Supplying bells and whistles in symmetric monoidal categories

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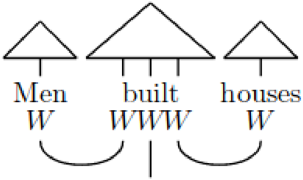
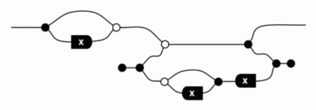
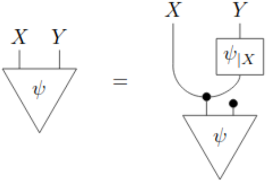
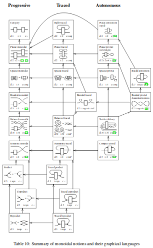
AMS Fall Western Section
UC Riverside
9 November 2019

Sometimes we want to add additional icons to string diagrams:









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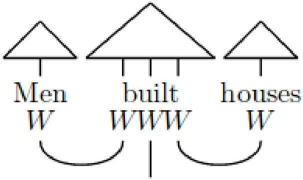
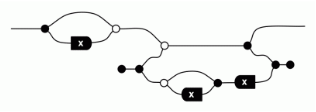
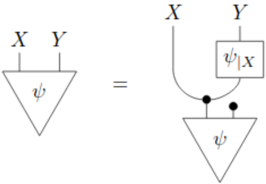
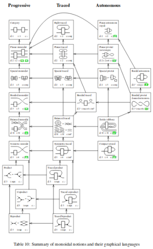




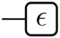


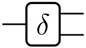


Every object is equipped with an algebraic structure, compatible with ⊗.

Example: **Set** supplies comonoids

In $(\mathbf{Set}, \times)$, we have commutative comonoids:

terminal: 

diagonal: 

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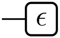
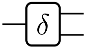
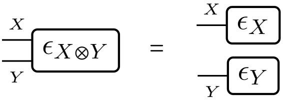
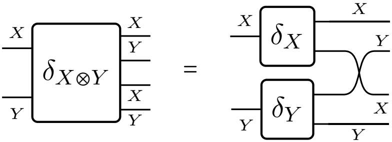
terminal: 

diagonal: 

There are compatible with $\times$:





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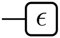
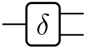
terminal: 

diagonal: 

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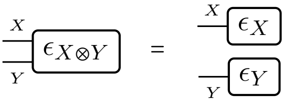
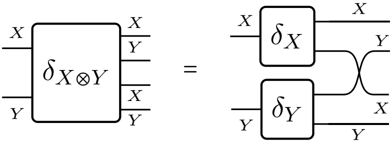
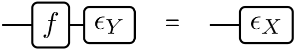
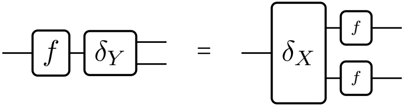




Moreover, morphisms are comonoid homomorphisms.





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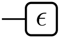
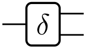
terminal: 

diagonal: 

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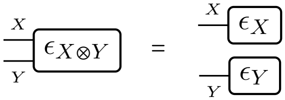

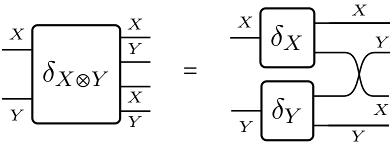

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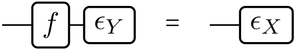

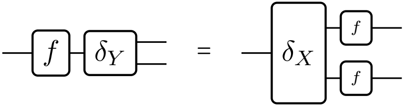

(In fact, we can recover products from this perspective.)

Outline

I. Props and theories
II. Definition
III. Examples
IV. An equivalent definition
V. Some fun facts

Props

A **prop** is a symmetric strict monoidal category where the monoid of objects is $(\mathbb{N}, +)$.

Examples:

$\mathbf{Bij}$: the prop of bijections

$\mathbf{Cob}$: the prop of unoriented 1-cobordisms

$\mathbf{FinSet}$: the prop of functions

$\mathbf{Cospan}$: the prop of cospans of functions

Props

Key idea: Props describe algebraic (monoidal) theories

For example, **FinSet** is the theory of commutative monoids.

$$\mathbf{SymMonCat}_{\mathbf{strong}}(\mathbf{FinSet}, \mathcal{C}) \cong \mathbf{CommMon}(\mathcal{C})$$

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For example, **FinSet** is the theory of commutative monoids.

$$\mathbf{SymMonCat_{strong}}(\mathbf{FinSet}, \mathcal{C}) \cong \mathbf{CommMon}(\mathcal{C})$$

Bij: the prop for objects

Cob: the prop for self-duality

FinSet: the prop for commutative monoids

Cospan: the prop for special commutative frobenius monoids

Definition of Supply

Let $\mathbb{P}$ be a prop, and $\mathcal{C}$ be a SMC.

A **supply** s of $\mathbb{P}$ in $\mathcal{C}$ is

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- for each $c \in \mathcal{C}$, a strong SMF $s_c\colon \mathbb{P} \to \mathcal{C}$

such that for all $m, n \in \mathbb{N}$, $c, d \in \mathcal{C}$, $\mu\colon m \to n \in \mathbb{P}$

(i) $s_c(m) = c^{\otimes m}$

(ii) the strongators are the unique coherence maps

$$c^{\otimes m} \otimes c^{\otimes n} \to c^{\otimes (m+n)}$$

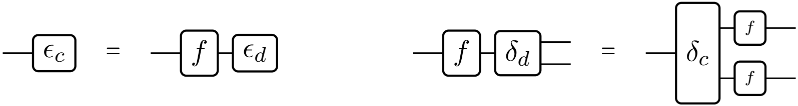
(iii)

$$\begin{array}{ccc} c^{\otimes m} \otimes d^{\otimes m} & \xrightarrow{s_c(\mu) \otimes s_d(\mu)} & c^{\otimes n} \otimes d^{\otimes n} \\ \sigma \downarrow & & \downarrow \sigma \\ (c \otimes d)^{\otimes m} & \xrightarrow[s_{c \otimes d}(\mu)]{} & (c \otimes d)^{\otimes n} \end{array} \qquad \begin{array}{ccc} I & = & I \\ \sigma \downarrow & & \downarrow \sigma \\ I^{\otimes m} & \xrightarrow[s_I(\mu)]{} & I^{\otimes n} \end{array}$$

Supply homomorphisms

A morphism $f\colon c \to d$ is an s-**homomorphism** if for all $\mu\colon m \to n$ in $\mathbb{P}$:

$$
\begin{array}{ccc}
c^{\otimes m} & \xrightarrow{s_c(\mu)} & c^{\otimes n} \\
{\scriptstyle f^{\otimes m}}\downarrow & & \downarrow{\scriptstyle f^{\otimes n}} \\
d^{\otimes m} & \xrightarrow[s_d(\mu)]{} & d^{\otimes n}
\end{array}
$$



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- A category $\mathcal{C}$ supplies $\mathbf{Cospan}$ iff it is hypergraph.
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- A category homomorphically supplies $\mathbf{FinSet}^{\mathrm{op}}$ iff the monoidal product is a categorical product.

A key theorem

Theorem
Let s be a supply of $\mathbb{P}$ in $\mathcal{C}$. Then all coherence maps of $\mathcal{C}$ are s-homomorphisms.

For example,

$$\begin{array}{ccc}
((a \otimes b) \otimes c)^{\otimes m} & \xrightarrow{s_{(a\otimes b)\otimes c}(\mu)} & ((a \otimes b) \otimes c)^{\otimes n} \\
{\scriptstyle \alpha^{\otimes m}} \downarrow & & \downarrow {\scriptstyle \alpha^{\otimes n}} \\
(a \otimes (b \otimes c))^{\otimes m} & \xrightarrow[s_{a\otimes(b\otimes c)}(\mu)]{} & (a \otimes (b \otimes c))^{\otimes n}
\end{array}$$

A key theorem

Define $\mathrm{inc}\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ to be the smallest subcategory of $\mathcal{C}$ containing all the coherence maps.

Corollary
The following are equivalent:

(a) A supply s of $\mathbb{P}$ in $\mathcal{C}$.

(b) A strong SMF $s\colon \mathbb{P} \to \mathbf{SymMonCat_{strong}}(\mathcal{C}_0, \mathcal{C})$ such that

  (i) $m \mapsto \mathrm{inc}^{\otimes m}$

  (ii) strongators are coherence maps

Preservation of supply

Let s, t respectively supply $\mathbb{P}$ in $\mathcal{C}$, $\mathcal{D}$.

Morphisms of categories supplying $\mathbb{P}$ are defined as follows.

A strong monoidal functor $(F, \varphi) \colon \mathcal{C} \to \mathcal{D}$ **preserves supply** iff for all μ in $\mathbb{P}$, $c \in \mathcal{C}$:

$$\begin{array}{ccc} F(c)^{\otimes m} & \xrightarrow{t_{F(c)}(\mu)} & F(c)^{\otimes n} \\ {\scriptstyle \varphi}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \varphi} \\ F(c^{\otimes m}) & \xrightarrow[F(s_c(\mu))]{} & F(c^{\otimes n}) \end{array}$$

Some fun facts

- If $A: \mathbb{P} \to \mathbb{Q}$ is a morphism of props, and s supplies $\mathbb{Q}$ in $\mathcal{C}$, then $A; s$ supplies $\mathbb{P}$ in $\mathcal{C}$.

- If $F: \mathcal{C} \to \mathcal{D}$ is a symmetric, essentially surjective strict monoidal functor, then if $\mathcal{C}$ supplies $\mathbb{P}$, so does $\mathcal{D}$.

- If $\mathcal{C}$ supplies $\mathbb{P}$, so does its strictification $\widetilde{\mathcal{C}}$, and the equivalence $\mathcal{C} \cong \widetilde{\mathcal{C}}$ preserves supply.

- If F preserves supply, it sends supply homomorphisms to supply homomorphisms.

- More in the paper...