Constructing symmetric monoidal bicategories functorially

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Outline

Symmetric monoidal bicategories

Symmetric monoidal bicategories are everywhere!

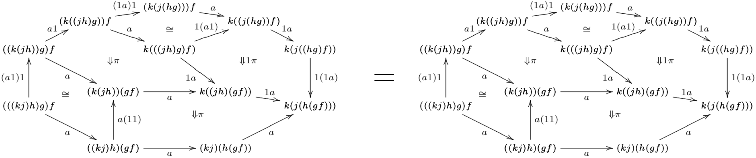
1. Rings and bimodules
2. Sets and spans
3. Sets and relations
4. Categories and profunctors
5. Manifolds and cobordisms
6. Topological spaces and parametrized spectra
7. Sets and (decorated/structured) cospans
8. Sets and open Markov processes
9. Vector spaces and linear relations

What is a symmetric monoidal bicategory?

A symmetric monoidal bicategory is a bicategory $\mathcal{B}$ with

1. A functor $\otimes \colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$
2. A pseudonatural equivalence $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$
3. An invertible modification

$$\begin{array}{ccccc} ((AB)C)D & \longrightarrow & (A(BC))D & \longrightarrow & A((BC)D) \\ \downarrow & & \pi \Downarrow & & \downarrow \\ (AB)(CD) & & \longrightarrow & & A(B(CD)) \end{array}$$

4. 

5. more data and axioms for units, braiding, syllepsis. . .

Surely it can't be that bad

In all the examples I listed before, the monoidal structures

1. tensor product of rings
2. cartesian product of sets
3. cartesian product of spaces
4. disjoint union of sets
5. . . .

are actually associative up to isomorphism, with strictly commuting pentagons, etc.

But a bicategory doesn't know how to talk about isomorphisms, only equivalences! We need to add extra data: ring homomorphisms, functions, linear maps, etc.

Double categories

A double category is an internal category in Cat. It has:

1. objects $A, B, C, \ldots$
2. loose morphisms $A \nrightarrow B$ that compose weakly
3. tight morphisms $A \to B$ that compose strictly
4. 2-cells shaped like squares:

$$\begin{array}{ccc} A & \overset{}{\relbar\!\!\mapsto} & B \\ \downarrow & \Downarrow & \downarrow \\ C & \relbar\!\!\mapsto & D \end{array} .$$

No one can agree on which morphisms to draw horizontally or vertically. But "loose" and "tight" are independent of that choice.

Symmetric monoidal double categories

Double categories, (pseudo) functors, and strictly-natural tight transformations form a 2-category $\mathcal{D}bl$.

Definition

A symmetric monoidal double category is a symmetric pseudo-monoid in $\mathcal{D}bl$.

- Coherences are isomorphisms, not equivalences, and diagrams commute strictly.
- Hardly more complicated than a pair of ordinary symmetric monoidal categories.

Symmetric monoidal double categories

Symmetric monoidal double categories are everywhere!

1. Rings, bimodules, and ring homomorphisms
2. Sets, spans, and functions
3. Sets, relations, and functions
4. Categories, profunctors, and functors
5. Manifolds, cobordisms, and diffeomorphisms
6. Topological spaces, parametrized spectra, and continuous maps
7. Sets, (decorated/structured) cospans, and functions
8. Sets, open Markov processes, and functions
9. Vector spaces, linear relations, and linear transformations

. . . but what if what we actually want is a monoidal bicategory?

Companions

A companion of a tight morphism $f : A \to B$ is a loose morphism $\widehat{f} : A \nrightarrow B$ and squares

$$\begin{array}{ccc} & \xrightarrow{\widehat{f}} & \\ f\downarrow & \Downarrow\epsilon_{\hat{f}} & \| \\ & = & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & = & \\ \| & \Downarrow\eta_{\hat{f}} & \downarrow f \\ & \xrightarrow[\widehat{f}]{} & \end{array}$$

such that the following equations hold.

$$\begin{array}{ccc} & = & \\ \| & \Downarrow\eta_{\hat{f}} & \downarrow f \\ & \xrightarrow{\widehat{f}} & \\ f\downarrow & \Downarrow\epsilon_{\hat{f}} & \| \\ & = & \end{array} \;=\; \begin{array}{ccc} & = & \\ f\downarrow & \Downarrow 1_f & \downarrow f \\ & = & \end{array} \qquad\qquad \begin{array}{ccccc} & = & & \xrightarrow{\widehat{f}} & \\ \| & \Downarrow\eta_{\hat{f}} & f\downarrow & \Downarrow\epsilon_{\hat{f}} & \| \\ & \xrightarrow[\widehat{f}]{} & & = & \end{array} \;=\; \begin{array}{ccc} & \xrightarrow{\widehat{f}} & \\ \| & \Downarrow 1_{\widehat{f}} & \| \\ & \xrightarrow[\widehat{f}]{} & \end{array}$$

There is also a dual notion of conjoint. The double categories listed before have companions and conjoints for all tight morphisms (i.e. they are framed bicategories).

Constructing symmetric monoidal bicategories

Theorem (S., 2010)

Let $\mathbb{D}$ be a symmetric monoidal bicategory whose coherence isomorphisms have loose companions. Then the underlying bicategory $\mathcal{L}\mathbb{D}$ of objects and loose morphisms is a symmetric monoidal bicategory. (And similarly for the monoidal and braided monoidal cases.)

Proof.

Lift all the coherences to the companions. □

In particular, all the examples listed before are symmetric monoidal bicategories.

Outline

What about functors?

But we'd also like to be able to construct

- monoidal functors and
- monoidal transformations,
- in a way that preserves composition,
- and hence preserves things like adjunctions,
- . . .

In other words, what we want is a functor

$$\mathcal{L} : \mathcal{M}onDbl \to \mathcal{M}onBicat.$$

Functors beget functors

Morally, we should have

$$\mathcal{M}on\mathit{Dbl} = \mathcal{M}on(\mathcal{D}\mathit{bl})$$
$$\mathcal{M}on\mathit{Bicat} = \mathcal{M}on(\mathcal{B}\mathit{icat})$$

and $\mathcal{M}on$ should be a functor, so that our desired functor

$$\mathcal{L} : \mathcal{M}on\mathit{Dbl} \to \mathcal{M}on\mathit{Bicat}$$

is simply induced by an easier functor

$$\mathcal{L} : \mathcal{D}\mathit{bl} \to \mathcal{B}\mathit{icat}.$$

But what kind of functors are these?

Tricategories?

First answer

$\mathcal{B}icat$ is a tricategory, and we can regard $\mathcal{D}bl$ as a tricategory with no nonidentity 3-cells. So $\mathcal{L}$ should be a tricategory functor.

First problem

- Constructing a tricategory functor is a lot of work.
- More importantly, a monoidal bicategory, as usually defined, is not just a "monoid object in the tricategory $\mathcal{B}icat$".

$\mathcal{B}icat$ is stricter than a general tricategory in several ways, and a definition that makes sense in an arbitrary tricategory would involve a lot of superfluous coherence data when specialized to $\mathcal{B}icat$.

Iconic tricategories?

Second answer

$\mathcal{B}icat$ and $\mathcal{D}bl$ are iconic tricategories: bicategories enriched over the 2-category of bicategories, pseudofunctors, and icons.

Definition (Lack 2010)

An icon (Identity Component Oplax Natural transformation) between pseudofunctors $F, G : \mathcal{A} \to \mathcal{B}$ consists of:

1. The assertion that $F(A) = G(A)$ for all objects $A \in \mathcal{A}$.
2. For all $\varphi : X \to Y$ in $\mathcal{A}$, a 2-cell

$$FX = GX \underset{G\varphi}{\overset{F\varphi}{\Downarrow}} FY = GY$$

Iconic tricategories?

An iconic tricategory is equivalently a tricategory in which composition of 1-cells is strictly associative and unital (though compostion of 2-cells along 0-cells need not be).

Second problem

$\mathcal{M}onBicat$ is not iconic!

Composition of monoidal functors of bicategories is not strictly associative. For $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$, the composite laxator is

$$GFX \otimes GFY \xrightarrow{G_\otimes} G(FX \otimes FY) \xrightarrow{F_\otimes} GF(X \otimes Y)$$

a composition in $\mathcal{C}$, which is a bicategory.

Third time's the charm

Final answer

$\mathfrak{Bicat}$, $\mathfrak{Dbl}$, $\mathfrak{MonBicat}$, $\mathfrak{MonDbl}$ are all locally cubical bicategories (Garner-Gurski 2009): bicategories enriched over the monoidal 2-category $\mathcal{D}bl$.

$\mathfrak{Bicat}$ has:

1. Objects: bicategories
2. 1-cells: pseudofunctors
3. loose 2-cells: pseudonatural transformations
4. tight 2-cells: icons
5. square 3-cells: cubical modifications

Third time's the charm

Final answer

$\mathfrak{Bicat}$, $\mathfrak{Dbl}$, $\mathfrak{MonBicat}$, $\mathfrak{MonDbl}$ are all locally cubical bicategories (Garner-Gurski 2009): bicategories enriched over the monoidal 2-category $\mathcal{D}bl$.

$\mathfrak{Dbl}$ has:

1. Objects: double categories (with companions)
2. 1-cells: pseudofunctors
3. loose 2-cells: strictly-natural tight transformations
4. tight 2-cells: only identities
5. square 3-cells: only identities

Third time's the charm

Final answer

$\mathfrak{Bicat}$, $\mathfrak{Dbl}$, $\mathfrak{MonBicat}$, $\mathfrak{MonDbl}$ are all locally cubical bicategories (Garner-Gurski 2009): bicategories enriched over the monoidal 2-category $\mathcal{D}bl$.

$\mathfrak{MonBicat}$ has:

1. Objects: monoidal bicategories
2. 1-cells: monoidal pseudofunctors
3. loose 2-cells: monoidal pseudonatural transformations
4. tight 2-cells: monoidal icons
5. square 3-cells: cubical monoidal modifications

In particular, composition of monoidal pseudofunctors is associative up to a *monoidal icon*.

Monoids in locally cubical bicategories

Theorem (Hansen–S.)

Let $\mathfrak{B}$ be a locally cubical bicategory with products, in which composition of 1-cells is strictly associative. Then there is a locally cubical bicategory $\mathrm{Mon}(\mathfrak{B})$ *of monoids in* $\mathfrak{B}$.

Just write down the ordinary definitions of monoidal bicategory, monoidal pseudofunctor, monoidal icon, etc. in "point-free style". A 1-strict locally cubical bicategory is (almost) exactly the correct structure in which they make sense.

Example

$$\mathrm{Mon}(\mathfrak{Dbl}) = \mathfrak{MonDbl}$$
$$\mathrm{Mon}(\mathfrak{Bicat}) = \mathfrak{MonBicat}$$

Constructing symmetric monoidal bicategories functorially

Theorem (Hansen-S.)

Let $F : \mathfrak{B} \to \mathfrak{C}$ *be a locally cubical functor that preserves products and also strict 1-cell composition. Then there is a locally cubical functor* $\mathsf{Mon}(F) : \mathsf{Mon}(\mathfrak{B}) \to \mathsf{Mon}(\mathfrak{C})$, *and similarly for braided and symmetric monoids.*

Example

$$\mathsf{Mon}(\mathfrak{Dbl}) \to \mathsf{Mon}(\mathfrak{Bicat})$$
$$\mathsf{BrMon}(\mathfrak{Dbl}) \to \mathsf{BrMon}(\mathfrak{Bicat})$$
$$\mathsf{SymMon}(\mathfrak{Dbl}) \to \mathsf{SymMon}(\mathfrak{Bicat})$$

The fine print, I

Monoidal pseudofunctors can be (monoidally) lax, colax, or strong — and so can monoidal pseudonatural transformations!

$$
\begin{array}{ccc}
FX \otimes FY & \xrightarrow{F_{\otimes}} & F(X \otimes Y) \\
{\scriptstyle \alpha\otimes\alpha}\downarrow & \swArrow & \downarrow{\scriptstyle \alpha} \\
GX \otimes GY & \xrightarrow[G_{\otimes}]{} & G(X \otimes Y)
\end{array}
$$

$$
\begin{array}{ccc}
FX \otimes FY & \xrightarrow{F_{\otimes}} & F(X \otimes Y) \\
{\scriptstyle \alpha\otimes\alpha}\downarrow & \neArrow & \downarrow{\scriptstyle \alpha} \\
GX \otimes GY & \xrightarrow[G_{\otimes}]{} & G(X \otimes Y)
\end{array}
$$

So we actually have nine functors.

The fine print, II

Bicategories also have a strictly associative whiskering operation.

$$(\alpha * G) * F = \alpha * (G \circ F) \qquad G * (F * \beta) = (G \circ F) * \beta$$

But a 1-strict locally cubical bicategory doesn't have such an operation: we have to use $\alpha \circ 1_F$, which is only isomorphic to $\alpha * F$. Thus a monoid in $\mathfrak{Bicat}$ still has a bit more coherence data than a monoidal bicategory.

Possible fix: Verity (1992) defined a multicategory $\underline{\mathcal{B}icat}$ such that $\underline{\mathcal{B}icat}$-enriched categories are "iconic tricategories with strictly associative whiskering". Perhaps there is a similar $\underline{\mathcal{D}bl}$.

The fine print, III

- A monoid in the category of monoids is a commutative monoid.
- A monoid in the 2-category of monoidal categories is a braided monoidal category (Joyal–Street 1993).
- A monoid in the 2-category of braided monoidal categories is a symmetric monoidal category (Joyal–Street 1993).

We should similarly have $\mathrm{Mon}(\mathrm{Mon}(\mathfrak{B})) \simeq \mathrm{BrMon}(\mathfrak{B})$, etc. — but our Mon can't be iterated, since $\mathrm{Mon}(\mathfrak{B})$ is no longer 1-strict!

The restriction of 1-strictness is surprisingly hard to lift. Without it, the coherence diagrams for defining monoids in a locally cubical bicategory end up trying to compose tight and loose 2-cells with each other. We need some kind of "local companions".