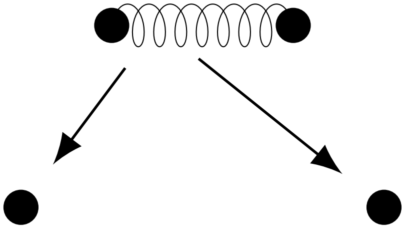

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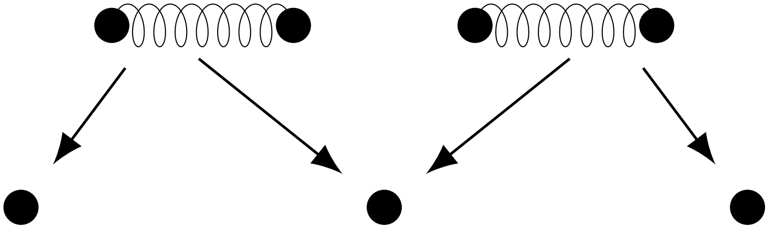

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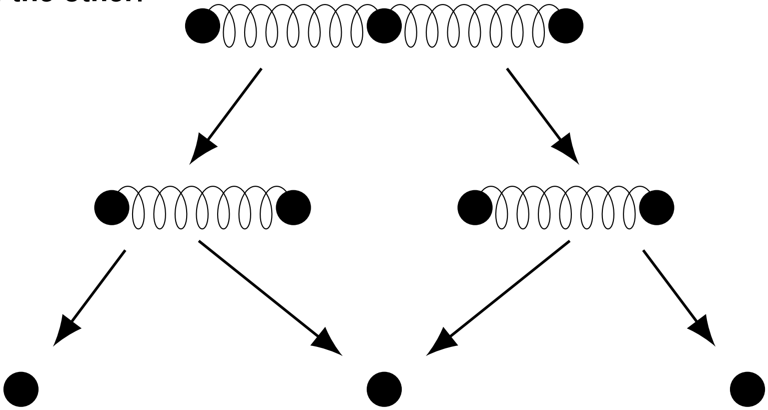

Using the framework of category theory, we formalize the heuristic principles that physicists employ in constructing the Hamiltonians for classical systems as sums of Hamiltonians of subsystems.

Definition (Poisson Manifold)

*A **Poisson manifold** is a manifold M endowed with a $\{\cdot,\cdot\}$ such that for any $f, g, h \in C^\infty(M)$ and $a, b \in \mathbb{R}$ with ordinary multiplication of functions, the following hold:*

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4. ***Leibniz Law***

$$\{fg,h\} = \{f,h\}g + f\{g,h\}$$

Symplectic Manifold

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A Poisson manifold of even dimension M equipped with a closed nondegenerate $2-$form ω satisfying $\{f, g\} = \omega(v_f, v_g)$ where v_f is the vector field with $v_f(h) = \{h, f\}$ is a ***symplectic manifold****.*

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Example

Let $\mathbb{R}^{2n}$ have standard coordinates $(x_1, ...x_n, y_1, ...y_n)$, the $2-$form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is closed and nondegenerate.

Definition (Poisson map)

Let $(M, \{\cdot,\cdot\}_M)$ *and* $(N, \{\cdot,\cdot\}_N)$ *be Poisson manifolds. We say that a map*

$$\Phi: M \to N$$

is a ***Poisson map*** *if, for any* $f, g \in C^\infty(N)$

$$\{f, g\}_N \circ \Phi = \{f \circ \Phi, g \circ \Phi\}_M .$$

Definition

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Definition

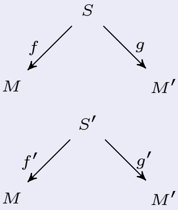
Tha subcategory $\mathbf{SympSurj}$ *of* $\mathbf{Symp}$ *has symplectic manifolds as objects and morphisms are surjective Poisson maps.*

Theorem (A.Y.)

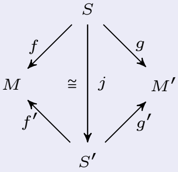
The morphisms of $\mathbf{SympSurj}$ *are pullbackable in* $\mathbf{Symp}$.

Definition

*A **map of spans** is a morphism $j\colon S \to S'$ in a category $\mathcal{C}$ between apices of two spans*



*such that both the following triangles commute. In particular, when j is an isomorphism, we have an **isomorphism of spans**.*



Theorem

Given a category $\mathcal{C}$ and a subcategory $\mathcal{D}$ such that every cospan in $\mathcal{D}$ is pullbackable in $\mathcal{C}$, then there exists a category Span$(\mathcal{C}, \mathcal{D})$ consisting of objects in $\mathcal{D}$ and whose morphisms are isomorphism classes of spans in $\mathcal{D}$ and composition is done using pullbacks in $\mathcal{C}$.

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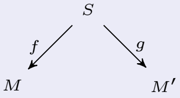

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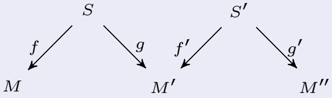

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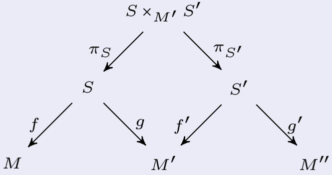

Remark

Now because pullbacks are unique up to isomorphism, we need to take isomorphism classes of spans to obtain a category.

Example

We can apply the theorem to the case $\mathcal{C} = \mathbf{Symp}$ *and* $\mathcal{D} = \mathbf{SympSurj}$ *as well as using the fact that the composition of surjective Poisson maps is surjective Poisson, to get that Span(*$\mathbf{Symp}, \mathbf{SympSurj}$*) is a category.*

Definition

Let M be a symplectic manifold of dimension $2n$. We define a ***Hamiltonian*** *to be a smooth function, H, with*

$$H\colon M \to \mathbb{R}.$$

1. In physics, the Hamiltonian corresponds to the total energy of the system.

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2. Often, the Hamiltonian is the sum of the kinetic energies of the all the particles,K, plus the potential energies of all the particles,V in the system. $H = K + V$.

We are now ready to state the main result, which will allow us to study Hamiltonian mechanics using category theory.

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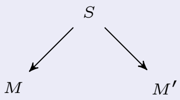

where the legs are surjective Poisson maps, together with a map $H\colon S \to \mathbb{R}$ *called the* ***Hamiltonian****.*

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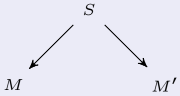
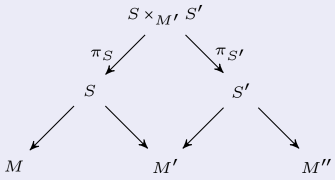
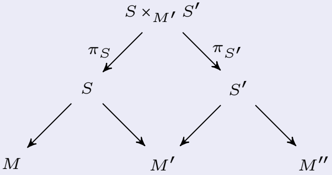


  where the legs are surjective Poisson maps, together with a map $H \colon S \to \mathbb{R}$ *called the* ***Hamiltonian****.*
- *we compose morphisms as follows:*

Theorem (Continued)



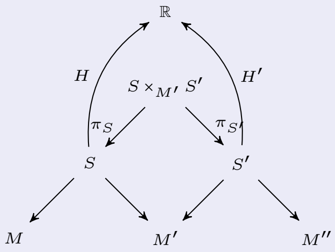
Theorem (Continued)



We have the following morphisms
$H \circ \pi_S \colon S \times_{M'} S' \to \mathbb{R}$
and
$H' \circ \pi_{S'} \colon S \times_{M'} S' \to \mathbb{R}$.
So we define the Hamiltonian on the pullback as

$$H'' = H \circ \pi_S + H' \circ \pi_{S'}.$$

Theorem (Continued)



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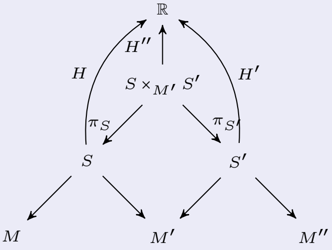

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Proof of Main Theorem

We use the theory of decorated cospans, developed in Fong's thesis:

- B. Fong, *The Algebra of Open and Interconnected Systems*, Ph.D. thesis, University of Oxford, 2016.

We adapt it to spans by working with the opposite categories.

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