Stone Duality for Relations

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Dual Relations

Given a topological space extended with

- an equivalence relation or partial order, what is the algebraic structure dual to the quotient of the space?
- a non-deterministic computation (relation), what is the dual structure of pre- and post-conditions?

Given an algebraic structure extended with

- relations, what is the topological dual?

Given an (in)equational calculus of logical operations extended with

- a Gentzen-style consequence relation, what is its dual semantics for which it is sound and complete?

A Motivating Example: Cantor Space

Cantor Space C : Excluded middle third subspace of the unit interval.

Equivalence relation $\equiv$ glues together the endpoints of the gaps.

Dual of C: Free Boolean algebra $\mathrm{Fr}_{BA}(\mathbb{N})$ over countably many generators.

How is $\equiv$ reflected in $\mathrm{Fr}_{BA}(\mathbb{N})$?
Does this give rise to $[0, 1]$ as the dual of $(\mathrm{Fr}_{BA}(\mathbb{N}), \ldots)$?

This talk concentrates on the first question.
The second question is the subject of another talk.

Example: Priestley Spaces

Consider C equipped with natural order $\leq$.

- Two clopens a, b are in the dual of $\leq$ if $\uparrow a \subseteq b$
- The reflexive elements $\uparrow a \subseteq a$ are the upper clopens
- $(C, \leq)$ is the coinserter of C wrt to $\leq$
- The dual of $(C, \leq)$ is the distributive lattice of reflexive elements
- The dual of $(C, \leq)$ is the inserter of the dual of C wrt to the dual of $\leq$

These are examples of general phenomena that require topological relations.

- Every compact Hausdorff space is a quotient of a Stone space
- Priestley spaces (the duals of distributive lattices) are strongly order separated ordered Stone spaces.

Example: Stralka's Ersatzkette

Suppose we start with an ordered Stone space that is not a Priestley space?

Consider C equipped with partial order $\sqsubseteq$ only linking the left to the right endpoint of each gap. (Stralka's example of a non-Priestley ordered Stone space).

Our analysis gives a new argument why $(C, \sqsubseteq)$ is not a Priestley space: The dual of $\sqsubseteq$ is the two element lattice, which is not dual to $(C, \sqsubseteq)$

Stralka (1980)

Algebraic Example: Sequent Calculi

Relations in **DL** (distributive lattices) are essentially sequent cacluli:

Taking subobjects of products in **DL** amounts to

$$\frac{}{0R0} \qquad \frac{}{1R1} \qquad \frac{aRb \quad a'Rb'}{(a \wedge a')R(b \wedge b')} \qquad \frac{aRb \quad a'Rb'}{(a \vee a')R(b \vee b')}$$

while weakening

$$\frac{a' \leq a \quad aRb \quad b \leq b'}{a'Rb'}$$

turns out arise naturally from the duality theory.

Again, an example of general phenomenon:

- “An $\mathcal{A}$-relation for a category $\mathcal{A}$ of ordered algebras is a sequent calculus”

The Dual of a Relation in the Case of Homming into $\mathbb{2}$

Let $\mathbb{2}^- : \mathcal{X} \to \mathcal{A}^{\mathrm{op}}$ be, for example, one of the functors

$\mathbb{2}^- : \mathbf{Pos} \to \mathbf{Pos}^{\mathrm{op}}$

$\mathbb{2}^- : \mathbf{Stone} \to \mathbf{BA}^{\mathrm{op}}$ $\quad$ $\mathbb{2}^- : \mathbf{BA} \to \mathbf{Stone}^{\mathrm{op}}$

$\mathbb{2}^- : \mathbf{Pri} \to \mathbf{DL}^{\mathrm{op}}$ $\quad$ $\mathbb{2}^- : \mathbf{DL} \to \mathbf{Pri}^{\mathrm{op}}$

The extension to binary relations is a functor

$$\begin{aligned} \overline{\mathbb{2}} : \mathbf{Rel}(\mathcal{X}) &\to \mathbf{Rel}(\mathcal{A}) \\ R &\mapsto \{\, (a,b) \mid R[a] \subseteq b \,\} \end{aligned}$$

We will see later why $\overline{\mathbb{2}}$ is an equivalence of categories whenever $\mathbb{2}^-$ is

Relations

Let $U : \mathcal{C} \to$ **Pos** be a category (with some good properties ...)

Definition: A **relation** $R : A \looparrowright B$ in $\mathcal{C}$ is a
- sub-object $R \subseteq A \times B$ that is also
- an order-preserving map $A^{\mathrm{op}} \times B \to \mathbb{2}$

where $\mathbb{2} = \{0 < 1\}$

Remark: Also called *monotone* or *weakening (closed)* relations

Examples: Stone-relations, BA-relations, Priestley-relations, DL-relations, ...

These requirements arise from the interplay of spans and cospans in **Pos**.

Relations as Spans and Cospans

Relations can be tabulated as

spans and as *cospans*

$$
\begin{tikzcd}
 & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[dr, "j"'] & & Y \arrow[dl, "k"] \\
 & C &
\end{tikzcd}
$$

with

$xRy \Leftrightarrow \exists w \,.\, x \leq pw \;\&\; qw \leq y$ $\qquad$ $xRy \Leftrightarrow jx \leq_C ky$

For spans the $\leq$ is not essential, it is for cospans:

- The order $\leq_C$ of C encodes the relation R
- C necessarily encodes a weakening relation

Relations as Equivalence Classes of (Co)Spans

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "j"'] & & Y \arrow[dl, "k"] \\ & C & \end{tikzcd}$$

Different spans, and different cospans, can represent the same relation

Each equivalence class has a normal form as a span and as a cospan.

- The span-normal form of a cospan as the ‘ordered pullback’ of the cospan
- The cospan-normal form of a span is the ‘ordered pushout’ of the span

Exact Squares

Exact squares were introduced by Hilton in the context of abelian categories and generalised by Guitart to 2-categories.
We apply these ideas to order enriched categories
A diagram in **Pos**

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A \arrow[dr, "j"'] \arrow[rr, phantom, "\leq"] & & B \arrow[dl, "k"] \\ & C & \end{tikzcd}$$

is called **exact** if $Rel(p, q) = Rel(j.k)$.
Proposition: Comma and cocomma squares in **Pos** are exact

Define: **Rel**(**Pos**) is the ordered category of spans (or cospans) modulo exact squares

Concretely Order-Regular Categories

Generalise **Rel**(**Pos**) to **Rel**($\mathcal{C}$) for suitable categories $\mathcal{C}$
In concretely-order regular categories relations behave as in **Pos**
Definition: $U : \mathcal{C} \to$ **Pos** is concretely-order regular if

- U is order faithful (injective and order-reflecting on homsets)
- $\mathcal{C}$ has and U preserves finite weighted limits
- $\mathcal{C}$ has and U preserves Onto-Embedding factorisations

The last item can be replaced by “existence of exact cocommas” and the last two items can be replaced by ”existence of enough exact squares”

Define: **Rel**($\mathcal{C}$) to have equivalence classes of weakening closed spans as morphisms. Composition is defined by order-pullback and Onto-Embedding factorization.

Main Theorem 1

If

$U : \mathcal{X} \to \mathbf{Pos}$ and $V : \mathcal{A} \to \mathbf{Pos}$ are concretely-order regular categories

$F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ are a dual equivalence preserving exact squares

Then

F and G extend to an equivalence of categories of relations

$$\begin{tikzcd} \mathbf{Rel}(\mathcal{A}) \arrow[r, bend left, "\mathbf{Rel}(F)"] & \mathbf{Rel}(\mathcal{X})^{\mathrm{co}} \arrow[l, bend left, "\mathbf{Rel}(G)"] \end{tikzcd}$$

Main Theorem 2

If

$U : \mathcal{X} \to \mathbf{Pos}$ and $V : \mathcal{A} \to \mathbf{Pos}$ are concretely-order regular categories

$F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ are a dual adjunction preserving exact squares and mapping surjections to embeddings

Then

F and G extend to an adjunction of framed bicategories of relations

$$\begin{tikzcd} (\S\mathcal{A})^{\mathrm{co}} \arrow[r, bend left, "\S F"] & \S\mathcal{X} \arrow[l, bend left, "\S G"] \end{tikzcd}$$

We cannot replace $\S\mathcal{A}, \S\mathcal{X}$ by $\mathbf{Rel}(\mathcal{A}), \mathbf{Rel}(\mathcal{X})$ because the unit and the counit of the extended adjunction are only natural wrt to maps, not wrt relations

Framed Bicategories

Shulman's framed bicategories are particular double categories in which the 'vertical' arrows behave like maps and the 'horizontal' arrows like relations

Framed bicategories organise themselves in a 2-category

2-categories come with a native notion of adjunction

Spelling out the details, one finds that this notion of adjunction requires naturality only wrt to vertical arrows (maps)

References, Background, Related Work

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Conclusion

Extend Stone duality from maps to relations

In preparation: Extending zero-dimensional dualities to continuous dualities

What we have done:

- category theory of *cat's enriched over cat's enriched over* $\mathbb{2}$
- examples with *dualising object* $\mathbb{2}$

Future work:

- (more of the above)
- *many-valued valuations*: general dualising poset of truth values (replacing $\mathbb{2}$)
- *many-valued relations*: enrich over lattice of truth values (replacing $\mathbb{2}$)

Other dualising objects could lead to new results for many-valued logic?

Ask me for a preprint if you are interested ...