

# Structured cospans

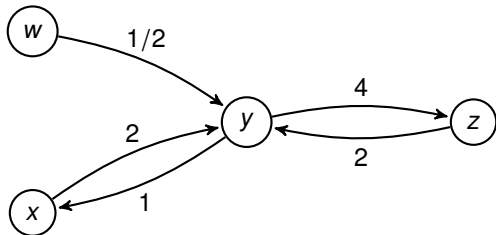
John Baez and Kenny Courser

University of California, Riverside

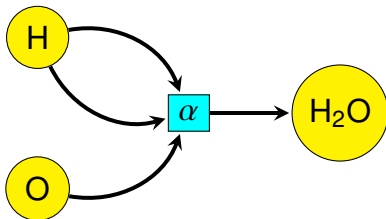
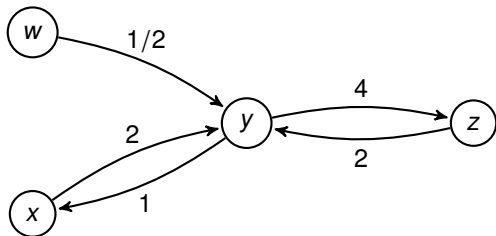
November 9, 2019



For example,

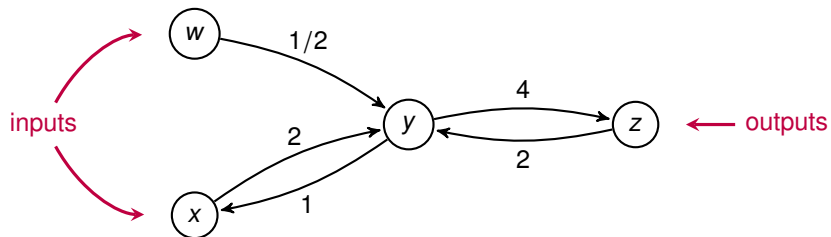


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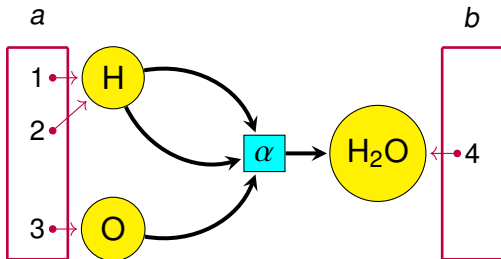
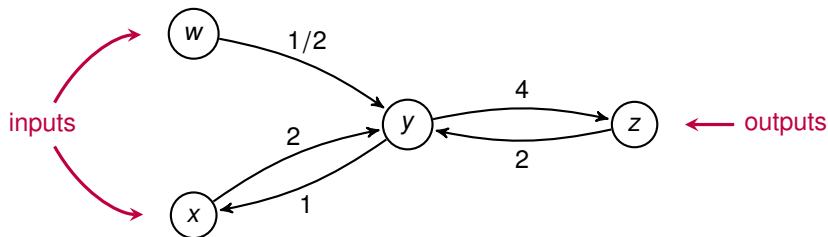


An *open* network is a network with prescribed inputs and outputs.

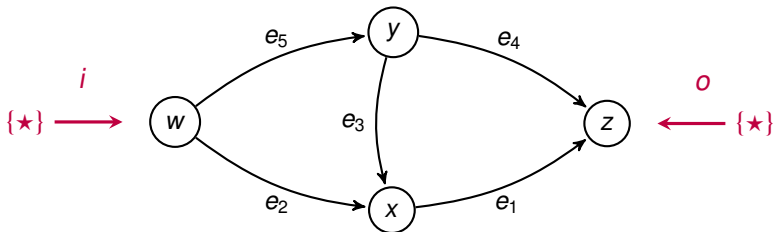
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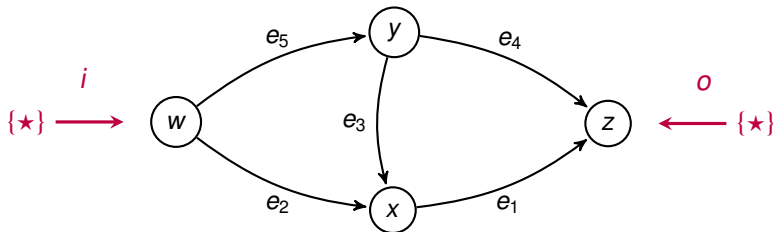


An easy example to have in mind is the example of open graphs:

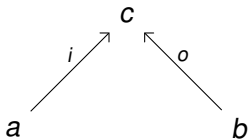




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The overall shape of this diagram resembles that of a **cospan**:



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### Theorem (B. Fong)

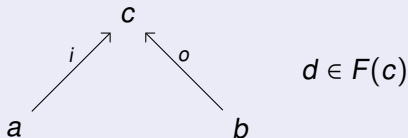
*Let  $A$  be a category with finite colimits and  $F: A \rightarrow \mathbf{Set}$  a symmetric lax monoidal functor. Then there exists a category  $FCospan$  which has:*

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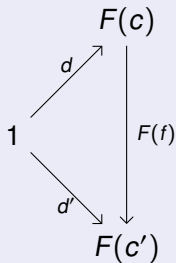
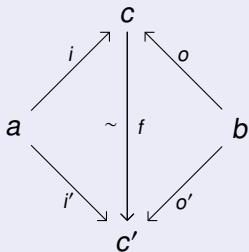
Let  $A$  be a category with finite colimits and  $F: A \rightarrow \text{Set}$  a symmetric lax monoidal functor. Then there exists a category  $FCospan$  which has:

- objects as those of  $A$  and
- morphisms as isomorphism classes of  $F$ -decorated cospans, where an  $F$ -decorated cospan is given by a pair:



## Theorem (B. Fong continued)

Two  $F$ -decorated cospans are in the same isomorphism class if the following diagrams commute:



## Theorem (B. Fong continued)

To compose two morphisms:

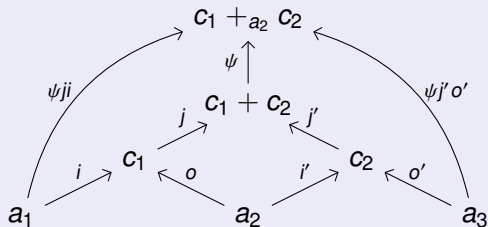
$$a_1 \xrightarrow{i} c_1 \xleftarrow{o} a_2$$

$$d_1 \in F(c_1)$$

$$a_2 \xrightarrow{i'} c_2 \xleftarrow{o'} a_3$$

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we take the pushout in  $\mathbf{A}$ :



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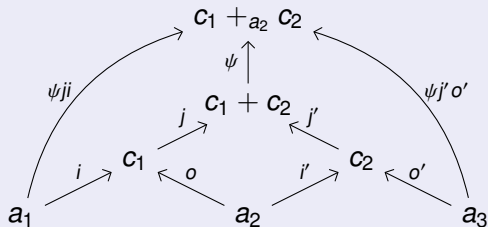
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$$d_1 \odot d_2: 1 \xrightarrow{d_1 \times d_2} F(c_1) \times F(c_2) \xrightarrow{F_{c_1, c_2}} F(c_1 + c_2) \xrightarrow{F(\psi)} F(c_1 +_{a_2} c_2)$$

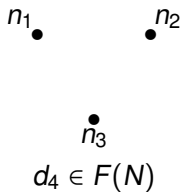
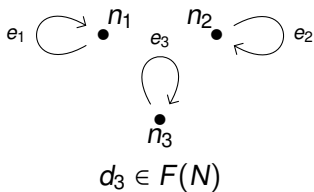
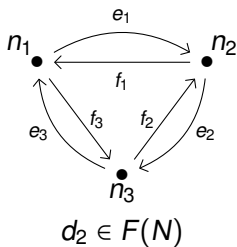
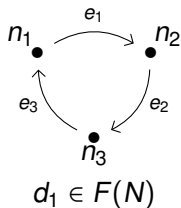
For example, if we let  $F: \text{Set} \rightarrow \text{Set}$  be the symmetric lax monoidal functor that assigns to a set  $N$  the (large) set of all graph structures having  $N$  as its set of vertices:

$$F(N) = \{E \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} N\}$$



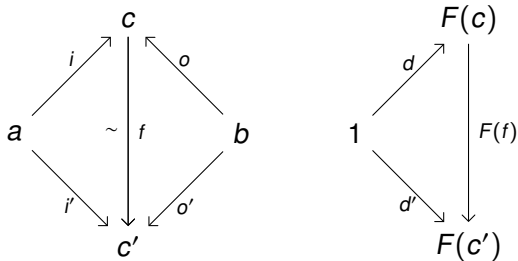
For an example of this example, if we take  $N = \{n_1, n_2, n_3\}$  to be a three element set, then some elements of the (large) set  $F(N)$  are given by:

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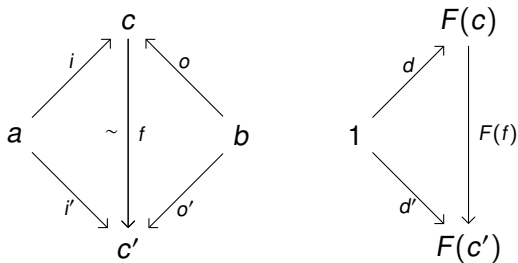
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The triangle on the right is in  $\text{Set}$  and commutes on the nose.

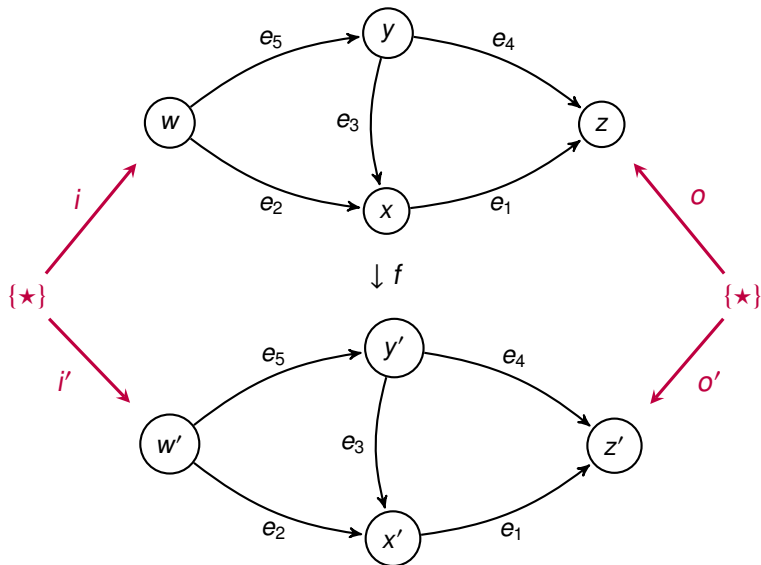
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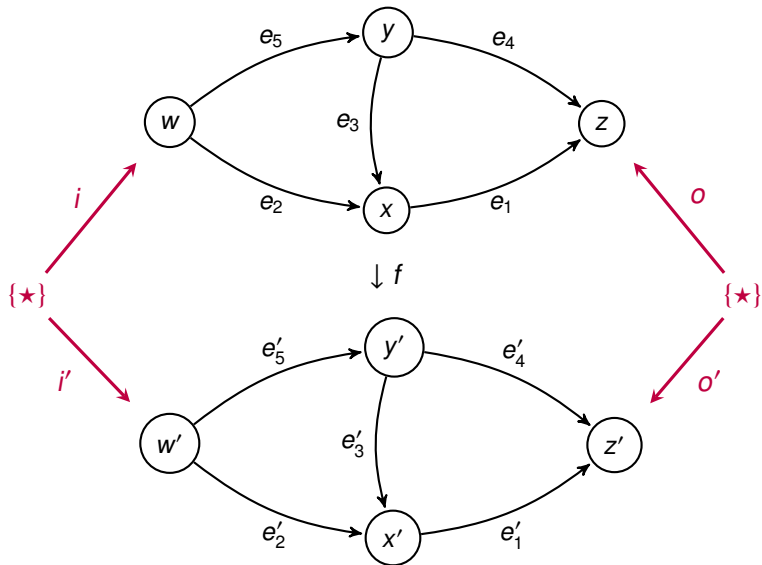
The triangle on the right is in  $\text{Set}$  and commutes on the nose.

This means that a decoration  $d \in F(c)$  together with a bijection  $f: c \rightarrow c'$  determines what the decoration  $d' \in F(c')$  must be.

In the context of open graphs, the following two open graphs would be in the same isomorphism class:



But the following two open graphs would *not* be in the same isomorphism class:



One remedy to this is to instead use ‘structured cospans’.



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### Theorem (Baez, C.)

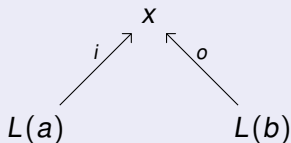
*Let  $A$  be a category with finite coproducts,  $X$  a category with finite colimits and  $L : A \rightarrow X$  a finite coproduct preserving functor. Then there exists a category  ${}_L Csp(X)$  which has:*

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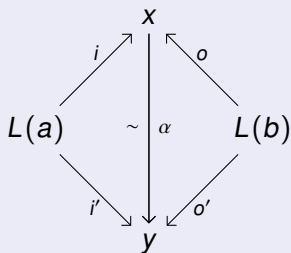
Let  $A$  be a category with finite coproducts,  $X$  a category with finite colimits and  $L : A \rightarrow X$  a finite coproduct preserving functor. Then there exists a category  ${}_L\text{Csp}(X)$  which has:

- objects as those of  $A$  and
- morphisms as isomorphism classes of  $L$ -**structured cospans**, where an  $L$ -structured cospan is given by a cospan in  $X$  of the form:



## Theorem (Baez, C. continued)

Two structured cospans are in the same isomorphism class if the following diagram commutes:

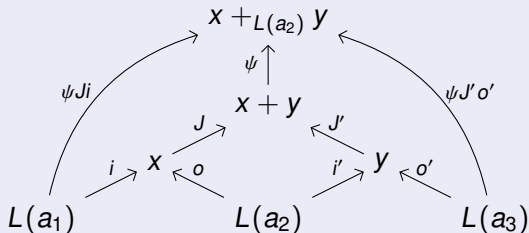


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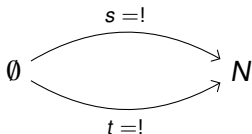
To compose two morphisms:

$$L(a_1) \xrightarrow{i} x \xleftarrow{o} L(a_2) \quad L(a_2) \xrightarrow{i'} y \xleftarrow{o'} L(a_3)$$

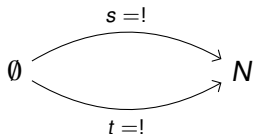
we take the pushout in  $X$ :



In the context of open graphs, we take  $L : \text{Set} \rightarrow \text{Graph}$  to be the discrete graph functor which assigns to a set  $N$  the edgeless graph with vertex set  $N$ .



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Both  $\text{Set}$  and  $\text{Graph}$  have finite colimits and  $L$  is a left adjoint, so we get the following:

## Corollary

Let  $L : \text{Set} \rightarrow \text{Graph}$  be the discrete graph functor. Then there exists a category  ${}_L\text{Csp}(\text{Graph})$  which has:

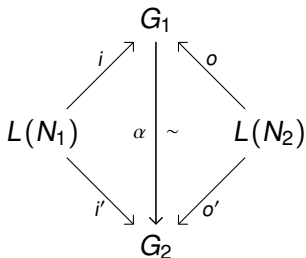
- sets as objects and
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Now, two open graphs are in the same isomorphism class if there exists an isomorphism of graphs  $\alpha : G_1 \rightarrow G_2$  making the following diagram commute:

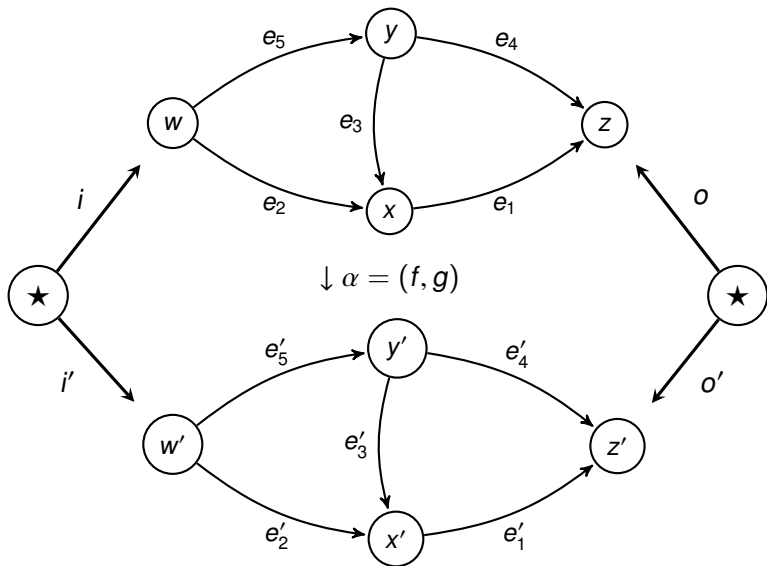




Here,  $\alpha: G_1 \rightarrow G_2$  is an isomorphism of graphs which is a *pair* of bijections  $(f, g)$  making the following squares commute:

$$\begin{array}{ccc}
 G_1 & = & E \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} N \\
 \alpha \downarrow & & g \downarrow \sim \quad \quad \quad \sim \downarrow f \\
 G_2 & = & E' \begin{array}{c} \xrightarrow{s'} \\ \xrightarrow{t'} \end{array} N'
 \end{array}$$

And now, the following two open graphs are in the same isomorphism class.



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You might be thinking that we should then use a bicategory... and we *could* do this.

But instead, we're going to use a 'double category'!

A double category has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ C & \xrightarrow{N} & D \end{array}$$

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Also, horizontal 1-cells, aka loose 1-cells, between objects, here denoted as  $M$  and  $N$ ,

and morphisms between horizontal 1-cells, called 2-morphisms, here denoted as  $\alpha$ .

These 2-morphisms can be composed both vertically and horizontally.

$$\begin{array}{ccc}
 A & \xrightarrow{M} & B \\
 \downarrow f & \Downarrow \alpha & \downarrow g \\
 C & \xrightarrow{N} & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 B & \xrightarrow{M'} & E \\
 \downarrow g & \Downarrow \beta & \downarrow h \\
 D & \xrightarrow{N'} & F
 \end{array}$$

$$\begin{array}{ccc}
 C & \xrightarrow{N} & D \\
 \downarrow f' & \Downarrow \alpha' & \downarrow g' \\
 G & \xrightarrow{P} & H
 \end{array}
 \qquad
 \begin{array}{ccc}
 D & \xrightarrow{N'} & F \\
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$$(\alpha \odot \beta)(\alpha' \odot \beta') = (\alpha\alpha') \odot (\beta\beta')$$

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- objects as those of  $A$ ,
- vertical 1-morphisms as morphisms of  $A$ ,
- horizontal 1-cells given by  $L$ -**structured cospans** which are cospans in  $X$  of the form:

$$\begin{array}{ccc} & X & \\ i \nearrow & & \nwarrow o \\ L(a) & & L(a') \end{array}$$

and

## Theorem (Baez, C. continued)

- 2-morphisms as maps of  $L$ -structured cospans given by commutative diagrams of the form:

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(a') \\ L(f) \downarrow & & \alpha \downarrow & & \downarrow L(g) \\ L(b) & \xrightarrow{i'} & y & \xleftarrow{o'} & L(b') \end{array}$$



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 \end{array}$$

The horizontal composite of two 2-morphisms:

$$\begin{array}{ccccc}
 L(a) & \xrightarrow{i_1} & x & \xleftarrow{o_1} & L(b) & & L(b) & \xrightarrow{i_2} & y & \xleftarrow{i_2} & L(c) \\
 L(f) \downarrow & & \downarrow \alpha & & \downarrow L(g) & & L(g) \downarrow & & \downarrow \beta & & \downarrow L(h) \\
 L(a') & \xrightarrow{i'_1} & x' & \xleftarrow{o'_1} & L(b') & & L(b') & \xrightarrow{i'_2} & y' & \xleftarrow{o'_2} & L(c')
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 \end{array}$$

is given by

$$\begin{array}{ccc}
 L(a) \xrightarrow{J\psi i_1} x +_{L(b)} y \xleftarrow{J\psi o_2} L(c) & & \\
 L(f) \downarrow & \alpha +_{L(g)} \beta \downarrow & \downarrow L(h) \\
 L(a') \xrightarrow{J\psi i'_1} x' +_{L(b')} y' \xleftarrow{J\psi o'_2} L(c') & &
 \end{array}$$

## Theorem (Baez, C. continued)

Monoidal structure:

$$\begin{array}{ccc}
 L(a_1) \xrightarrow{i_1} x_1 \xleftarrow{o_1} L(b_1) & & L(a'_1) \xrightarrow{i'_1} x'_1 \xleftarrow{o'_1} L(b'_1) \\
 L(f) \downarrow \quad \quad \alpha \downarrow \quad \quad \downarrow L(g) & \otimes & L(f') \downarrow \quad \quad \alpha' \downarrow \quad \quad \downarrow L(g') \\
 L(a_2) \xrightarrow{i_2} x_2 \xleftarrow{o_2} L(b_2) & & L(a'_2) \xrightarrow{i'_2} x'_2 \xleftarrow{o'_2} L(b'_2)
 \end{array}$$

$$\begin{array}{ccc}
 L(a_1 + a'_1) \xrightarrow{(i_1 + i'_1)\phi} x_1 + x'_1 \xleftarrow{(o_1 + o'_1)\phi} L(b_1 + b'_1) & & \\
 = L(f + f') \downarrow \quad \quad \alpha + \alpha' \downarrow \quad \quad \downarrow L(g + g') & & \\
 L(a_2 + a'_2) \xrightarrow{(i_2 + i'_2)\phi} x_2 + x'_2 \xleftarrow{(o_2 + o'_2)\phi} L(b_2 + b'_2) & &
 \end{array}$$

We can also tweak decorated cospans by instead of using a functor  $F: A \rightarrow \text{Set}$ , using a *pseudofunctor*  $F: A \rightarrow \text{Cat}$ .

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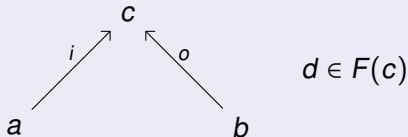
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- objects as those of  $A$ ,
- vertical 1-morphisms as morphisms of  $A$ ,
- horizontal 1-cells as  $F$ -decorated cospans, which are again pairs:





## Theorem (Baez, Vasilakopoulou, C. continued)

- 2-morphisms given by maps of cospans in  $A$ :

$$\begin{array}{ccccc} a & \xrightarrow{i} & c & \xleftarrow{o} & b \\ g \downarrow & & f \downarrow & & \downarrow h \\ a' & \xrightarrow{i'} & c' & \xleftarrow{o'} & b' \end{array}$$

$$\begin{array}{ccc} & & F(c) \\ & \nearrow d & \downarrow F(f) \\ 1 & & F(c') \\ & \searrow d' & \end{array}$$

## Theorem (Baez, Vasilakopoulou, C. continued)

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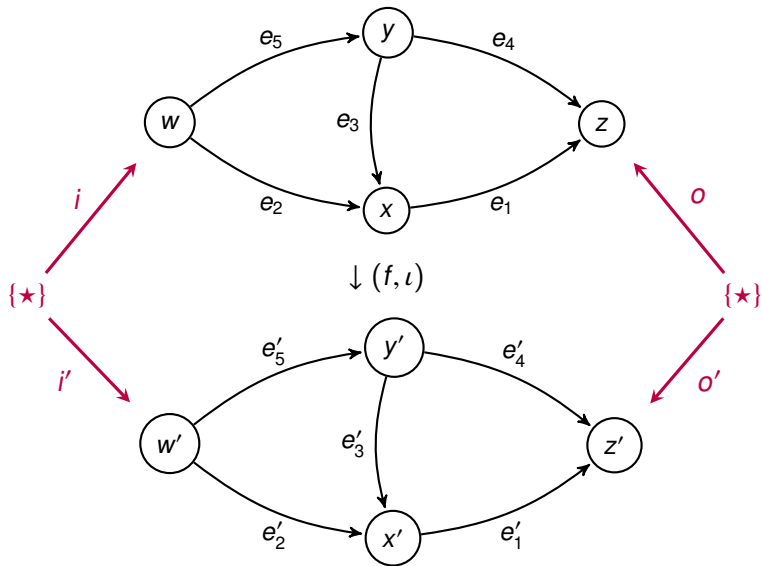
together with a 2-morphism  $\iota$  which can be viewed as a morphism

$$\iota: F(f)(d) \rightarrow d'$$

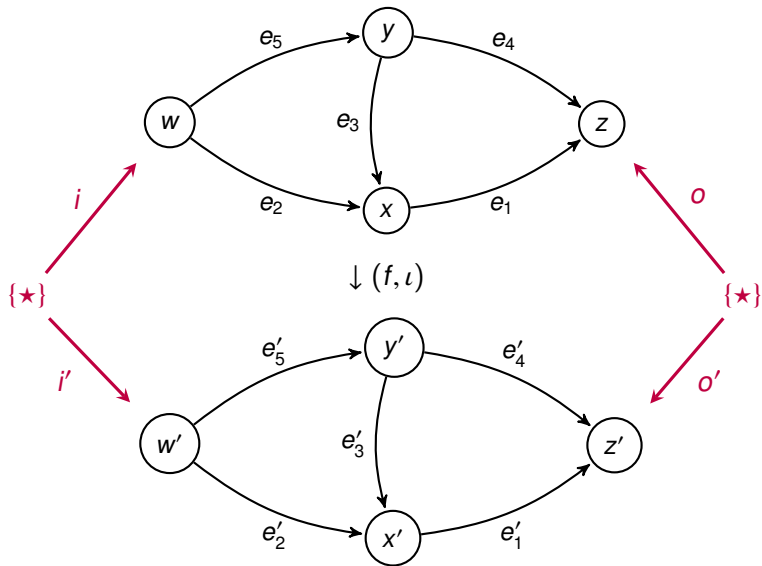
in  $F(c')$ .

In the context of open graphs:

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the morphism  $\iota: F(f)(d) \rightarrow d'$  is the map of edges.

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### Theorem (Baez, Vasilakopoulou, C.)

*Given a category  $A$  with finite colimits and a symmetric lax monoidal pseudofunctor  $F: A \rightarrow \text{Cat}$  such that  $F$  factors as  $F \rightarrow \text{CoCat} \hookrightarrow \text{Cat}$ , then there is an equivalence of symmetric monoidal double categories*

$${}_{\perp}\mathbb{C}\text{sp}(f F) \simeq F\mathbb{C}\text{sp}.$$

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$$L\mathbb{C}sp(\int F) \simeq F\mathbb{C}sp.$$

The functor  $L$  used to obtain the structured cospans double category is left adjoint to the Grothendieck construction of the pseudofunctor  $F$ :

$$R: \int F \rightarrow A.$$



We've used the framework of structured cospans to create syntax categories for black box functors.

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There exists a left adjoint  $L : \mathbf{FinSet} \rightarrow \mathbf{Mark}$  which we can use to obtain a symmetric monoidal category

$${}_L \mathbf{Csp}(\mathbf{Mark})$$

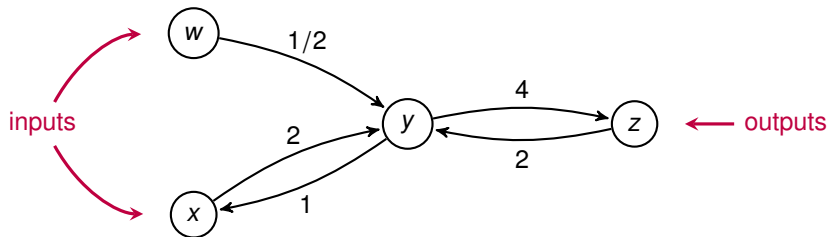
of finite sets and open Markov processes.

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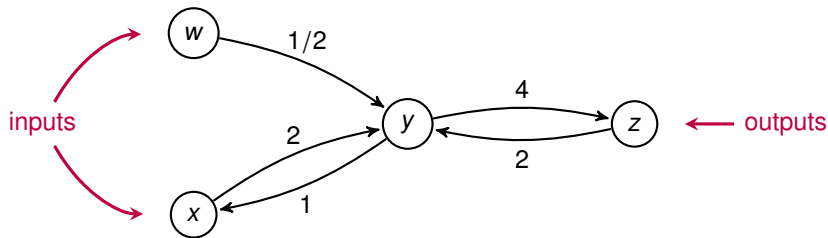


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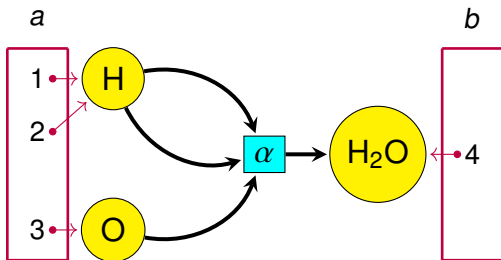
of finite sets and open Markov processes.



From this, we can obtain a black box functor

$$\blacksquare : {}_L \text{Csp}(\text{Mark}) \rightarrow \text{Rel}.$$

And likewise for open Petri nets.



$L : \text{Set} \rightarrow \text{Petri}$

$\blacksquare : {}_L\text{Csp}(\text{Petri}) \rightarrow \text{Rel.}$

For more, see my thesis on Dr. Baez's website:

<https://tinyurl.com/courser-thesis>