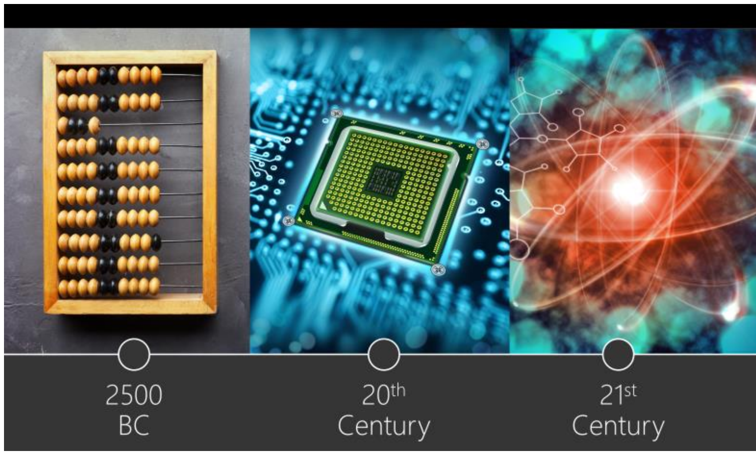
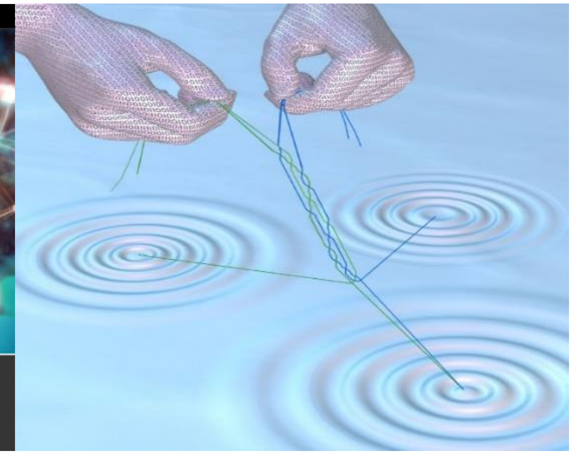
Mathematics for Second Quantum Revolution





Zhenghan Wang
Microsoft Station Q & UC Santa Barbara

AMS Riverside, Nov. 10, 2019

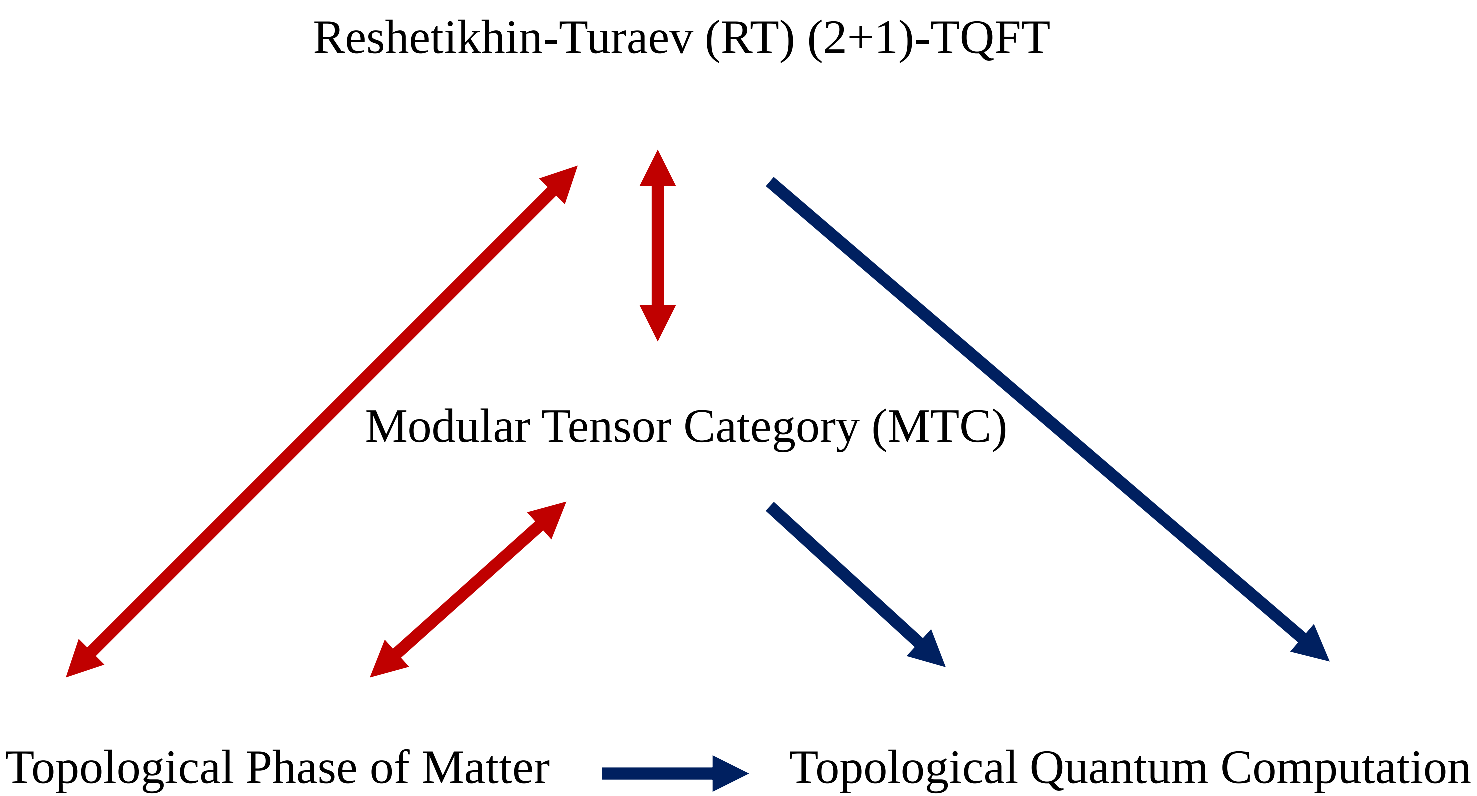
Driving Force of Convergence of Math & Sciences

Physics	Mathematics
Newtonian Mechanics	Calculus (arranged marriage)
General Relativity and Gauge theory	Differential Geometry
Quantum Mechanics	Linear Algebra and Functional Analysis
Many-body Entanglement Physics	???

Tip of iceberg: 2D Topological Phases of Matter

(Microscopic physics?) **Topological Quantum Field Theory (TQFT)**
Conformal field theory (CFT)

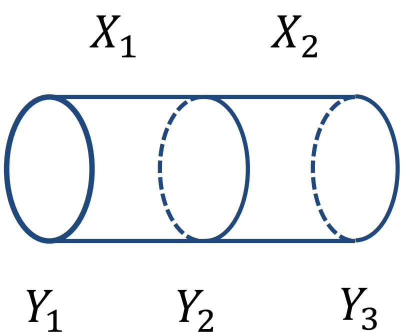
What is quantum field theory mathematically?
New calculus for the second quantum revolution and future.



Topological phases of matter are TQFTs in Nature and hardware for hypothetical topological quantum computers.

Atiyah Type (2+1)-TQFT

A symmetric monoidal "functor" (V, Z):
category of 2-3-mfds → Vec
2-mfd Y → vector space $V(Y)$
3-bord X from Y_1 to Y_2 → Z(X): $V(Y_1) \to V(Y_2)$



- $V(S^2) \cong \mathbb{C} \longrightarrow V(\emptyset) = \mathbb{C}$
- $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y \times I) = \mathrm{Id}_{V(Y)}$
- $Z(X_1 \cup X_2) = \kappa^m \cdot Z(X_1) \cdot Z(X_2)$ (anomaly=1)

(2+1)-TQFTs ~ Modular tensor categories (Turaev)

“Categorification” and “Quantization” of Group

Let G be a finite group, **e.g.** $\boldsymbol{S_n}$

- A finite set of elements: a, b, c,…
- A binary operation: a×b=c
- A unital element e: e × a=a
- An inverse for each element g: $g^{-1} \times g = e$
- Associativity: (a × b) × c=a × (b × c)

- **Categorification:**

Kapranov and Voevodsky’s main principle in category theory:

“**In any category it is unnatural and undesirable to speak about equality of two objects**.”

Equality should be simply some canonical isomorphism.

- **Quantization:**

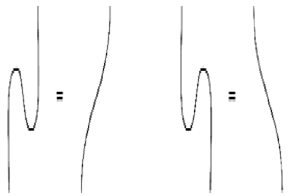
Every set S should span a complex Hilbert space V(S).

MTC = Fusion Category with a non-degenerate Braiding (Abelian)

A fusion category is a categorification of
a based ring $\mathbb{Z}[\boldsymbol{x_0}, \dots, \boldsymbol{x_{r-1}}]$ /categorification and quantization of a finite group

finite rigid $\mathbb{C}$-linear semisimple monoidal category with a simple unit

monoidal: $(\otimes, \mathbf{1})$, $\mathrm{X_i} \otimes X_j = \sum_k N_{ij}^k \mathrm{X_k}$



semisimple: $X \cong \bigoplus_i m_i X_i$,
linear: $\mathrm{Hom}(X, Y) \in \mathrm{Vec}_{\mathbb{C}}$,
rigid: $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$

finite rank: $\mathrm{Irr}(\mathcal{C}) = \{\mathbf{1} = X_0, \dots, X_{r-1}\}$

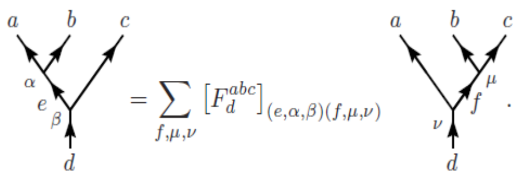
$\boldsymbol{X}$ **simple if** $\mathbf{Hom}(\boldsymbol{X}, \boldsymbol{X}) = \mathbb{C}$
Rank of $\boldsymbol{\mathcal{C}}$: $\mathbf{r}(\boldsymbol{\mathcal{C}}) = \boldsymbol{r} = \mathbf{dim}\, \boldsymbol{V}(\boldsymbol{T}^2)$

Examples

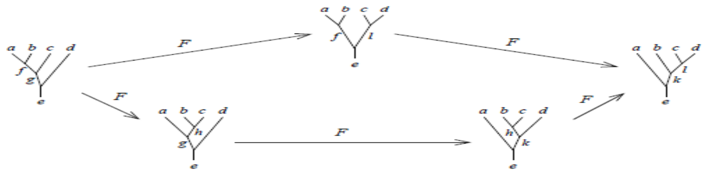
- **Pointed: $\mathcal{C}(A, q)$, A finite abelian group, q non-degenerate quadratic form on A.**
- $\mathrm{Rep}(D^{\omega}G)$, ω a 3-cocycle on G a finite group.
- Quantum groups/Kac-Moody algebras: subquotients of $\mathrm{Rep}(U_q\mathfrak{g})$ at $q = e^{\pi i/l}$ or level k integrable $\hat{\mathfrak{g}}$-modules, e.g.
  - $\mathrm{SU}(N)_k = \mathcal{C}(\mathfrak{sl}_N, N + k)$,
  - $\mathrm{SO}(N)_k$,
  - $\mathrm{Sp}(N)_k$,
  - for $\gcd(N, k) = 1$, $\mathrm{PSU(N)_k} \subset \mathrm{SU}(N)_k$ “even half”
- Drinfeld center: $\mathcal{Z}(\mathcal{D})$ for spherical fusion category $\mathcal{D}$.

Modular Tensor Category $\mathcal{C}$

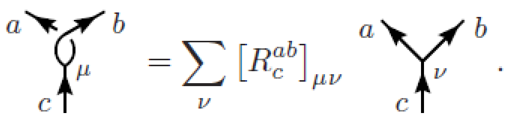
Modular tensor category (=anyon model if unitary): a collection of numbers {L, N_{ab}^{c}, $F_{d;nm}^{abc}$, R_{c}^{ab}} that satisfy some polynomial constraint equations including pentagons and hexagons.



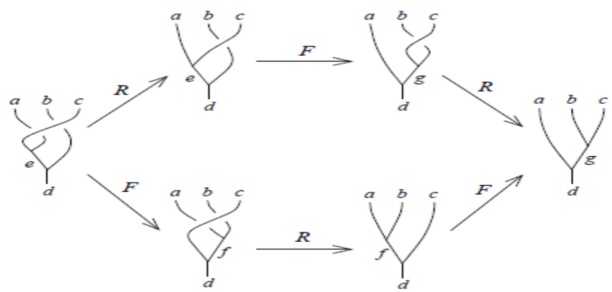
6j symbols for recoupling



Pentagons for 6j symbols



R-symbol for braiding



Hexagons for R-symbols

Invariants of Modular Tensor Category

MTC $\mathcal{C}$ $\rightleftarrows$ RT (2+1)-TQFT (V, Z)

- Pairing $<Y^2, \mathcal{C}> = V(Y^2; \mathcal{C}) \in \mathrm{Rep}(\mathcal{M}(Y^2))$ for an surface Y^2, $\mathcal{M}(Y^2)$=mapping class group
- Pairing $Z_{X,L,\mathcal{C}} = <(X^3, L_c), \mathcal{C}> \in \mathbb{C}$

for colored framed oriented links L_c in 3-mfd X^3

fix $\mathcal{C}$, $Z_{X,L,\mathcal{C}}$ invariant of (X^3, L_c)

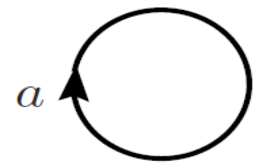
fix (X^3, L_c), $Z_{X,L,\mathcal{C}}$ invariant of $\mathcal{C}$

fix Y^2, $V(Y^2; \mathcal{C})$ invariant of $\mathcal{C}$

Quantum Dimensions and Twists: Unknot

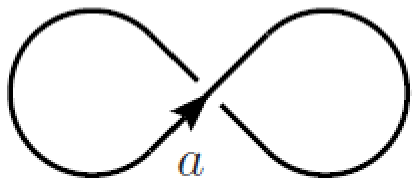
- Label set $L =$ isomorphism classes of simple objects
- Quantum dimension of a simple/label $a \in L$:

$$d_a = d_{\bar{a}} =$$



- Topological twist/spin of a: finite order by Vafa's thm

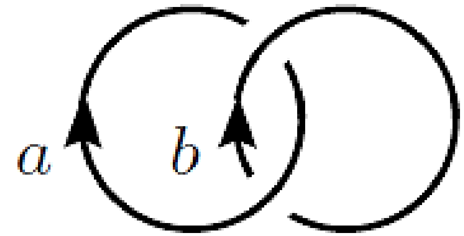
$$\theta_a = \theta_{\bar{a}} = \frac{1}{d_a}$$



- Dimension D^2 of a modular category:

$$\dim(\mathcal{C}) = \boldsymbol{D^2} = \boldsymbol{\sum_{a \in L} d_a^2}$$

Modular S-Matrix: Hopf Link

- Modular S-matrix: $S_{ab} =$ 

- Modular T-matrix: $T_{ab} = \delta_{ab}\theta_a$-diagonal
- (S, T)-form a projective rep. of $SL(2, \mathbb{Z})$:

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \mathrm{S}$$

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \mathrm{T}$$

Finite Group Analogue I

Theorem (E. Landau 1903)

For any $r \in \mathbb{N}$, there are finitely many groups G with $|\mathrm{Irr}(G)| = r$.

Rank-finiteness Theorem (Bruillard-Ng-Rowell-W., JAMS 2016, Alexanderson Award 2019):

For a fixed rank, there are only finitely many equivalence classes of modular categories.

Analogue II: Cauchy Theorem

Cauchy Theorem:

The prime factors of the order and exponent of a finite group form the same set.

Theorem (Bruillard-Ng-Rowell-W., JAMS 2016)

The prime factors of $|D|^2$: Galois norm $|D| = \prod \sigma(D)$ and $N = \mathrm{ord}(T)$ form the same set.

“Periodic Table” of Topological Phases of Matter

Classification of symmetry enriched topological order (TQFT) in all dimensions

Too hard!!!

Special cases:

Symmetry	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

Symmetry classes	Physical realizations	$d=1$	$d=2$	$d=3$
D	SC	p-wave SC	$(p+ip)$-SC	0
DIII	TRI SC	Z_2	$(p+ip)(p-ip)$-SC	He^3-B
AII	TRI ins.	0	HgTe Quantum well	$Bi_{1-x}Sb_x$, Bi_2Se_3, etc.
CII	Bipartite TRI ins.	Carbon nanotube	0	Z_2
C	Singlet SC	0	$(d+id)$-SC	0
CI	Singlet TRI SC	0	0	Z
AI	TRI ins. w/o SOC	0	0	0
BDI	Bipartite TRI ins. w/o SOC	Carbon nanotube	0	0

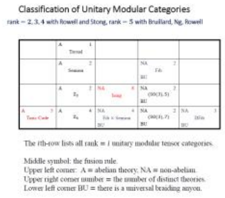
1): short-range entangled (or symmetry protected--SPT) including topological insulators and topological superconductors: X.-G. Wen (Group Cohomology), …, A. Ludwig et al (random matrix theory) and A. Kitaev (K-theory)---generalized cohomologies,…

2): **Low dimensional: spatial dimensions D=1, 2, 3, n=d=D+1**

2a: **classify 2D topological orders without symmetry**

2b: enrich them with symmetry

2c: 3D much more interesting and harder



Classification of Unitary Modular Categories

rank = 2, 3, 4 with Rowell and Stong, rank = 5 with Bruillard, Ng, Rowell

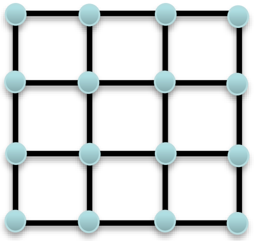
	A 1 Trivial			
	A 2 Semion		NA 1 Fib BU	
	A 2 Z_3	NA [illegible] Ising	NA 2 $(SO(3),5)$ BU	
A [illegible] Toric Code	A 4 Z_4	NA 4 Fib × Semion BU	NA 2 $(SO(3),7)$ BU	NA [illegible] DFib BU

The ith-row lists all rank = i unitary modular tensor categories.

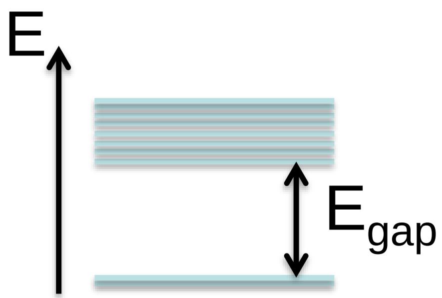
Middle symbol: the fusion rule.
Upper left corner: A = abelian theory, NA = non-abelian.
Upper right corner number = the number of distinct theories.
Lower left corner BU = there is a universal braiding anyon.

Model Topological Phases of Quantum Matter

Local Hilbert Space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$



Local, Gapped Hamiltonian $H : \mathcal{H} \to \mathcal{H}$

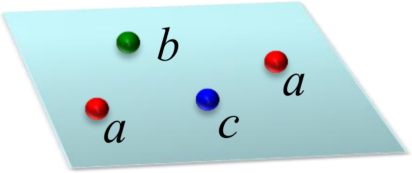


Two gapped Hamiltonians H_1, H_2 realize the same topological phase of matter if there exists a continuous path connecting them without closing the gap/a phase transition.

A topological phase, to first approximation, is a class of gapped Hamiltonians that realize the same phase. Topological order in a 2D topological phase is encoded by a TQFT or anyon model.

Anyons in Topological Phases of Matter

Finite-energy topological quasiparticle excitations=anyons



Quasiparticles a, b, c

Two quasiparticles have the same topological charge or anyon type if they differ by local operators

Anyons in 2+1 dimensions described mathematically by a Unitary Modular Tensor Category $\mathcal{C}$

Bulk-edge Connection of Topological Phases

- Edge physics of fractional quantum Hall liquids:

∂Witten-Chern-Simons theories$\sim$Wen's chiral Luttinger liquids
∂TQFTs/UMTCs $\sim$ χCFTs/Chiral algebras

Chiral algebras $\longrightarrow$ UMTCs=Rep(chiral algebras)
Injective? No. e.g. all holomorphic ones goes to trivial.
Onto?
Conjecture: Yes

- Tannaka-Krein duality (Gannon):
  Reconstruct chiral algebras from UMTCs=Rep(Chiral algebras)

Symmetric fusion categories are 1-1 correspondence with pairs (G, μ)

Chiral and Full Conformal Field Theories

The BPZ definition of conformal field theory is that it is an inner product space $\mathscr{H}$ which can be decomposed into a direct sum

$$\mathscr{H} = \bigoplus_{h,\bar{h}} V(h,c) \otimes \bar{V}(\bar{h},\bar{c}) \tag{2.1}$$

of irreducible highest weight modules of $Vir_c \times \overline{Vir}_{\bar{c}}$ such that

1. There is a unique $SL_2(R) \times SL_2(R)$ invariant states $|0\rangle$ with $(h,\bar{h}) = (0,0)$.
2. For each vector $\alpha \in \mathscr{H}$ there is an operator $\phi_\alpha(z)$ on $\mathscr{H}$, parametrized by $z \in C$. Also, for every operator ϕ_α there exists a conjugate operator ϕ_{α^*} (partially) characterized by the requirement that the operator product expansion $\phi_\alpha \phi_{\alpha^*}$ contains a descendant of the unit operator.
3. For $\alpha = i$ a highest weight state we have $[L_n, \phi_i(z,\bar{z})] = \left(z^{n+1}\frac{d}{dz} + \Delta_i(n+1)z^n\right)\phi_i$.
4. The inner products $\langle 0|\phi_{i_1}(z_{i_1},\bar{z}_{i_1}) \ldots \phi_{i_n}(z_{i_n},\bar{z}_{i_n})|0\rangle$ exist for $|z_{i_1}| > \ldots |z_{i_n}| > 0$ and admit an unambiguous real-analytic continuation, independent of ordering, to C^n minus diagonals. This is called the assumption of duality.
5. The one-loop partition function and correlation functions, computed as traces exist and are modular invariant.

We now discuss the notion of chiral algebras, or vertex algebras [26]. The fields in a conformal field theory form a closed operator product expansion. An important subset of the fields are the holomorphic fields. Since the operator product expansion of two holomorphic fields is holomorphic, these form a closed subalgebra of the operator product algebra called the "chiral algebra," $\mathscr{A}$, of the theory.[2] Every conformal field theory has at least two holomorphic fields given by the unit operator and the stress tensor: 1, $T(z)$ and thus every chiral algebra contains the (enveloping algebra of the) Virasoro algebra. We can choose a basis $\{\mathcal{O}^i(z)\}$ for $\mathscr{A}$ such that each field has a well-defined dimension Δ_i. By the axiom of duality, fields in a conformal field theory have no relative monodromy, in particular, the weights Δ_i are integers. Defining modings $\mathcal{O}^i(z) = \sum_n \mathcal{O}^i_n z^{-n-\Delta_i}$ we can write the operator product algebra in two equivalent ways:

$$\begin{aligned} \mathcal{O}^i(z)\mathcal{O}^j(w) &= \sum_k \frac{c_{ijk}}{(z-w)^{\Delta_{ijk}}}\mathcal{O}^k(w), \\ [\mathcal{O}^i_n, \mathcal{O}^j_m] &= \sum_k c_{ijk}(n,m)\mathcal{O}^k_{n+m}, \end{aligned} \tag{2.2}$$

$(\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k)$. Using the moding one can define Verma modules and irreducible quotients and, therefore, one can speak of the irreducible representations $\mathscr{H}_i$ of $\mathscr{A}$.

Classical and quantum conformal field theory

G Moore, N Seiberg - Communications in Mathematical Physics, 1989

Chiral CFT (χCFT) or chiral algebra = mathematically vertex operator algebra (VOA).

A full CFT is determined by a VOA V plus an indecomposable module category over Rep(V).

Vertex Operator Algebra (VOA)

A VOA is a quadruple (V, Y, 1, ω), where $V = \bigoplus_n V_n$ is Z-graded vector space and

$$Y : V \to \mathfrak{F}(V),\ v \mapsto Y(v,z) = \sum v_n z^{-n-1}$$
$$\mathbf{1}, \omega \in V,\ \mathbf{1} \neq 0.$$

The fields $Y(v,z)$ are mutually local and creative, and the following hold:

$$Y(\omega, z) = \sum L_n z^{-n-2} \text{ with a constant } c \text{ such that}$$
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n} c \mathrm{Id}_V$$
$$V_n = \{v \in V_n \mid L_0 v = nv\}$$
$$\dim V_n < \infty,\ V_n = 0 \text{ for } n \ll 0$$
$$Y(L_{-1}u, z) = \partial Y(u,z)$$

$$\mathit{locality} : Y(u,z) \sim Y(v,z)$$
$$\mathit{creativity} : Y(u,z)\mathbf{1} = u + O(z)$$

$\mathfrak{F}(V) = \{a(z) \in \mathrm{End}(V)[[z, z^{-1}]] \mid a(z) \text{ is a field}\}$.

I.e. for $v \in V$ there is an integer N (depending on v) such that $a_n(v) = 0$ for all $n > N$.

a(z) is a field if it satisfied the truncation condition,

Genus of Vertex Operator Algebra

Genus of lattice:

genus of a lattice Λ is equivalent to (q, G, c),

G=discriminant Λ^*/Λ, $q: G \to U(1), c =$signature.

Genus of VOA:

genus of VOA=the pair $(\mathcal{C}, c)$. Recall $\frac{p_+}{D} = e^{\frac{\pi i c}{4}}$

$\mathcal{C}$=MTC of Rep(VOA) (**Huang**), c=central charge

Zhu's Theorem

Given a good VOA and an irreducible module M,
the character χ_M of M is

$$\mathrm{Tr}_M\, q^{L_0^M - c/24} = q^{h-c/24} \sum_{n \geq 0} \dim M_{h+n} q^n$$

The vector X=$(\chi_1, \ldots, \chi_r)^t$ is a vector-valued modular form for the modular representation of the corresponding modular tensor category.

Conjectures

Given a UMTC $\mathcal{C}$

1. **Existence** (W., Gannon):

There is a genus ($\mathcal{C}$, c) that can be realized by a VOA(χCFT).

e.g. (Toric code, 8) is realized by $SO(16)_1$

2. **Genus finiteness conjecture** (Hoehn):

There are only finitely many different realizations of any genus

$$D_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 2 & -1 & 0 & 0 \\ 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Holomorphic VOAs: Trivial UMTC

- (Vec, 0) trivial
- (Vec, 8) E_8
- (Vec, 16), $E_8 \oplus E_8, D_{16}$
- **Monster Moonshine** VOA has genus (Vec, 24)

John McKay’s remark: 196 884=196883+1

$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$

- There are at least 71 VOAs in the genus (Vec, 24)

$n = 0, 24, 36, \ldots, 168, 192, 216, 240, 264, 288, 300, 312, 336,$ $360, 384, 408, 456, 552, 624, 744, 1128$ are values of actual VOA characters (at $c = 24$)

- Classify VOAs modulo holomorphic ones

Vector-Valued Modular Forms

$\mathcal{C}$—MTC with real s-matrix and irreducible $\rho: SL(2,\mathbb{Z}) \to U(r)$.
$\tilde{s}$ un-normalized s-matrix and $s = \frac{\tilde{s}}{D}$, normalized
$\tilde{t} = (\theta_i)$ un-normalized t-matrix, and $t = e^{-\frac{2\pi i c}{24}}\tilde{t}$, normalized
A vector-valued modular form is a holomorphic function
$X : H \to \mathbb{C}^r$ with finite poles at infinity such that

$$X(\gamma\tau) = \rho(\gamma)X(\tau), \tau \in H, \gamma \in SL(2,\mathbb{Z}),$$

where $H = \{z, Imz > 0\}$.

Gannon, T. The theory of vector-valued modular forms for the modular group. *Conformal Field Theory, Automorphic Forms and Related Topics*. Springer Berlin Heidelberg, 2014. 247-286.

$\mathcal{M}_\rho$=all VVMFs with ρ as multiplier---infinite dim vector space over $\mathbb{C}$.

We will fix an MTC as above.

Quantum Mathematics

- **Quantum inspired mathematics**

  Quantum topology and algebra

  Quantum analysis?

  Applications:

  1. Simulation of QFTs by quantum computers
  2. Witten conjecture that Donaldson inv.=SW inv.
  3. Volume conjecture

- **Quantum logic-based mathematics**

  Do we need a new logic?