Right adjoints to operadic restriction functors

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Lawvere's Question (1963)

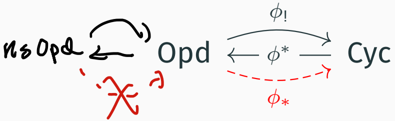
- Given an algebraic functor $f : T \to T'$ between algebraic theories, there is an adjoint pair

$$f_! : Fun^\times(T, Set) \rightleftarrows Fun^\times(T', Set) : f^*$$

  When does f^* admit a *right* adjoint f_*?
- Gavin Wraith, *Algebraic theories. Lectures Autumn 1969*. Lecture Notes Series, No. 22, Matematisk Institut, Aarhus Universitet, 1970

Motivation: Operads and Cyclic Operads

There is an unexpected right adjoint (Templeton 2003)



which may be described at an operad P by

$$(\phi_* P)(n) = \prod_{i=0}^{n} P(n) = \mathrm{hom}_{\Sigma_n}(\Sigma_{n+1}, P(n)).$$

When do such operadic right Kan extensions exist?

Main theorem (Monochrome version)

If P is an operad, let $|P|$ denote the underlying monoid.

Monoidal extension
An operad map $P \to Q$ is a *monoidal extension* just when

$$P \circ_{|P|} |Q| \to Q \circ_{|Q|} |Q| \cong Q$$

is an isomorphism.

Theorem (H & Drummond-Cole 2019)
Let $\phi : P \to Q$ be a map between (monochrome) operads. The restriction functor

$$\phi^* : \mathbf{Alg}(Q) \to \mathbf{Alg}(P)$$

admits a right adjoint if and only if ϕ is a monoidal extension.

Main theorem (Monochrome version)

Monoidal extension
An operad map $P \to Q$ is a *monoidal extension* just when

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Isomorphism of underlying monoids
If $|P| \to |Q|$ is an isomorphism, then $P \to Q$ is a monoidal extension if and only if it is an isomorphism.

Standard non-example
The inclusion functor from commutative monoids to associative monoids does not admit a right adjoint.

New Example: Little Disks, Framed Little Disks

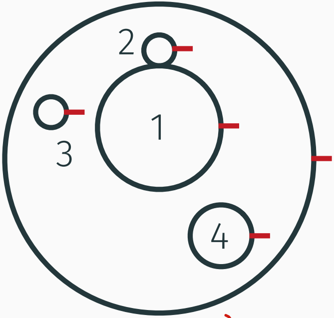
Let $\mathbb{D} \subseteq \mathbb{R}^2$ be the closed unit disk.

$$D_2(n) \subseteq D_2^{fr}(n) \subseteq \left\{ f : \coprod_{k=1}^{n} \mathbb{D} \to \mathbb{D} \right\}$$

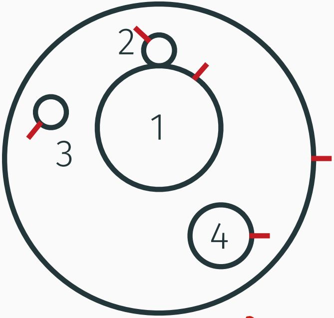
- Each $f_k : \mathbb{D} \to \mathbb{D}$ is an embedding.
- $f_k(\mathbb{D}) \cap f_j(\mathbb{D}) \subseteq f_k(\partial(\mathbb{D}))$ for $k \neq j$
- $D_2(n) \subseteq (\mathbb{R}_{>0} \times \mathbb{R}^2)^n$: f_k has has the form $f_k(\mathsf{x}) = a\mathsf{x} + \mathsf{b}$
- $D_2^{fr}(n) \subseteq (SO(2) \times \mathbb{R}_{>0} \times \mathbb{R}^2)^n$ rotation–dilation–translation

Observation
The inclusion $D_2 \to D_2^{fr}$ is a monoidal extension.

New Example: Little Disks, Framed Little Disks



$\in D_2(4)$



$\in D_2^{fr}(4)$

The inclusion $D_2 \to D_2^{fr}$ is a monoidal extension.

$$D_2(n) \circ D_2^{fr}(1) \longrightarrow D_2^{fr}(n)$$

New Example: Little Disks, Framed Little Disks

If X is a D_2-algebra, then the free loop space $LX = Map(S^1, X)$ realizes the right adjoint.

$D_2^{fr}(1)$

- $D_2^{fr}(n) \times (LX)^{\times n} \to LX = Map(SO(2), X)$
- The adjoint to the level n action takes the form:

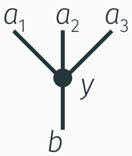
$$
\begin{array}{ccc}
SO(2) \times D_2^{fr}(n) \times (LX)^{\times n} & & \\
\downarrow & & \\
D_2^{fr}(n) \times (LX)^{\times n} & \overset{\cong}{\dashleftarrow} & D_2(n) \times SO(2)^{\times n} \times (LX)^{\times n} \\
 & & \downarrow \\
 & & D_2(n) \times X^{\times n} \\
 & & \downarrow \\
 & & X
\end{array}
$$

Bicategory of colored collections

Objects: Sets named A, B, C, etc.

(A, B) Collections:

- $S_A = \{\sigma : \underline{a} = (a_1, \ldots, a_n) \to (a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = \underline{a}\sigma\}$
- (A, B) collection Y: functor $S_A \times B \to Set$



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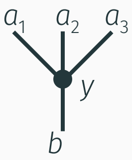
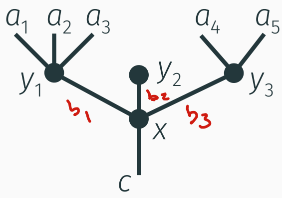


Horizontal Composition

- $\circ : (B, C)\text{-}Coll \times (A, B)\text{-}Coll \to (A, C)\text{-}Coll$
- Elements of $X \circ Y$



Adjoints at level of collections

- Let Y be an (A, B)-collection. Then

$$(-) \circ Y : (B, C)\text{-}Coll \to (A, C)\text{-}Coll$$

has a right adjoint $[Y, -]$ (Kelly, 1972)

- If X is a (B, C)-collection, then

$$X \circ (-) : (A, B)\text{-}Coll \to (A, C)\text{-}Coll$$

only has a right adjoint, denoted by $\langle X, - \rangle$, when X is concentrated in arity one

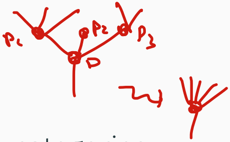
$$\langle X, Z \rangle (\underline{a}; b) = \prod_{c \in C} \hom \left(X(b; c), Z(\underline{a}; c) \right)$$

$P \to Q$

$\hom_{|P|}(|Q|, -) \leq \langle |Q|, - \rangle$

Colored operads

- An *A-colored operad P* is a monoid in the monoidal category of (A, A)-collections:



$$\mu : P \circ P \to P \qquad\qquad \eta : \mathbf{1}_A \to P$$

- Colored operads concentrated in arity one are categories.
- A map of A-colored operads $\phi : P \to Q$ is a map of monoids.

Colored operads

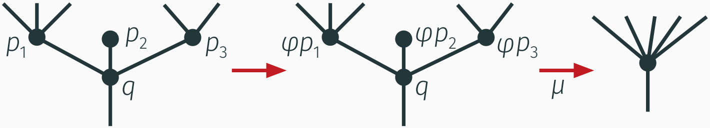
- An *A-colored operad P* is a monoid in the monoidal category of (A, A)-collections:

$$\mu : P \circ P \to P \qquad \eta : 1_A \to P$$

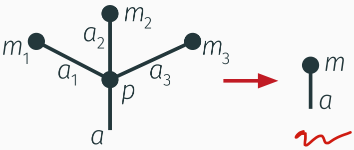
- Colored operads concentrated in arity one are categories.
- A map of A-colored operads $\phi : P \to Q$ is a map of monoids.
- If Q had a different color set B, then we also have maps $\phi : (A, P) \to (B, Q)$ lying over each function $A \to B$

Actions

- If $\phi : P \to Q$ is a map of A-colored operads, then Q is a P-Q bimodule. $P \circ Q \to Q$



- An *algebra* over P is nothing but an $(\varnothing, A)$-collection M along with a left P-action: $P \circ M \to M$



Main theorem (Fixed-color version)

Definition
A map $\phi : P \to Q$ of A-colored operads is a *categorical extension* when

$$P \circ_{|P|} |Q| \to Q$$

is an isomorphism of (A, A)-collections.

Theorem (H & Drummond-Cole 2019)
Let $\phi : P \to Q$ be a map of A-colored operads. The restriction functor

$$\phi^* : \mathsf{Alg}(Q) \to \mathsf{Alg}(P)$$

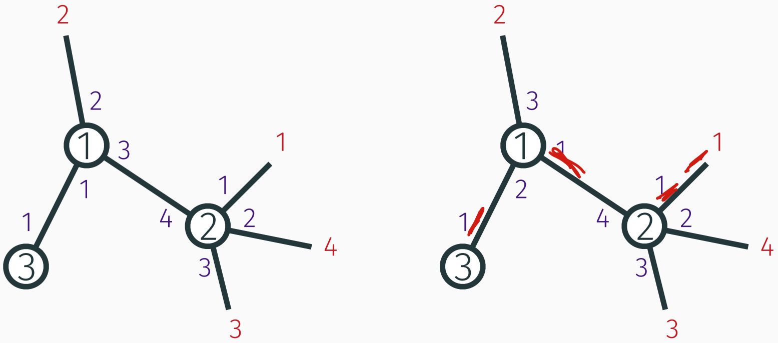
admits a right adjoint ϕ_* if and only if ϕ is a categorical extension.

Example (Operads and Cyclic Operads)

- R and T are $\mathbb{N}$-colored operads
- Operations in T are trees with total orderings on
  - set of vertices
  - vertex neighborhoods
  - boundaries
- $R \subseteq T$ consists of *rooted* trees: root of tree is first edge of boundary, root of vertex is first edge in the vertex neighborhood, and these are compatible
- $R(n; n) = \Sigma_{n-1}$ and $T(n; n) = \Sigma_n$
- $\mathsf{Alg}(R) = \mathbf{Opd}$ and $\mathsf{Alg}(T) = \mathbf{Cyc}$
- $R \subseteq T$ is a categorical extension

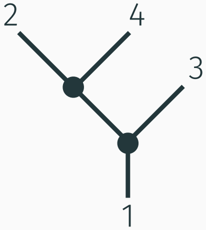
Example (Operads and Cyclic Operads)

Elements of $T(3, 4, 1; 4)$ and $R(3, 4, 1; 4)$



Non-Example (Non-Symmetric Operads and Operads)

- $P \subseteq R$ are the *planar* rooted trees.
- $P(n; n) = *$ and $R(n; n) = \Sigma_{n-1}$
- $\mathbf{Alg}(P) = \mathbf{nsOpd}$ and $\mathbf{Alg}(R) = \mathbf{Opd}$
- *Not* a categorical extension:



Necessity (Idea)

Consider the composite $F : A \to \mathbf{Alg}(Q)$

$$A \hookrightarrow (\varnothing, A)\text{-}Coll \xrightarrow{\text{free}} \mathbf{Alg}(Q)$$

Suppose that $\phi^* : \mathbf{Alg}(Q) \to \mathbf{Alg}(P)$ is a left adjoint.

If $\underline{a} = (a_1, \ldots, a_n)$ is any tuple of elements of A, then

$$\coprod_{i=1}^{n} \phi^* F(a_i) \to \phi^* \coprod_{i=1}^{n} F(a_i)$$

is an isomorphism of P-algebras, hence of $(\varnothing, A)$-collections.

Necessity (Idea)

$$\coprod_{i=1}^{n} \phi^* F(a_i) \cong \left(P \circ \coprod_{i=1}^{n} Q \circ a_i \right) / \sim$$

$$\phi^* \coprod_{i=1}^{n} F(a_i) \cong \left(Q \circ \coprod_{i=1}^{n} Q \circ a_i \right) / \sim$$

Careful analysis:

$$\left(P \circ_{|P|} |Q| \right) \binom{a}{a} \subseteq \left[\coprod_{i=1}^{n} \phi^* F(a_i) \right] \binom{\varnothing}{a}$$

$\cong$

$$Q \binom{a}{a} \subseteq \left[\phi^* \coprod_{i=1}^{n} F(a_i) \right] \binom{\varnothing}{a}$$