Strings for Cartesian Bicategories

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[1] Steve Vickers is reporting on closely related joint work at SYCO 6 in Leicester next month.

Motivation

- Partially ordered (or pre-ordered) sets are the basic structures of algebraic logic:
  - A set (of "propositions")
  - An "entailment" relation between them: $p \Rightarrow q$
- Additional logical structure provided by connectives with rules.
- Want to put this onto a category-theoretic footing.
- The idea is to regard morphisms between logics as generalized entailment relations.

Categories of logics

- Two obvious options for making a category of abstract logics:
  - Use $\Rightarrow$ (and connective) preserving functions between objects.
    Propositions in one language are directly translated to propositions of another.
  - Or, since $\Rightarrow$ (the "identity " of a logic) is itself a relation, generalize these to "entailment" morphisms.
- This becomes interesting because:
  - We discover lots of other models
  - We have a calculus (a way to construct free categories of the right kind)
  - The calculus has an interesting geometric interpretation via string diagrams.
  - There are connections (via the diagrams) to sequent calculus. – [N.B. Related to Alexander Kurz' and my report earlier today.]

The Compact Structure of Pos*

Pos* consists of posets and weakening relations. [We only use this as a reference model.]
Cartesian products of posets are posets, but *not* the categorical product.

- In this category, disjoint union is product and coproduct [Exercise]
- Let $A \otimes B$ denote cartesian product. Then
  - $\mathbb{I}$ (singleton $\{\star\}$) is unit (up to natural iso) for $\otimes$
  - $\otimes$ is symmetric, monoidal: $A \otimes B \simeq B \otimes A$, etc.
  - $A \otimes B \simeq B^\partial \otimes A^\partial$
  - $\mathrm{hom}(A \otimes B, C) \equiv \mathrm{hom}(A, B^\partial \otimes C)$
  - $A = A^{\partial\partial}$
- These say that Pos* is a compact closed category.

But wait! There's more.

Cartesian Bicategories

Definition (Carboni & Walter)

A cartesian bicategory is

- Poset enriched
- Symmetric monoidal: $\otimes$, $\mathbb{I}$ with the usual natural isos
- $\otimes$ is monotonic on hom sets
- Every object is equipped with a comonoid:
  - $\hat{\delta}_A \colon A \to A \otimes A$
  - $\hat{\kappa}_A \colon A \to \mathbb{I}$
- all morphims are lax homomorphisms for the comonoid:

$$R; \hat{\delta}_B \leq \hat{\delta}_A; (R \otimes R)$$
$$R; \hat{\kappa}_B \leq \hat{\kappa}_A$$

- $\hat{\delta}_A$ and $\hat{\kappa}_A$ are maps [they have lower adjoints $\check{\delta}_A$ and $\check{\kappa}_A$].
- $\hat{\delta}_A; \check{\delta}_A = \mathrm{id}_A$

$\mathbf{Pos}^*$, $\mathbf{Lat}^*$, $\mathbf{DLat}^*$, $\mathbf{BA}^*$ and $\mathbf{Set}^*$ are compact cartesian bicategories

In Pos^*

- $\otimes$ and $\mathbb{I}$ determine the symmetric monoidal structure.
- Obviously $\otimes$ is monotonic on hom-sets.
- The relations
  - $a\ \hat{\delta}_A\ (b, c)$ if and only if $a \leq b$ and $a \leq c$
  - $a\ \hat{\kappa}\ \star$ (all a)

  determine cartesian bicategory structure.
- $-^\partial$ makes Pos^* also compact closed.

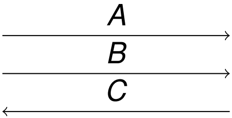
Moreover,

- In Set^* and BA^*, $A^\partial \simeq A$.
- In Lat^*, DLat^* same as in Pos^*.

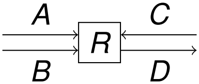
String Diagrams For Symmetric Monoidal Categories (review)

Symmetric monoidal (and compact closed) categories have a coherence theorem based on string diagrams

A diagram of $A \otimes B \otimes C^\partial$:



A diagram of $R\colon A \otimes B \nrightarrow C^\partial \otimes D$:

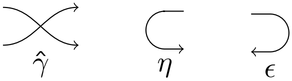


Theorem (Joyal & Street)

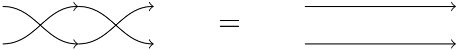
Two diagrams denote the same morphism in all compact closed categories iff they are homotopically equivalent (in $\mathbb{R}^4$).

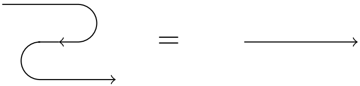
Some Details of Diagrams (review)

Symmetry is “crossed wires”. Empty diagram is $\mathbb{I}$. Unit and counit are “u-turns”.



So the compact closed structure is reflected in various *equations*:





Bicartesian Enrichment

Diagrams for the diagonals

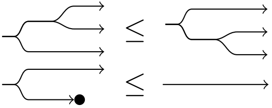
$\hat{\delta}$ $\check{\delta}$ $\hat{\kappa}$ $\check{\kappa}$

Map axioms

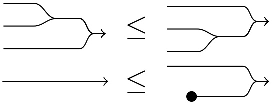
More Axioms (and Lemmas)

Comonoid/monoid

Comonoid Axioms



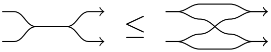
Monoid lemmas



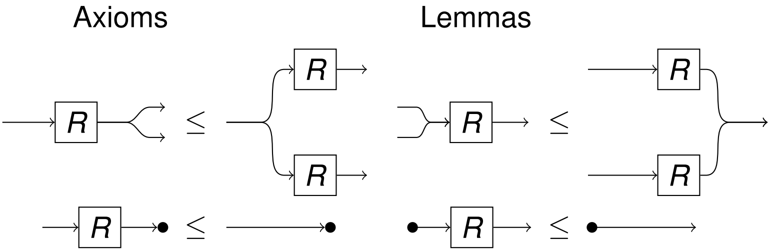
Split monicity axiom for δ

Lax Naturality Axioms and lemmas

Weak Frobenius Axiom (laxity for $\check{\delta}$ wrt $\hat{\delta}$)



Laxity for basic morphisms



Coherence Theorems

Theorem

Let $\leq$ be the least pre-order on string diagrams including the axioms and closed under composition and $\otimes$ (stacking). Then the poset reflection of $\leq$ determines an initial cartesian bicategory (for a given set of basic objects and morphisms).

Theorem

The same construction works for compact closed cartesian bicategories.

Theorem

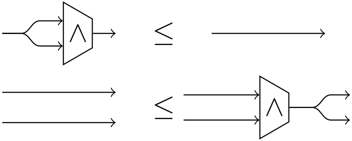
The same construction also works when $\leq$ is augmented with an inequational theory (a set of pairs of diagrams).

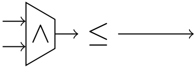
Lattice-like Objects in Cartesian Bicategories

Meets and joins

- And object is meet semilattice-like if $\hat{\delta}_A$ is a comap (it is already a map).

  That is, there is a morphism  satisfying

  It is easy to show that $\wedge$ is idempotent, associative and commutative and deflating: 
- Dually, A is join semilattice-like if $\check{\delta}_A$ is a map.

More on Lattices

Lemma

In Pos*:

- *A poset P is an actual meet semilattice iff it is meet semilattice-like.*
- *A poset P is an actual join semilattice iff it is join semilattice-like.*

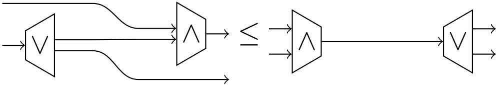
Moreover

- Boundedness is characterized by $\hat{\kappa}_A$ being a comap ($\top$) or $\check{\kappa}_A$ being a map ($\bot$).
- What about distributivity?

Distributivity

Lemma

A lattice-like object in a cartesian bicategory is distributive, i.e., $\wedge$ distributes over $\check{\delta}$ (the map corresponding to comap $\vee$), if and only if

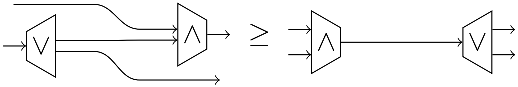


The proof is entirely “stringy”. That is, we use only string rewriting in the initial bicartesian category of string diagrams.

Complementedness

Lemma

In Pos*, if an object is a distributive lattice, then it is complemented if and only if



Remark

- This condition is dual to the Frobenius Law (FL) for the bialgebra $(\hat{\delta}, \check{\delta}, \hat{\kappa}, \check{\kappa})$.
- If FL holds for all objects, the bicartesian category is a regular allegory (objects ares “discrete”).
- “Complemented distributive lattice” is dual to “discrete”.

Other Examples and Constructions

Examples

- Compact pospaces by taking *closed weakening relations* as morphisms. Then maps are bijective with continuous monotonic functions.
- Proximity lattices (not quite discussed yesterday).
- Rel – all objects satisfy Frobenius Law

Constructions

- Map-comma: Objects are maps into a base poset B. Morphisms are lax homomorphisms.
- Karoubi envelope of a given cartesian bicategory (Steve Vickers is investigating these for particular applications).
- $(\mathrm{Pos}^*)^{\mathcal{A}^{\mathrm{op}}}$ – “presheaves” over the base Pos^*?

Thanks