Metrics on Functor Categories & Reeb Graph Operations

Vin de Silva
Pomona College

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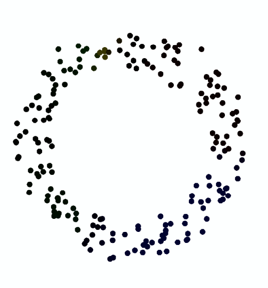
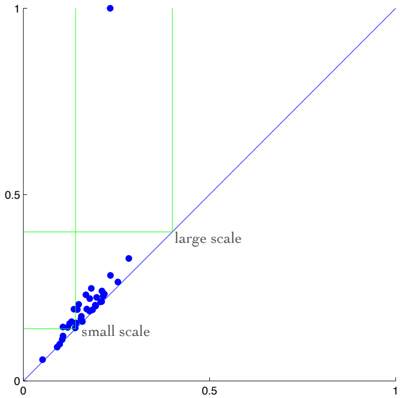
Persistence diagrams

Edelsbrunner, Letscher, Zomorodian 2000

Persistent homology takes a filtered space $\mathbb{X} = \{X_t \mid t \in \mathbf{R}\}$ and returns a **barcode** of intervals $[p, q) \subset \mathbf{R}$ or a **persistence diagram** of points $(p, q) \in \mathbf{R}^2$.





Persistence diagrams

Using commutative algebra (Zomorodian, Carlsson 2003).

- Discretize the t-variable to integers: $t = 0, 1, 2, \ldots$
- Present $\mathbb{X}$ as an increasing sequence:

$$\mathbb{X} : \quad X_0 \subset X_1 \subset X_2 \subset \ldots$$

- Apply a homology functor $\mathrm{H} = \mathrm{H}(-; \mathbf{k})$ to obtain a **persistence module**:

$$\mathrm{H}(\mathbb{X}) : \quad \mathrm{H}(X_0) \to \mathrm{H}(X_1) \to \mathrm{H}(X_2) \to \ldots$$

- Observe that $\mathrm{H}(\mathbb{X})$ is a graded module over the polynomial ring $\mathbf{k}[z]$, where z acts by shifting to the right.
- Decompose this graded module as a direct sum of cyclic submodules.
- Summands $z^s\mathbf{k}[z]/(z^{t-s})$ are recorded as intervals $[s, t)$.
- Summands $z^s\mathbf{k}[z]$ are recorded as intervals $[s, +\infty)$.

Interleaving of Persistence Modules

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

The map {persistence modules} $\rightarrow$ {diagrams} is 1-Lipschitz.

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Relators

The metrics on the two spaces are defined in terms of 'relators'.

- Two persistence modules may be related by an **interleaving**.
- Two diagrams may be related by a **matching**.

Every relator, of each type, has a size associated with it. The metrics are defined by finding the infimum of the size of relators between a given pair of objects. (Compare the geodesic distance in a Riemannian manifold.)

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

If two persistence modules admit an ϵ-interleaving, then their persistence diagrams admit an ϵ-matching.

Interleaving of Persistence Modules

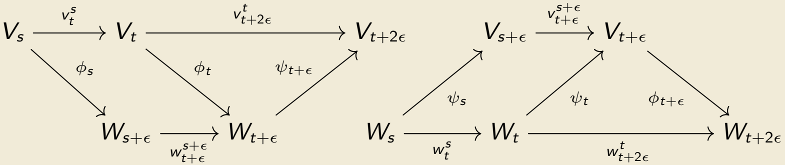
Definition

Let $\mathbb{V}, \mathbb{W}$ be persistence modules. An ϵ-interleaving between $\mathbb{V}, \mathbb{W}$ is a pair (Φ, Ψ) where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

$$\phi_t : V_t \to W_{t+\epsilon} \qquad \psi_t : W_t \to V_{t+\epsilon}$$

such that [various conditions].

The [various conditions] require the diagrams



to commute for all $s < t$.

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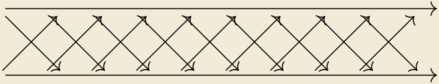
such that [various conditions].

Interleavor categories (Chazal, dS, Glisse, Oudot 2016)

An ϵ-interleaved pair of modules $(\mathbb{V}, \mathbb{W}, \Phi, \Psi)$ is 'the same thing' as a persistence module defined over the set $\mathbf{I} = \mathbf{R} \times \{0, \epsilon\}$ (two copies of the real line) with the partial order

$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$

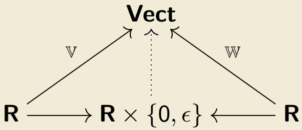
$\mathbf{R} \times \{0, \epsilon\}$:



Interleaving of Persistence Modules

Interleavings for classical persistence modules

Two classical persistence modules $\mathbb{V}, \mathbb{W}$ are ϵ-interleaved iff the following functor extension problem has a solution:



Here $\mathbf{R} \times \{0, \epsilon\}$ has the partial order

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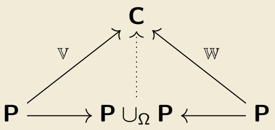
Interleaving Metrics on Functor Categories

Interleavings for generalized persistence modules over a poset

Two persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$ are Ω-interleaved iff the following functor extension problem has a solution:



Here $\mathbf{P} \cup_\Omega \mathbf{P}$ has the partial order

$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ \Omega s \leq t & \text{if } a \neq b \end{cases}$$

where $\Omega : \mathbf{P} \to \mathbf{P}$ is a translation.

Translations (Bubenik, dS, Scott 2015)

$\mathbf{Trans_P}$ is the poset of functions $\Omega : \mathbf{P} \to \mathbf{P}$ that are order-preserving and satisfy $x \leq \Omega x$ for all $x \in \mathbf{P}$.

Superlinear Families

A **superlinear family** is a 1-parameter family of translations of $\mathbf{P}$

$$(\Omega_\epsilon \mid \epsilon \in [0, \infty))$$

such that

$$\Omega_{\epsilon_1}\Omega_{\epsilon_2} \leq \Omega_{\epsilon_1+\epsilon_2}$$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Sublinear Projections

A **sublinear projection** is a map $\pi : \mathbf{Trans_P} \to [0, \infty]$ such that

$$\pi(\Omega_1\Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)$$

for all $\Omega_1, \Omega_2 \in \mathbf{Trans_P}$.

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Examples of superlinear famlies

- $\mathbf{P} = \mathbf{R}$,
  $\Omega_\epsilon(t) = t + \epsilon$.
- $\mathbf{P} = \{$compact intervals in the real line$\}$,
  $\Omega_\epsilon([a, b]) = [a - \epsilon, b + \epsilon]$.
- $\mathbf{P} = \{$closed subsets of a metric space $X\}$,
  $\Omega_\epsilon(V) = V^\epsilon = \{x \in X \text{ such that } \mathrm{d}(x, V) \leq \epsilon\}$.

Interleaving Metrics on Functor Categories

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for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Interleaving distance (Bubenik, dS, Scott 2015)

Given a superlinear family (Ω_ϵ) of translations of $\mathbf{P}$, we define the interleaving distance

$$\mathsf{d_i}(\mathbb{V}, \mathbb{W}) = \inf\left(\epsilon \mid \mathbb{V}, \mathbb{W} \text{ are } \Omega_\epsilon\text{-interleaved}\right)$$

between generalized persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$.

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Interleaving Metrics on Functor Categories

Functoriality

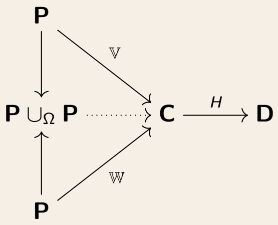
Suppose $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$ and $H : \mathbf{C} \to \mathbf{D}$ are functors. Then

$$\mathrm{d_i}(H\mathbb{V}, H\mathbb{W}) \leq \mathrm{d_i}(\mathbb{V}, \mathbb{W})$$

for any superlinear family or sublinear projection.

Proof.

An Ω-interleaving of $\mathbb{V}, \mathbb{W}$ gives an Ω-interleaving of $H\mathbb{V}, H\mathbb{W}$:



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Interleaving Metrics on Functor Categories

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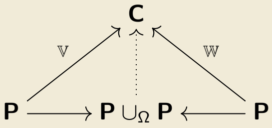

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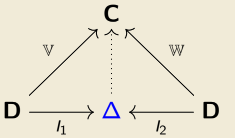
where $\Omega : \mathbf{P} \to \mathbf{P}$ is a translation.

Interleaving Metrics on Functor Categories

Interleavings for generalized persistence modules over an arbitrary category

Two persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{D} \to \mathbf{C}$ are Δ-interleaved iff the following functor extension problem has a solution:



Here Δ is a **cospan**. The two functors I_1, I_2 are full-and-faithful. Every object of Δ is of the form $I_1(d)$ or $I_2(d)$.

Bubenik, dS, Scott

Interleaving and Gromov–Hausdorff distance: arXiv:1707.06288

Interleaving Metrics on Functor Categories

Example: dynamical system interleavings

Let **D** be the category defined by the directed graph

Thus **D** has one object and morphisms $\{0, 1, 2, 3, \dots\}$.

- Functors $\mathbf{D} \to \mathbf{Top}$ are **discrete dynamical systems**.

Let Δ_n be the category with two objects $\bullet_1$ and $\bullet_2$ and morphisms

$$\mathrm{Mor}(\bullet_1, \bullet_1) = \mathrm{Mor}(\bullet_1, \bullet_1) = \{0, 1, 2, 3, \dots\}$$
$$\mathrm{Mor}(\bullet_1, \bullet_2) = \mathrm{Mor}(\bullet_2, \bullet_1) = \{n, n+1, n+2, n+3, \dots\}$$

with addition as composition.

- Δ_n-interleavings are **shift-equivalences**.

Categories with a flow (dS, Munch, Stefanou 2018)

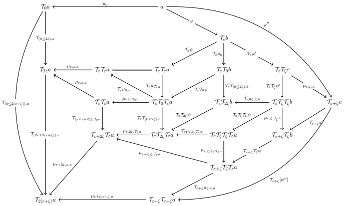
Interleaving distance defined on categories with a coherent $[0, \infty)$-action.

Examples

- Functor categories $\mathbf{C}^{\mathbf{P}}$, equipped with a superlinear family (Ω_ϵ) on $\mathbf{P}$.
- Poset $\mathbf{S}$ of subsets of a metric space X; ‘thickening’ action on $\mathbf{S}$:

$$A \mapsto A^\epsilon = \{x \in X \mid d(x, A) \leq \epsilon\}$$

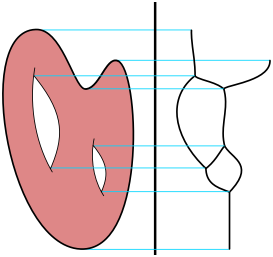
Interleaving distance = Hausdorff distance.



Reeb Graphs & Reeb Cosheaves

Reeb graphs

- An **R-space** (X, f) is a topological space X with a map $f : X \to \mathbf{R}$.
- An **R**-space is a **Reeb graph** if each $f^{-1}(t)$ is finite.



Reeb functor

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$$(X, f) \longmapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff x, y are in the same component of the same levelset of f.

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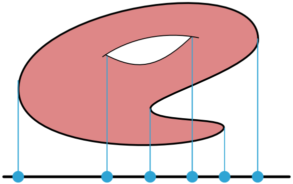

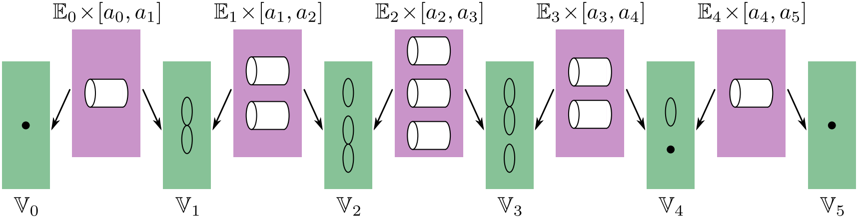

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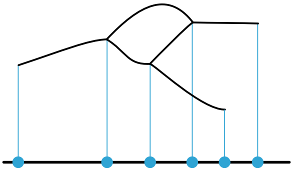
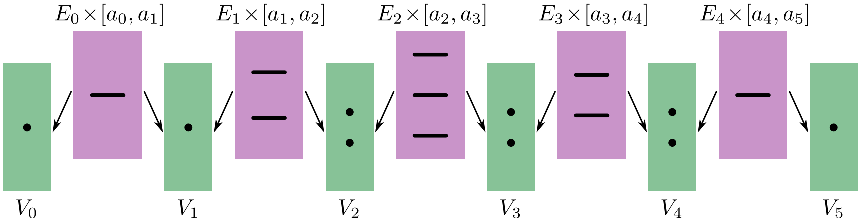
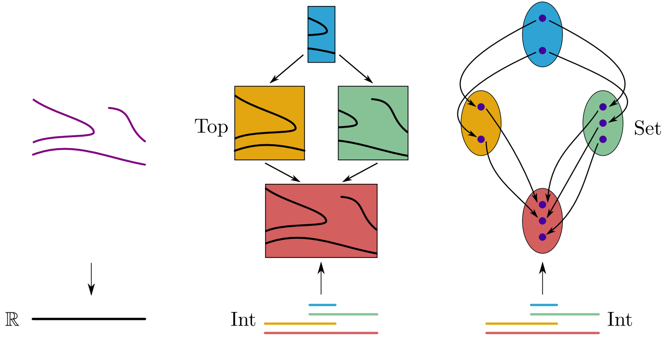




Reeb cosheaves (dS, Munch, Patel 2016)

- Let **Int** denote the poset of open intervals, $\subseteq$.
- A Reeb graph gives rise to a functor $\mathbf{F} = \pi_0 f^{-1} : \mathbf{Int} \to \mathbf{Set}$ that is constructible and satisfies the cosheaf condition for unions of intervals.



$$\mathbf{F}(I = \bigcup I_\alpha) = \operatorname{colim}\left[\coprod_{\alpha,\beta} \mathbf{F}(I_\alpha \cap I_\beta) \rightrightarrows \coprod_\alpha \mathbf{F}(I_\alpha)\right]$$

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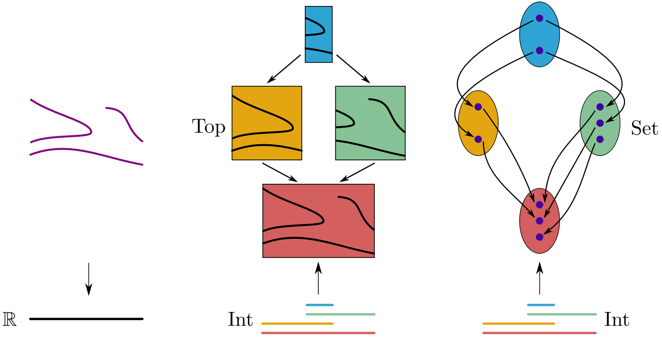

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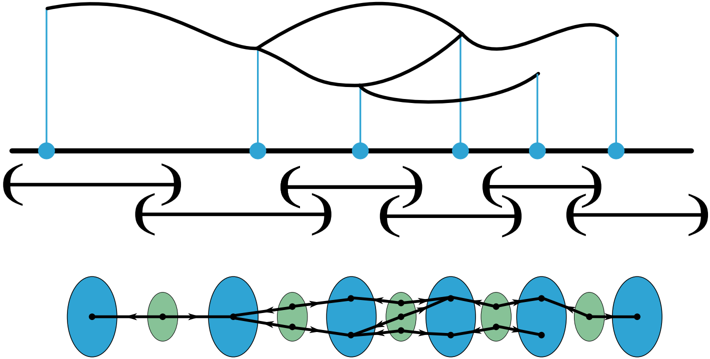
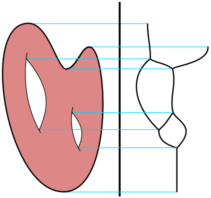


Reeb Graphs & Reeb Cosheaves



Reeb functor (two versions)

- The **Reeb functor** converts a (constructible) **R**-space into a Reeb graph:

$$(X, f) \longmapsto ((X/\sim), \overline{f})$$

where $x \sim y$ iff x, y are in the same component of the same levelset of f.

or

- The **Reeb functor** converts a constructible **R**-space into a Reeb cosheaf:

$$\mathbf{F}(I) = \pi_0 f^{-1}(I)$$

$$\mathbf{G}[I \subseteq J] = \pi_0 \left[f^{-1}(I) \subseteq f^{-1}(J) \right]$$

Translation operators on **Int**

We define a 1-parameter semigroup (Ω_ϵ) of functors $\mathbf{Int} \to \mathbf{Int}$ by setting

$$\Omega_\epsilon(I) = I^\epsilon = \text{“}\epsilon\text{-neighbourhood of } I\text{”}$$

Reeb interleaving distance (dS, Munch, Patel 2016)

An ϵ-**interleaving** between $\mathbf{F}, \mathbf{G}$ is given by two families of maps

$$\phi_I : \mathbf{F}(I) \to \mathbf{G}(I^\epsilon), \quad \psi_I : \mathbf{G}(I) \to \mathbf{F}(I^\epsilon)$$

which are natural with respect to inclusions $I \subseteq J$, and such that for all I

$$\psi_{I^\epsilon} \circ \phi_I = \mathbf{F}[I \subseteq I^{2\epsilon}], \quad \phi_{I^\epsilon} \circ \psi_I = \mathbf{G}[I \subseteq I^{2\epsilon}].$$

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Stability Theorem

If $f, g : X \to \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $\mathrm{d_i}(\mathbf{F}, \mathbf{G}) \leq \epsilon$.

Universal Reeb Metric (Bauer, Landi, Mémoli 2018)

The **universal metric** $\mathrm{d_u}(\mathbf{F}, \mathbf{G})$ is the largest that satisfies the stability theorem.

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Cosheaf Smoothing Theorem

If $\mathbf{F} : \mathbf{Int} \to \mathbf{Set}$ is a (constructible) cosheaf, then so is $\mathbf{F}\Omega_\epsilon : \mathbf{Int} \to \mathbf{Set}$.

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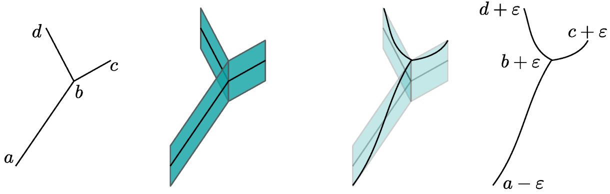
Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.



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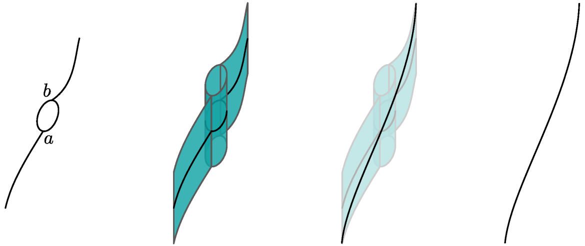

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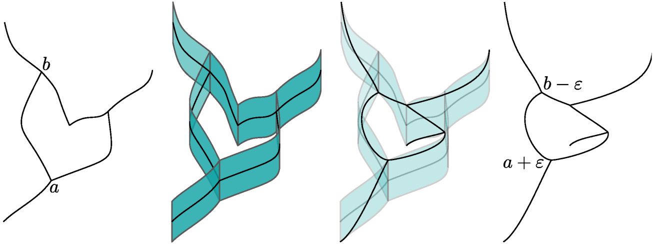
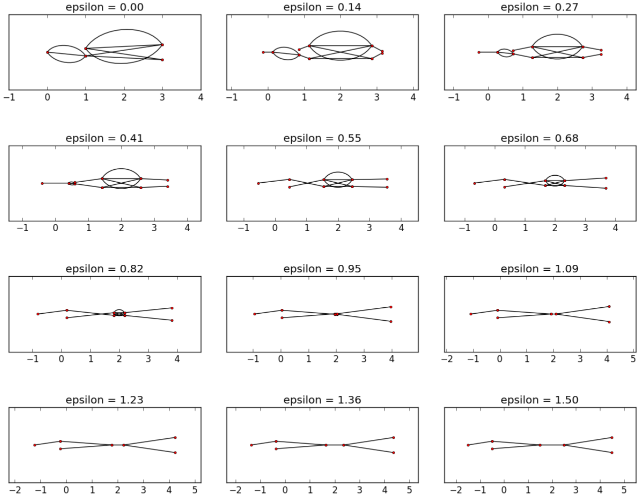


Reeb smoothing algorithm by Dmitriy Smirnov & Song Yu



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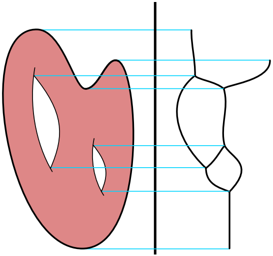

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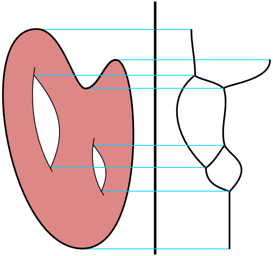
Reeb space operations

Reeb spaces

- A **B-space** (X, f) is a topological space X with a map $f : X \to \mathbf{B}$.
- A **B**-space is a **Reeb B-space** if each $f^{-1}(t)$ is finite.



Reeb functor

- The **Reeb functor** converts a (constructible) **B**-space into a Reeb **B**-space:

$$(X, f) \longmapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff x, y are in the same component of the same fiber of f.

Example: Universal Cover

Let **B** be a (locally well-behaved) topological space. Then

$$\mathrm{Path}(\mathbf{B}, b_0) = \{\text{paths } \gamma : [0,1] \to \mathbf{B} \text{ with } \gamma(0) = b_0\}$$

is a **B**-space with respect to the evaluation map

$$e : \mathrm{Path}(\mathbf{B}, b_0) \longrightarrow \mathbf{B};\ \gamma \longmapsto \gamma(1).$$

Then

$$\boxed{\mathrm{Univ}(\mathbf{B}) = \mathrm{Reeb}\left[\mathrm{Path}(\mathbf{B}, b_0), e\right]}$$

is the universal cover of **B**.

Example: Universal Reeb Metric (Bauer, Landi, Mémoli 2018)

Let $X = (X, f)$ and $Y = (Y, g)$ be Reeb graphs.

- A **relator** for X, Y is an $(\mathbf{R} \times \mathbf{R})$-space

$$W \xrightarrow{F} \mathbf{R} \times \mathbf{R}$$

  such that

$$\mathrm{Reeb}\left[W, p_1 \circ F\right] \cong (X, f),$$
$$\mathrm{Reeb}\left[W, p_2 \circ F\right] \cong (Y, g).$$

- The **deviation** of a relator

$$\mathrm{dev}(W) = \sup_{w \in W} |p_1(F(w)) - p_2(F(w))|$$

  measures how far $F(W)$ deviates from the diagonal.

- The **universal distance** between X, Y is defined

$$\mathrm{d_u}(X, Y) = \inf \left(\mathrm{dev}(W) \mid W \text{ is a relator for } X, Y \right)$$

Reeb space operations

Let **B** be a topological semigroup with operation $\odot$.

Reeb **B**-space convolutions

The π_0-**convolution** of Reeb spaces

$$X \xrightarrow{f} \mathbf{B}, \quad Y \xrightarrow{g} \mathbf{B}$$

is defined to be

$$(X, f) * (Y, g) = \text{Reeb}\left[X \times Y, f \odot g\right]$$

Reeb graph convolutions

$$(X, f) * (Y, g) = \text{Reeb}\left[X \times Y, f + g\right]$$

Reeb space operations

Reeb graph convolutions

$$(X, f) * (Y, g) = \text{Reeb}\left[X \times Y, f + g\right]$$

Examples

- The σ-smoothing of a Reeb graph $X = (X, f)$ is given by the formula
$$X^\sigma = X * [-\sigma, \sigma].$$
- The intervals $[-\sigma, \sigma]$, for $\sigma \geq 0$, form a semigroup under $*$.
- More generally, the convolution of intervals is their Minkowski sum:
$$[m_1 - \sigma_1, m_1 + \sigma_1] * [m_2 - \sigma_2, m_2 + \sigma_2] = [m - \sigma, m + \sigma]$$
where $m = m_1 + m_2$ and $\sigma = \sigma_1 + \sigma_2$.
- The **merge-tree** and **split-tree** of X are given by the formulas
$$\text{Merge}(X) = X * [0, +\infty), \quad \text{Split}(X) = X * (-\infty, 0].$$
- Thus $X * [-R, R]$, when $R \gg 0$, combines the merge and split trees of X.

Reeb space operations

Metric properties of Reeb graph smoothing

Let X, Y be Reeb graphs. Then

$$\mathrm{d_i}(X^\sigma, Y^\sigma) \leq \mathrm{d_i}(X, Y), \quad \mathrm{d_u}(X^\sigma, Y^\sigma) \leq \mathrm{d_u}(X, Y)$$

and

$$\mathrm{d_i}(X, X^\sigma) \leq \mathrm{d_u}(X, X^\sigma) \leq \sigma$$

for all $\sigma \geq 0$.

Reeb space operations

Metric properties of Reeb graph smoothing

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and

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for all $\sigma \geq 0$.

Analogy: Gaussian kernel smoothing

Is there a theory of π_0 signal processing?

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