

sketch of pieces of

A synthetic ~~introduction to~~ probability and statistics

Tobias Fritz

November 2019

I will give a very short introduction to the **mathematical foundations of probability and statistics**.

This will be based on measure theory, as usual.

I will give a very short introduction to the **mathematical foundations of probability and statistics**.

category theory

This will be based on ~~measure theory, as usual~~.

⇒ An unconventional introduction!

References

- ▷ Tobias Fritz,
A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics,
[arXiv:1908.07021](#).
- ▷ Tobias Fritz and Eigil Fjeldgren Rischel,
The zero–one laws of Kolmogorov and Hewitt–Savage in categorical probability, in preparation.
- ▷ Peter V. Golubtsov,
Axiomatic description of categories of information converters,
Problemy Peredachi Informatsii, 35(3):80–98 (1999).
(And other similar papers by Golubtsov.)
- ▷ Kenta Cho and Bart Jacobs,
Disintegration and Bayesian inversion via string diagrams,
Math. Struct. Comp. Sci. 29:938–971 (2019). [arXiv:1709.00322](#).

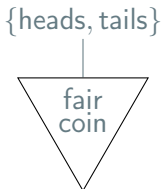
Probability theory is about *processes*



which can be composed in series and in parallel.

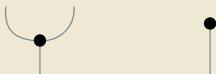
Intuition: process = function with random output.

A *probability distribution* is a process with no input, like this:

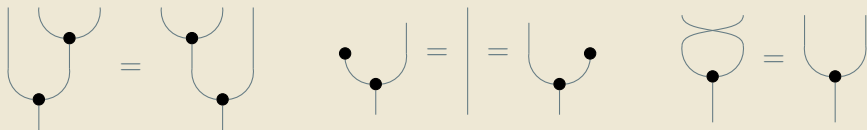


Definition

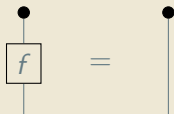
A **Markov category** \mathcal{C} is a symmetric monoidal category with **copying** and **deleting** operations on every object,



giving commutative comonoid structures

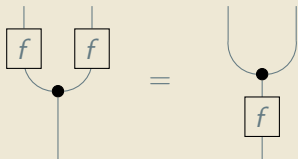


which interact well with the monoidal structure, and such that



Definition

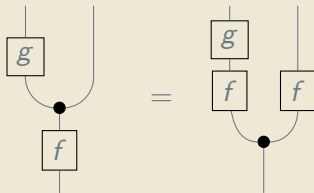
A morphism $f : X \rightarrow Y$ is **deterministic** if it commutes with copying,



- ▷ **Intuition:** Applying f to copies of input = copying the output of f .
- ▷ The deterministic morphisms form a cartesian monoidal subcategory.

Definition

\mathcal{C} is **positive** if whenever gf is deterministic for composable f and g , then also

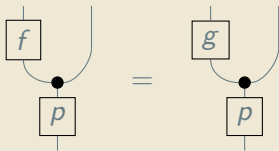


- ▷ **Intuition:** If a deterministic process has a random intermediate result, then that result can be computed independently from the process.
- ▷ Not every Markov category is positive.

Definition

Let $p : A \rightarrow X$ and $f, g : X \rightarrow Y$.

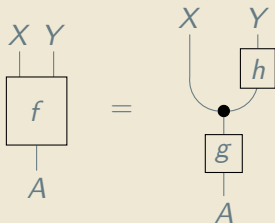
f and g are **equal p -almost surely**, $f =_{p\text{-a.s.}} g$, if



- ▷ **Intuition:** f and g behave the same on all inputs produced by p .
- ▷ Other concepts (besides equality) also relativize with respect to p -almost surely.
- ▷ In particular, C is **strictly positive** if it satisfies a relativized positivity axiom.

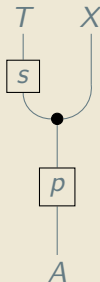
Definition

$f : A \rightarrow X \otimes Y$ displays the conditional independence $A \perp Y \mid X$ if there are g and h such that



Definition

- ▷ A **statistical model** on X is a morphism $p : A \rightarrow X$.
- ▷ A **statistic** for p is a deterministic morphism $s : X \rightarrow T$.
- ▷ The statistic is **sufficient** if



displays $A \perp X \mid T$.

There is a version of the **Fisher–Neyman factorization theorem**.

Theorem

Suppose that C is strictly positive.

A statistic $s : X \rightarrow T$ is sufficient for $p : A \rightarrow X$ if and only if there is $\alpha : T \rightarrow X$ with $\alpha s p = p$.

There are versions of other classical theorems of statistics.

Basu's theorem

A complete sufficient statistic for p is independent of any ancillary statistic.

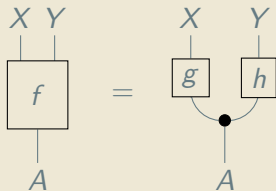
Bahadur's theorem

If a minimal sufficient statistic exists, then a complete sufficient statistic is minimal sufficient.

Explaining these would first require stating the relevant additional definitions, for which I don't have time.

Definition

$f : A \rightarrow X \otimes Y$ displays the conditional independence $X \perp Y \parallel A$ if there are g and h such that



- ▷ **Intuition:** The outputs X and Y can be produced independently.
- ▷ Note the difference from the earlier definition of conditional independence!

Definition

Let $(X_i)_{i \in I}$ be a family of objects. The **infinite tensor product**

$$X_I := \bigotimes_{i \in I} X_i$$

is the cofiltered limit of the finite tensor products $X_F := \bigotimes_{i \in F} X_i$, if this limit exists and is preserved by every $- \otimes Y$.

Definition

An infinite tensor product X_I is a **Kolmogorov product** if the limit projections $\pi_F : X_I \rightarrow X_F$ are deterministic.

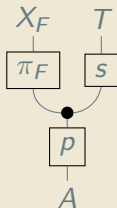
- ▷ This additional condition fixes the comonoid structure on X_I .

Theorem (Kolmogorov zero–one law)

Let X_I be a Kolmogorov product of a family $(X_i)_{i \in I}$.

If

- ▷ $p : A \rightarrow X_I$ makes the X_i independent and identically distributed, and
- ▷ $s : X_I \rightarrow T$ is such that

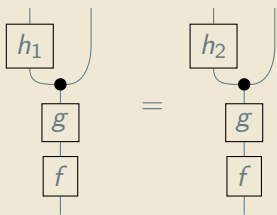


displays $X_F \perp T \parallel A$ for every finite $F \subseteq I$,

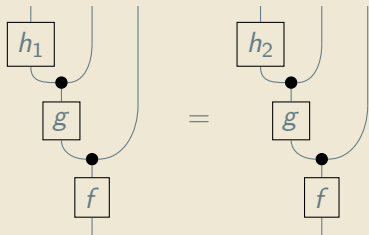
then sp is deterministic.

Definition

C is **causal** if



implies



- ▷ **Intuition:** The choice between h_1 and h_2 in the “future” of g does not influence the “past” of g .
- ▷ Not every Markov category is causal.

Theorem (Hewitt–Savage zero–one law)

Suppose that C is causal, I infinite and $X_I := \bigotimes_{i \in I} X$ a Kolmogorov product of the same X with itself.

If

- ▷ $p : A \rightarrow X_I$ makes the X_i independent and identically distributed, and
- ▷ $s : X_I \rightarrow T$ is deterministic and invariant under finite permutations,

then ps is deterministic.

Example

If $\prod_{i \in I} X$ is an infinite product of the same topological space, Y a Hausdorff space and $f : \prod_i X \rightarrow Y$ continuous and invariant under finite permutations, then f is constant.

The End

Please join us for

“Categorical Probability and Statistics”

at 75th Anniversary Summer Meeting, Canadian Mathematical Society,

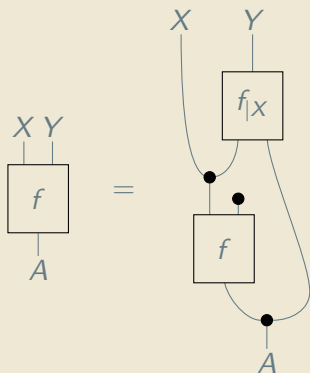
5–8 June 2020, University of Ottawa.

Organizers: Rory Lucyshyn-Wright and myself.



Definition

C has conditionals if for $f : A \rightarrow X \otimes Y$ there is $f_{|X} : X \otimes A \rightarrow Y$ with



- ▷ If C has conditionals, then it is both strictly positive and causal.
- ▷ The positivity and causality axioms (partly?) eliminate the relevance of conditionals!