Quantitative Equational Reasoning

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November 2019,
AMS Sectional Meeting, Riverside, California
Special Session on Applied Category Theory

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- Enriched Lawvere theories?

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- Theories: set of equations closed under deduction.

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- Substitution

$$\frac{Ax \vdash t = s}{Ax \vdash t[u/x] = s[u/x]}$$

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- As Ω-algebra $\mathcal{A}$ is a set A to interpret the basic sort and, for each operation f of arity n a function $f_{\mathcal{A}} : A^n \longrightarrow A$.

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- The elements of X are in TX.
- If $t_1, \ldots, t_n$ are in TX and f has arity n then $f(t_1, \ldots, t_n)$ is in TX.

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- Let S be a set of equations between pairs of terms in TX. We define a *congruence relation* $\sim_S$ on TX in the evident way.

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- Easy to check that if $t_1 \sim_S s_1, \ldots, t_n \sim_S s_n$ then $f(t_1, \ldots, t_n) \sim_S f(s_1, \ldots, s_n)$ we can define $f_{\sim_S}$ on $TX/\sim_S$.

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- When are a set of equations bad? If we can derive $x = y$ from S then the only algebras have one element.

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- Stacks are equationally definable but queues are not.

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- Axioms:

$$(B_1) \vdash t +_1 t' = t$$
$$(B_2) \vdash t +_\epsilon t = t$$
$$(SC) \vdash t +_\epsilon t' = t' +_{1-\epsilon} t$$
$$(SA) \vdash (t +_\epsilon t') +_{\epsilon'} t'' = t +_{\epsilon\epsilon'} (t' +_{\frac{\epsilon' - \epsilon\epsilon'}{1-\epsilon\epsilon'}} t'')$$

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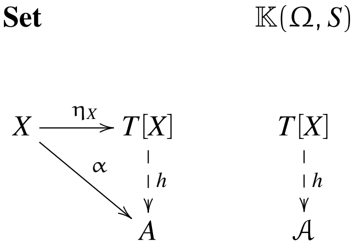

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- Quantitative equations give monads on $\mathbf{Met}$.

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- $W^{(p)}(\delta_x, \delta_y) = d(x, y)$.

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(NExp) For $f : n \in \Omega$,
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(Subst) If $\sigma \in \Sigma(X)$, $\Gamma \vdash t =_\varepsilon s$ implies $\sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(s)$.

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- (Inf) For all $\varepsilon \geqslant 0$, $\{t =_{\varepsilon'} s \mid \varepsilon' > \varepsilon\} \vdash t =_\varepsilon s$. Infinitary!
- (NExp) For $f : n \in \Omega$,
  $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1, ..t_i, ..t_n) =_\varepsilon f(s_1, ..s_i, ..s_n)$
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- (Cut) If $\Gamma \vdash \phi$ for all $\phi \in \Gamma'$ and $\Gamma' \vdash \psi$, then $\Gamma \vdash \psi$.

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- $\mathbb{T}X$ is an Ω-algebra. $\sigma : \mathbb{T}X \longrightarrow A$, Ω-homomorphism.

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- We write $\{s_i =_{\varepsilon_i} t_i / i = 1, \ldots, n\} \models_{\mathcal{A}} s =_{\varepsilon} t$.
- We write $\mathbb{K}(\mathcal{U}, \Omega)$ for the algebras satisfying $\mathcal{U}$.

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- We can do this for any set M of generators and produce a "free" quantitative algebra.

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- Left to right is by a model construction.
- The proof needs to deal with quantitative aspects and uses the infinitary rule in a crucial way.

Free construction from a metric space

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- Given any $\alpha : M \longrightarrow A$ non-expansive we can turn $\mathcal{A} = (A, d)$ into an algebra in $\mathbb{K}(\Omega_M, \mathcal{U}_M)$ by interpreting each $m \in M$ as $\alpha(m) \in A$.

Universal property

Met $\qquad\qquad\qquad\qquad$ $\mathbb{K}(\Omega, \mathcal{U})$

$$\begin{array}{ccc} (M, d^M) & \xrightarrow{\eta_M} & T[M] \\ & \searrow^{\alpha} & \downarrow h \\ & & (A, d^A) \end{array} \qquad\qquad \begin{array}{c} T[M] \\ \downarrow h \\ \mathcal{A} \end{array}$$

We have a monad on **Met**.

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- The total variation metric on probability distributions.

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- There is a duality theorem that gives it as a minimum rather than a maximum.

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- $\mathcal{C}(\mu, \nu)$ is the set of couplings for (μ, ν).

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- Splitting lemma: If μ, ν are Borel probability measures on M and $e = tv(\mu, \nu)$. There are μ', ν', θ such that

$$\mu = e\mu' + (1-e)\theta \text{ and } \nu = e\nu' + (1-e)\theta.$$

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- We endow it with the total-variation metric to make it a quantitative algebra.

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- The axioms give rise to the total-variation metric.

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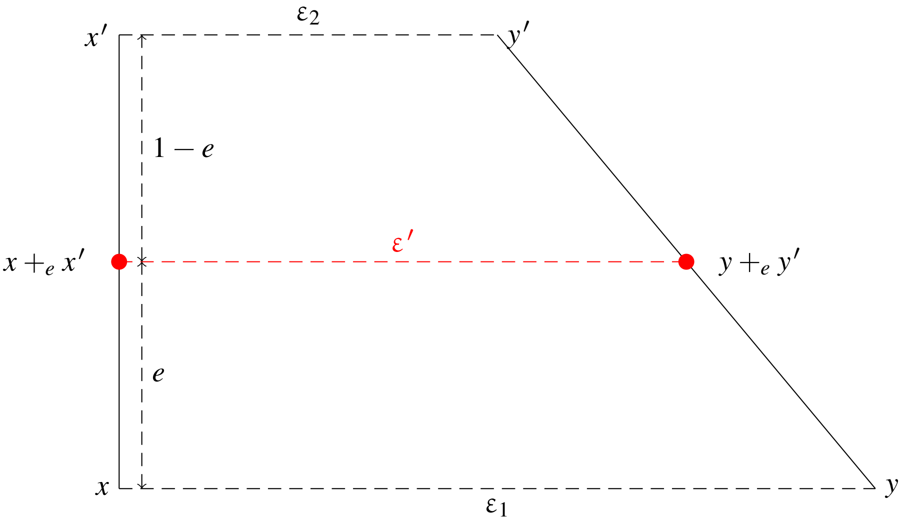
Picture of IB_1



Kantorovich metric

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$$K_d(\mu,\nu) = \sup\left\{ \left| \int f\mathrm{d}\mu - \int f\mathrm{d}\nu \right| \right\}$$

Kantorovich metric

Let (M, d) be a complete separable metric space and $p \geqslant 1$.

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- How do we lift it to the continuous case?

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- The finitely supported probability measures are *dense* in the space of all probability measures with weak topology.

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- In this case we get a monad on $\mathbf{CSMet}_1$: complete separable 1-bounded metric spaces.

Conclusions

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- Would like many more examples: e.g. Choquet capacities and games.
- Fit it into the Lawvere theory picture: new notions of arity.