Syllabus for the mathematician and physicist and a historical review of some the main, unavoidable limitations of mathematics and physics.

What set of core ideas should physicists, mathematicians, and engineers be aware of in terms of key historical developments, essential foundational texts, and key research literature? In general, there are no formal paths leading students to solid foundations, and most students go through considerable pain and effort just to attain peaceful satisfaction with a minimal mastery of their required skill sets—which is not the same as mastering fundamental understanding. One can master the art of flying without ever understanding the physics of flight. It is natural that it should take longer to acquire the maturity of deep understanding. The reality, however, is that it takes far longer than it should to attain foundational maturity, and most simply give up, accepting defeat, that there will be some theoretical constructs they will never understand no matter what—which is shamefully wrong.

In my case, I did not begin to feel that I was reasonably accomplished in my trade as a mathematician and physicist until the passage of five additional years of self-imposed studies beyond my masters degree in mathematics and my doctoral degree in physics. This is how long it took me to eliminate the majority of the concepts in physics and mathematics that I had been previously forced to accept as true in order to press on with the business of taking the next classes and ultimately finishing school. Fixing these cracks in my foundations was a painstaking, arduous, nonlinear, fuzzy process resulting in notebooks and textbooks full of marginal notes penned during long nights. The garbage pile grew tall with journal articles and books pedagogically worthless. Another five years of additional work were required before I began to feel comfortable with how current physics and mathematics operate in their most general formulations and descriptions of possible existences.

It needn't have taken me this long. There aren't that many deep, fundamental ideas in physics, nor that many mathematical technologies in mathematics—just enough of them, apparently, to have been scattered to the winds, variously disentangled and re-entangled, and re-scattered over a few centuries into a disparate mess of incomprehensible jargon comprehensible only to a rare super genius. Speaking pedagogically, our physics constructs and mathematical toolsets, intertwined by their histories and intent to describe and categorize, should not be voodoo. There should be, for example, a logical path from the work of Sophus Lie applying group theory to differential equations through to the extraction of irreducible representations in particle physics. The former is unknown work to most of us, and the latter is a collection of prescriptions that may as well be incantations handed to us by wizards who alone truly understand what they are trying to do. We need to weave the histories and intent of mathematics and physics into our undergraduate and graduate programs to eliminate false, pedagogical mysteries from the true mysteries that currently await theoretical resolution, like dark matter in physics and open problems in mathematics such as in quasigroup theory. The aim of the following is to give students of mathematics and physics and engineering at least one path to good foundations in terms of books, articles, history, core ideas, key methods, and inherent limitations. One day, time permitting, I would like to fill in the material with my step-by-step notes, subject by subject, that cost me so much grief, exasperation, and ultimately peace.

Foreword

The best suggestion I ever received from a professor was to get into the habit of collecting toy models of core ideas.

I wish I would have learned the key ideas in physics and the essential methods of mathematics underlying our current understanding of existence properly—with little or no magic—the first time around. This is that syllabus in three parts.

Part 1—A list of core mathematics and physics courses and reference material.

Part 2—Explosion of Part 1 into a description of WHY the material in Part 1 is important to a solid foundation, along with the concomitant historical developments and motivations.

Part 3—An approach for describing the relative likelihood of theories. (More at the speculation level.)

For those who have already finished an undergraduate degree, let me first address what appear to be fundamental limitations to the perfection of knowledge. Some of these limitations will only become sensible after you attain a threshold level of mastery of the courses, subjects, ideas, and methods identified below, often only after acquiring some real world experience with physical systems and their mathematical descriptions.

Some limits of mathematics, physics, and engineering—

One—historical paths to insanity and mathematical limitations. When I was a young Air Force lieutenant, I followed the following chain to Gödel's incompleteness theorem and nearly fell into a regress of uselessness. Fourier: By investigating heat conduction, Fourier "discovered" that functions $f(x)$—the functions can be multivariable—can be represented by infinite series of trigonometric functions. Cauchy: Cauchy investigated convergence of Fourier series and "discovered" a few restrictions on which particular functions $f(x)$ can be represented by Fourier series. Dirichlet: Dirichlet also searched for restrictions on which functions $f(x)$ can be represented by Fourier series, and ended up establishing sufficiency conditions. Riemann: Riemann searched for necessary conditions and considered the meaning of $\int f(x)dx$. Hankel: Hankel tried to answer what Dirichlet and Riemann couldn't, namely, ¿is a function that has an infinite number of oscillations in an arbitrarily small neighborhood representable by Fourier series? Cantor: Along these lines, Cantor looked at the consequences of the domain of a function, and looked at the meaning of sets. He "discovered" the transfinite numbers, an infinity of ever larger infinities and their hierarchies. Cantor believed that the transfinites had been communicated to him by God. Leading mathematicians of the time called his work a grave disease, and called Cantor a corrupter of youth. Predictably, Cantor suffered bouts of depression. Bertrand Russell and Alfred North Whitehead tried to reduce all of mathematics to sets, but they ran into paradoxes (see the Barber paradox below). They were trying to use set theory and purely logical systems to derive all of mathematics.

Their three volume *Principia Mathematica,* which began with set theory, and never got past the real numbers, ultimately went down in flames. Hilbert: David Hilbert wanted to determine the consistency of the axioms in the *Principia Mathematica*. Actually, Hilbert had a greater ambition. Hilbert wanted to develop an ultimate mechanical means of testing whether a set of axioms (think of high school geometry as a good example) could be proven to be consistent (not contradictory) in a finite number of logical steps. A lightning bolt then struck down the so-called queen of the sciences: mathematics. Kurt Gödel came along with his Incompleteness Theorem: No finistic, logical algorithm can be constructed to carry out Hilbert's program even for a system of axioms as rudimentary as arithmetic, *e.g.*, as system containing statements like: if $x = y$ then $y = x$. Before committing suicide, Alan Turing picked this result up in the language of software and computers. Computers operate with logic and set theory, and Hilbert's problem of finding a mechanical, terminating algorithm to determine if a formal mathematical system is consistent in finite steps (finite time) was shown by Turing to be equivalent to the problem of knowing whether a computer will ever accept more user input. Think about composing a magnum opus of romantic poetry on your word processor, when suddenly the computer freezes and you haven't backed up your gift to humanity. How long do you wait to see if the computer, working with logic and set theory, starts to respond, or you give up and pull the plug in frustration? You face a potential infinity of wait time. There is no *a priori* way for you to know if the problem with the computer will resolve in a finite amount of time (finite logical operations). This is Incompleteness in the sense of Gödel. The book, "*Gödel, Escher, Bach, an Eternal Golden Braid*" by physicist Douglas Hofstadter won a Pulitzer Prize on Gödel's Incompleteness Theorem. This book is entertaining, puzzles you with some of the great paradoxes humans have faced, but ultimately failed me in helping me understand Gödel's Incompleteness Theorem without first digging into old mathematics literature over years of my life. Putting aside Löwenheim-Skolem theory (see historical notes section at the end of this work), Gödel's incompleteness theorem is a fundamental restriction finite beings cannot escape.

Was this effort to understand these particular foundations worthwhile? No, at least not in the sense of making my operational proficiency in mathematics any better. But yes in helping me frame existential thoughts. Gödel's Incompleteness Theorem is a restriction finite beings cannot escape, and helps me avoid wasting any more time on certain kinds of existential questions, a kind of insurance against some types of insanity. Also, the time I wasted on this path between Fourier and Gödel, made me feel very comfortable with the underpinnings of mathematics at the working level of physicists and engineers.

Today's underpinnings of mathematics based on the axioms Zermelo-Fraenkel set theory can be another path over the edge. These axioms give most of today's mathematics a solid foundation, but they are also infected with some weird paradoxes. Using the Zermelo-Fraenkel axioms, one can, for example, divide the surface of a sphere into parts which can then be reassembled to construct two spheres of the same radius as the original sphere—Scientific American played a great April fool's hoax based on this odd result [Dewdney, A. K. (1989). "A matter fabricator provides matter for thought". *Scientific American* (April): 116–119]. Unfortunately, the proof of the Banach-Tarski paradox suffers what students of differential equations suffer. Existence proofs that a solution to a differential equation exists are plentiful enough, but these proofs provide no guide in actually finding a solution. Nuts. There is active work on creating a set of axioms that are less paradoxical than the Zermelo-Fraenkel axioms. More on whether any mathematical construct is "true" in the sense that Greeks thought of geometry follows later below.

To finish up on the theory of integration, we go to a little physics. The theory of integration advanced greatly from Riemann's time. One motivation came from statistical physics concerning the random (stochastic) motion of pollen particles in a glass of water: Brownian motion. Einstein's statistical physics paper on Brownian motion ended all doubts about whether atoms were real, or merely mathematical conveniences. The differential equations which describe the motion of pollen particles, photons diffusing in a star, and in principle, the returns of stocks in efficient stock markets, are stochastic. To integrate these stochastic differential equations, one is faced with difficult choices. In ordinary integral calculus, one adds up the sum of areas of rectangles whose width goes to zero in the limit: Area equals an infinite sum of function height times width (infinite sum of $f(x)\cdot\Delta x$ as Δx goes to zero from x= a to x = b). But when the value (height) of the function $f(x)$ at x, is randomly drawn from some probability distribution, the function's height at some x is stochastic. A drunken sailor will zigzag his way home differently most of the time. One solution to the limitation of the theory of integration was to generalize the concept of integration, and Lebesgue (along with others) correspondingly developed measure theory and Lebesgue integration, material which seniors majoring in mathematics might be exposed to, but can't escape in first year graduate work.

Lebesgue made choices, other choices have been made. Itô made choices when he developed the Itô calculus to treat stochastic differential equations, a calculus which is used today in pricing stock options. A physicist, Fischer Black, and an economist Myron Scholes adapted results from statistical physics to this problem. Scholes won the Nobel Prize in Economics for his work. Black didn't. The prize is not given posthumously.

Physicists typically prefer to solve stochastic differential equations by so-called Monte Carlo numerical methods which were invented at Los Alamos National Laboratory to deal with the transport of neutrons inside atomic weapons. ¿Again I ask if I gained anything over Monte Carlo methods by tracking down the key threads of measure theory, probability, the Itô calculus, etc.? I'm not sure yet. At a minimum, I don't have that empty feeling in the pit of my stomach that I'm missing something important when I deal with stochastic processes in my job. I thank finance first, then particle transport later on, for giving meaning to what I had thought were completely useless analysis courses I was required to take to earn my non-thesis masters degree in mathematics. I had entertained notions of pursuing a doctoral degree in mathematics until I could not bear the dryness anymore. A great, entry level introduction to stochastic processes is "*Options, Futures, and Other Derivatives*," 5th ed. (or above) by John C. Hull. You need to know what partial derivatives are. Hull will take you into the heart of the Itô calculus, the book only falling short in failing to present a better development of quadratic variation. There is more about these matters below, in the Analysis section.

Two—Barber's Paradox. Imagine a town with only one barber. Every man in town keeps clean-shaven. Some shave themselves, others attend the barber. Therefore, the barber only shaves those men who do not shave themselves. ¿So who shaves the Barber? If he does shave himself, then he will not shave himself. If the barber does not shave himself, then he must shave himself—an infinite loop due to the self reference inherent in the paradox. This problem with self reference was the seed to Gödel's work. In computer science, we call this an infinite loop. Such loops could easily hide in the millions, or more of lines of code our computers run.

This is a perfect place to interrupt the numbered list of limitations and talk a little about the perfectibility of mathematics and physics in the sense of Plato's ideal plane, a topic that often entraps people today as easily as it did the greatest minds since the early 1800s. The aforementioned Zermelo-Frankel axioms, Godel incompleteness and the Barber's paradox go to the impossibility of mathematics as we use it issuing from a perfect (Plato ideal plane) foundation, but these are only the tips of icebergs. The book, "<u>Mathematics: The loss of Certainty</u>" by Morris Kline, Galaxy Books, 1982, goes into far more expansive cracks in mathematics.

Klein goes into the history of whether mathematics exists independently of self aware beings, on Plato's plane, or is invented. Both notions appeal to me. At times my intuition feels that Plato's idea of mathematics is correct. It seems logical that carefully constructed axiomatic systems like those of arithmetic and plane geometry, are pure, perfect and independent of our existence. ¿Is the Pythagorean theorem not eternally true within the codification of plane geometry no matter what the nature of existence may be? Yet at other times I feel that all we do is invent crap. Physicists have gotten very skilled at mental masturbation, capable of creating tons of contradictory, competing high energy physics models that are tuned to converge to what we observe at low energy in our accelerators. (What we call high energy physics, the physics produced by high energy accelerators, is actually very low energy physics relative to the energies we believe existed much earlier in time. Accelerators in this sense are thus time machines, and every modest advance in energy allows us to probe physics earlier in time, and into smaller volumes of space.)

Klein and many others argue that mathematics issues from our imperfect sense perceptions (from physics). Thus mathematicians who interpret mathematics as pure, free of contact with the "real" world, as did the Greeks, are just simply full of it. Clearly then, physics expressed in mathematical language, is even more full of it. After all, Bertrand Russell, along with many other mathematicians and philosophers, rendered reality a fuzzy illusion. Sit at a wooden table. Look at it. ¿Is it the same table at breakfast when sunlight pours into the room, or when lit by evening lamplight? Both times you tell me that it is brown. ¿But doesn't the brown change depending on the spectrum of light reflecting off of it, and how can I be certain that what you call brown is the same as what I call brown? ¿Is this table even there when I'm sleeping? ¿Is it only an imperfect representation of an ideal table?

According to Kline's history, this kind of self questioning by mathematicians began in earnest during the early 1800s. For two thousand years, Euclid's geometry had been considered to be the geometry of the universe, logical, pure, perfect, unlike the physical sciences. There are imperfect physical triangles drawn on sheets of paper, then there are ideal, perfect triangles in Euclidean geometry. What's funny to me is that no one thought of the existence of other geometries, even though they were staring people in the face—spheres. On a flat plane, the sum of the interior angles of a triangle add to 180 degrees, but on a sphere, the sum exceeds 180 degrees, and the geometry of the surface of a sphere is as much a geometric system as is Euclidean plane geometry. So people belatedly realized that there is more than one geometry, many more in fact. Around this period, Gauss asked his student Riemann to investigate if there was anything physical predisposing the universe to choose one geometry over another. Riemann's answer was no. A closer inspection of Euclidean geometry made things even worse, far worse. For example, Euclid's geometry uses the concept of the side of a line, a self-evident concept no? No. Russell points out that one must first carefully specify the topological nature of the "line". Once mathematicians started along this line, it was easy pickings to find ever more problems with Euclid's geometry. Geometry, it turned out, was more an invention of mankind, with infallible senses, than a system of pure truth to be discovered by thinking beings.

Mathematicians then turned their thoughts to the arithmetic of the real numbers. Surely this system was not the creation of mankind. Surely the properties of real numbers had to be eternal, absolute truths with no inconsistencies. Some of the early lessons learned were a need to be careful with our language. How does one add 1/2 to 1/3? By finding a common denominator of course. The answer is 5/6. Not if you're doing baseball arithmetic in which player A has one hit at two at bats in game 1, and 1 hit in three at bats in game 2. Player A then had 2 hits from 5 at bats, or an average of 2/5. This is how we add fractions in baseball. What does the infinite sum of 1 - 1 + 1 - 1 + 1… add up to? Daniel Bernoulli showed the answer to be 1/2. Then he showed that 1 + 0 - 1 + 0 - 1… would add to 1/3. Yet the only difference between that latter and the former series is the presence of the zero terms. ¿How can adding zeroes to the first infinite sum change its value? Mathematicians learned that they had to be careful with how they worded their mathematics, and this path to more carefulness led to deep schisms that still lay wide open: logicism, intuitionism, formalism, and set theory. I quote from Kline. "To top all the disagreements and uncertainties about which foundation is best, the lack of proof of consistency still hangs over the heads of all mathematicians like the sword of Damocles. No matter which philosophy of mathematics one adopts, one proceeds at the risk of arriving at a contradiction."

Excepting those of us with perfect faith in perfect gods, there is no perfect truth. It is confusing. I don't know if what you call brown is what I would call brown. I don't even know what brown is. ¿Yet what is the conspiracy of mass delusion that orchestrates descriptive physics? ¿Why does it seem that you and I can independently use Newtonian gravity to compute the same orbits of the planets in solar systems? I could have asked why it seems that you and I can use general relativity to enable a common global positioning system. Einstein's theory of general relativity is a generalization of

Newtonian gravity consistent with small corrections to small deviations between Newtonian gravity and experimental observation. It doesn't matter to my point. General relativity is still descriptive physics. It doesn't tell us what gravity is. And my point remains. We suffer an uncanny mass delusion. Everywhere we look, gravity works the same—I can't speak to gravity inside black holes with any certainty, but neither can anyone else. Everywhere we look, stars follow a path of nucleosynthesis consistent with our understanding of quantum mechanics, nuclear and particle physics, very good descriptive theories of atoms, atomic nuclei, and subatomic particles. ¿Doesn't this point to a universal truth? Just because we can't frame even a small portion of existence into a self-consistent, self-evident construct, not even a system encompassing real number arithmetic, doesn't mean there isn't universal, eternal, perfect, divine truth. I honestly can't say. It certainly seems to me for now that finite beings can't attain even a small slice of perfection no matter how advanced they get. I even have my doubts as to what kind of perfection if any infinite beings might be privy to. ¿Do infinite beings exist? I don't presume to know. I do know that some mathematicians don't believe in the infinity of the real numbers, stopping at the smaller infinity of the integers. Consult Wikipedia. It is not difficult to follow how Georg Cantor showed that the infinity of numbers between zero and one is much larger than the infinity of the integers via diagonalization.

¿So where are we left? It seems that we can't even put together a perfectly worded system for arithmetic, evolved from a simple, but perfect grammar producing incontrovertible, perfect, absolutely true statements, nor can we escape the handcuffs of an imperfectly described physical universe we seem to share. The former tells us there is no ultimate truth whatsoever, but the latter instills in us the shared delusion that there is an ultimate truth with ultimate causes. John Archibald Wheeler in his article, "Law Without Law" comes up with a pretty good middle ground, but aspects of it are hard to swallow. The universe is in a Darwinian feedback loop with time as it evolves life that ponders that origin of the universe.

There is also one more handcuff I find interesting. It comes not from physics, but from mathematics. As we progress from the natural numbers to the integers, we can "solve" more problems. We cannot express debt with natural numbers. We can express debt (negative numbers) with integers. With integers, we cannot express ratios, be we can with fractions (rational numbers). Rational numbers are richer than integers. Between any two rational numbers, we can always find another rational number. Rational numbers are dense. However, rational numbers have holes. The square root of two is not a rational number. Neither is pi. The real numbers, roughly speaking, fill in the gaps between rational numbers. This is what makes real numbers a far larger infinite set than the integers, which are the same size as the natural numbers and rational numbers—read Cantor. Yet we can't solve $x^2 = -1$ with the real numbers. So we adjoin the complex numbers. The process continues. To describe a three dimensional arithmetic (think modern vector algebra), Hamilton invented the quaternions. However, if A and B are quaternions, $AB \neq BA$? Oh crap. Nothing weird had happened as we progressed to ever more useful systems of numbers: natural, integer, rational, real and complex. The quaternions force us to lose multiplicative commutativity, but we need them. Quaternions describe the relativistic, quantum mechanical electron (and positron) of

Dirac. More generalized numbers loose even more "nice" properties. Here seems to be a set of mathematical handcuffs more fundamental in origin than physics. ¿Do these handcuffs limit physics, or only our ability to create improved descriptive physics models of our shared delusion? For the latter question, there are potential ways around the limits. Life as we know it is written in a handful of nucleotides. Yet there seems to be no limit to how life can take these few notes and create an opera of diverse life from monkey to amoeba to chimera in the labs of mad biologists. That is to say, we can always form combinations of our number systems by direct product, etcetera.

Three—Bifurcation in mathematics and physics. My toy model is from "*Newtonian Dynamics*," by Ralph Baierlein, McGraw-Hill Inc., 1983, section 4.5. A bead is free to move frictionlessly on a circular wire that is rotating about its vertical axis with constant angular frequency. Imagine the bead resting at the very bottom of the spinning wire loop. At a certain angular frequency, there are two possibilities for the bead: it can stay at the bottom, or it can fly up to the middle of the hoop. One or the other of these cases happens in the real world. With negligible friction unavoidable, some perturbation from the real world, say a small vibration from a car passing nearby will lead to a choice. But in the idealized physics model, the mathematics runs into a dead end. There is nothing in the equations which gives preference of one outcome over the other. This toy problem exemplifies the key aspects of stability as studied by physicists from junior level mechanics up to the required training in graduate school for a masters degree or beyond. The topic of stability touches a very broad range of physics ranging from the stability of planets orbiting a star, to weather, population dynamics, economics, etc. Bifurcation occurs at a much more elemental level in the mathematics of one-dimensional iterative maps, *e.g.*, a simple recursion relationship such as $x_n = (1 + \mu)x_n + x_n^2$, for $\mu > 0$, see "*Chaos*", ed. by Arun V. Holden, Princeton University Press, 1986. Bifurcation is tied to period doubling, chaos, fractal dimension, and strange attractors. Baierlein's *Newtonian Dynamics* and Holden's *Chaos*, make these subjects (which limit our ability to model nature already at the classical level) accessible to the junior level student. Much more material resides on the internet. Some key words in a book/literature search: chaos, complexity, complex adaptive systems, graph theory and network theory as studied by physicists who have made great strides in what had been static for decades under the care of mathematicians ignoring dynamics. A good first sophomore or junior level course in ordinary differential equations will touch on stability and Lyapunov exponents. A great mathematician to look up is Henri Poincare.

An additional toy concept very important to the physicist and engineer is the concept (covered very well in *Chaos*) of the Lyapunov exponent, a parameter which characterizes the sensitivity of the solutions to differential equations to initial conditions. The toy model is known as Caesar's last breath. It's easier to imagine cigarette smoke rising smoothly from a lit cigarette. Two tiny particles of smoke starting off very near each other as they rise, all of a sudden begin to separate very rapidly. In relatively short time, one smoke particle may end up in Europe, and one in Latin America. Beyond early times, a simulation of such a process by a computer solving differential equations by numerical methods would never be able to discretize the space coordinates into a fine enough grid to track particles of smoke even when they begin rising arbitrarily near each other. So much for the watchmaker god of the Newtonian era.

Bifurcation is tied to the concept of fractional dimension, a fractal. A line has one dimension, a square has two dimensions, a cube has three dimension, and so on into hyperspace. But what in the world is a fractal? Again, I have a couple of toy models. The simplest is the Cantor set. Take the interval [0, 1] and remove the open middle third, (1/3, 2/3). Remove the open middle thirds of the two remaining segments. Continue to remove the open middle thirds of the remaining segments infinitely many more times. The set of points that survives is the Cantor set, and its dimension is $\ln(2)/\ln(3) \approx 0.63$. The other toy example I have is taken from the Baierlein text and involves a sophomore level differential equation: Duffing's equation, named after Georg Duffing who studied nonlinear mechanical oscillators in 1918. Plotted in a graph of x verses t, the evolution of the trajectory can become chaotic and develop a strange attractor. A strange attractor is usually associated with a fractal curve. If you imagine a line laying on the plane, the trajectory of $x(t)$ will cross that line at a set of points with fractal dimension (*Deterministic Chaos, An Introduction*" by H. G. Schuster, Physik-Verlag, 1984) Fluid turbulence is related to the emergence of a strange attractor. There are other types of fractals, *e.g.*, the Sierpinski triangle fractal.

Four—The uncertainty principle in Fourier transform pairs, otherwise known as the Heisenberg uncertainty principle to physicists. Imagine the graph of $f(t) = A$, where A is a constant (just some number) and t is defined on $[-\infty, \infty]$. It's just a line stretching between negative and positive infinity on plot of $f(t)$ versus t in the time domain. The Fourier transform $F(\omega)$ of this graph $f(t)$ is a spike of height $2\pi A$ (look up delta function) located at ω in the frequency domain where we graph $F(\omega)$ versus ω. Conversely, the inverse Fourier transform of the spike in the frequency domain is A in the time domain. The graph of $f(t) = A$ is of infinite width: $[-\infty, \infty]$, the graph of the spike is of zero width. In this case, the product of the two widths (look up a good junior level text book covering the Fourier transform, *e.g.*, a book on partial differential equations) is a finite, non-zero number. This situation is famous in quantum physics in two well known versions. One, if you know where you are (your location in space), the uncertainty in your momentum is infinite, or conversely because momentum and location are Fourier transform pairs in quantum mechanics. Two, if you know your energy, then your uncertainty in time is infinitely unknown, or conversely because energy and time are Fourier transform pairs. In either case, mathematics or physics, the product of the such "noncommuting observables" is a finite number. In physics this finite number is a fixed fraction of Planck's constant. For the less extreme case in which the uncertainty in one variable is large, the uncertainty principle (of Fourier transform pairs) states that the uncertainty in the other variable is small. Think about lightning. It is of very short temporal duration, but it spits out electromagnetic radiation across a very wide band of frequencies you hear on your radio no matter what frequency you are tuned to. There are great resources for studying this on the internet. Planck's derivation of the quantum nature of light (photons) has at least a few versions requiring little more than algebra and basic calculus.

Five—Propagation of error. In probability theory, under the topic of propagation of error, we have to Taylor series expand and keep only the linear terms of a function of several random variables more complex than a sum of random variables to compute the variance. I think of propagation of error from a physical point of view as well. Even in classical physics treatments of physical systems with no mathematical pathologies, our inability to measure perfectly the positions and velocities of particles quickly leads to error between simulations and the real world. In my mind, I think of billiard balls undergoing elastic collisions on a frictionless pool table with no friction and no pockets. We cannot measure the positions and velocities of the billiard balls with zero experimental error. Then I think of the pool table I inhabit: a planet and its billiard ball atmosphere of molecules. Will it rain this weekend? Will the ski hill open by Christmas?

Six—N-body problems. In general, classical mechanics cannot solve a three body problem of three comparable bodies interacting via gravity. Note the word comparable. The masses of the planets in our solar system are incomparably small relative to the sun. Quantum mechanics adds to our difficulties.

Seven—Closing the loopholes to Bell's inequality, if ever: In Newtonian gravity, we would instantaneously feel the vanishing of the sun, and this is called "spooky" action-at-a-distance. "Spooky" action-at-a-distance is often expressed as nonlocality. With our current general relativistic understanding of gravity, if the sun were to vanish, it would take us several minutes to know this, as the speed at which information flows is limited to the speed of light. While quantum mechanics doesn't violate useful information flowing faster than the speed of light, Einstein recognized that certain results of quantum mechanics require superluminal physics. Einstein, Podolsky, and Rosen produced the EPR paradox to show that because of this superluminosity, quantum mechanics cannot be a complete theory of physics. In the 1960s, Bell produced a theorem in the form of an inequality that could, in principle, lead to experiments capable of detecting the difference between quantum mechanics and its nonlocality and local, hidden variable theories that can explain spooky action-at-a-distance without nonlocality. Early experimental work seemed to settle the issue: quantum nonlocality was a fact of life. Not long afterwards however, people subsequently pointed out loopholes of various kinds that bring Bell's work and its experimental methods under question. Today, experimentalists have greatly pushed back these so-called loopholes, but as of early 2011, they have not been able to simultaneously close all of the loopholes, and the struggle continues on closing all loopholes in the lab. For most physicists, the evidence in favor of quantum mechanics (as we get better at quantum communication and computation) is good enough. It simply may be the case that closing the all of the loopholes simultaneously in the lab will never be fully experimentally feasible, and even if this is achieved, some will still point out that superdeterminism (everything is predetermined) could still underlie quantum mechanics, that is that there is no nonlocality with superdeterminism; we press on. The Wheeler delayed choice experiment is a variant of the EPR paradox. Look it up. Wheeler's paper, "*Law without law*" should be read by everyone.

I think the emphasis on "hidden variable theories" is misleading. Nonlocality—which is what the lab is showing to be real—simply means that not all physical phenomenology propagate on a smooth (rubber) spacetime manifold. However, if we can imagine an extra dimension in terms of say a string, two points that are far apart along the length of the string can be placed arbitrarily near each other (in a knot, which can be made by closing braids). Indeed, work on braids and nonlocality has shown that that this might a plausible approach to describing quantum nonlocality without anything spooky about it. The string is continuous along its length. It just so happens to be knotted up in a higher dimension, which to us makes quantum entanglement look like superluminal, nonlocal, action-at-a-distance. [*On a suggestion of relating topological and quantum mechanical entanglements*, M. Asoudeh, V. Karimipour, L. Memarzadeh, and A. T. Rezakhani, Physics Letters A, 327 (2004) 380-390.] There are actually other promising approaches to nonlocality, including sheaf theory, and even proposed experimental approaches to look for extra dimensions big and/or small. [*The Sheaf-Theoretic Structure Of Non-Locality and Contextually*, S. Abramsky and A. Brandenburger, 21 June 2011] (Our physics and mathematical technology should be always be plausible to us, not voodoo.)

Eight—Thermodynamics, the second law and its variants. When we see a film of a porcelain cup being dashed to bits by a bullet, no laws of physics as we know them are violated when we run the movie backwards. The arrow of time, however, seems to prefer a universe with increasing entropy (read increasing disorder, or decreasing energy to do useful work). It's true that biological evolution bucks this trend. A living human being represents a lot more molecular order than the same chemicals scattered to the winds, but this kind of increase in order is at the cost of a greater macroscopic disorder. We are definitely depleting the Earth's useful energy (stored energy like fossil fuel energy) into useless energy like hot tailpipes. Stars are doing the same thing, burning up potential energy and radiating it out into space. The second law of thermodynamics can be stated as no process is possible whose sole result is the transfer of heat from a body of lower temperature to a body of higher temperature—Clausius statement. It is a fundamental limit engineers face everyday. The thermodynamic measure of disorder, known as entropy is a point of departure leading to the formulation of statistical physics and Shannon information theory, which is crucial to today's computer and communications engineer. Among many areas of application, Shannon theory provides us limits on how efficiently we may encode and compress signals. Information theory is now framing fundamental notions in physics, *e.g.*, the Holographic Principle tying the black hole physics of Hawking with quantum gravity and string theories, though a good friend of mine tells me this importance is likely overstated.

More limitations can be listed. These are some of the main ones I carry with me.

Part 1—Mathematics Courses

Calculus I: Analytic geometry, limits, differentiation, integral calculus. Almost any text will do. (If there is a great text for the course, I'll cite it.)

Calculus II: More integration, series convergence tests. Again, almost any text will do.

Calculus III: Partial derivatives, vector algebra, vector calculus, method of Lagrange multipliers—see most any text. Calculus III is **elaborated on in an Part 2 below on what aspects of it are very important, and why this is so**—this elaboration will apply to many of the classes in the list.

Introductory Linear Algebra concurrent with Ordinary Differential Equations: This pair of courses is elaborated in a Part 2 below. If I had to pick out something that stands out in my mind, it's that numerical methods to solve ordinary differential equations and systems of differential equations lead to systems of linear equations, which can be treated by the methods of linear algebra.

Ordinary Differential Equations: The ordinary derivatives covered in Calculus I lead to the study of differential equations. The laws of physics are expressed as ordinary or partial differential equations. A common method of partial differential equations, separation of variables, can decompose partial differential equations into a system of ordinary differential equations. Ordinary Differential Equations have application to all areas of science that I can think of, from biology to cliodynamics, the new field of mathematical history. Of course, ordinary differential equations are the bread and butter of the physicist and engineer.

Complex Analysis: This course has a lot of constructs parallel to vector algebra and vector calculus. One has to learn the meaning of ordinary functions now as functions of a complex variable, or complex variables. As functions of complex variables, powers, roots (of unity), trigonometric functions, exponential functions, logarithmic functions all take on very powerful generalizations to applications in pure mathematics as well as to applied subjects in physics and engineering. The theory of integration takes on a much more general set of tools connecting integrals with series, analyticity, etc. From the junior level and upwards, a physicist cannot survive without complex variables. I thoroughly recommend "*Complex Variables and Applications*," 5th ed. by R. Churchill and J. W. Brown.

Partial Differential Equations: The partial derivatives covered in Calculus III lead to the study of partial differential equations. Maxwell's electromagnetic equations and Einstein's general relativity field equations are partial differential equations, as are equations dealing with heat and diffusion. So too is the Schrodinger wave equation. I taught myself much of undergraduate partial differential equations from an early edition of "*Applied Differential Equations with Fourier Series and Boundary Value Problems*" by Haberman. Later on, I stumbled pleasantly into, "*Transform Methods for Solving Partial Differential Equations*" by Duffy. I like "*Applied Fourier Series*" by H. P. Hsu, Harcourt Brace Jovanovich, 1984.

Advanced undergraduate Linear Algebra: You cannot get enough of this stuff. A great place to see why is "*Numerical Recipes*" in Fortran 77 or C or C++, which has also been printed without any computer code, and is available free online. Linear algebra is the basis of many statistical methods, *e.g.*, Principle Component methods such as Singular Value Decomposition (SVD), one of the most important statistical methods I've ever run into. It offers one of the best approaches to solving a system of linear equations. I've used SVD to figure out the most prevalent configurations protein molecules fold into from a molecular dynamics code. I've used SVD to help a climatologist detect when El Niño is transitioning to La Niña to help energy traders place educated bets on natural gas contracts six months out. A little research shows that S`VD is very widely applied across wildly disparate fields. You can teach yourself this stuff with paper and pencil, or a PC.

Abstract (or Modern) Algebra (or simply Algebra): First the books:

1. Sophomore level mathematics (recommended for the physicist who may never take such a course). "Modern Algebra, An Introduction," by John R. Durbin, 2nd ed. It is very readable and easy to do the homework problems. The whole concept of elaborating on the subgroups of a group is very important to the physicist who uses group theory. This takes up the first four chapters of Durbin, and the physicist will get some ideas of the pure mathematics approach.
2. "Groups, Representations and Physics," by H. F. Jones, 2nd ed. This should be read by the physicists concurrently, or shortly after the one year series in graduate quantum mechanics. For reinforcing Jones, I strongly recommend "Modern Quantum Mechanics" by Sakurai.
3. For physicists looking for deeper applications, I recommend "Lie Algebras in Particle Physics, From Isospin to Unified Theories," Georgi, 2nd ed. It's difficult reading but doable if you've mastered Jones. Reconciling Georgi to Jones is a great exercise. Read my notes in Lie.PDF p. 251-293 else Georgi is illegible.
4. For the mathematician looking to see what Sophus Lie was up to regarding applications of group theory and topology to partial differential equations, I recommend "Lie Groups, Lie Algebras, and Some of their Applications," by Robert Gilmore. A MUST READ for the physicist. This is fundamental stuff for the exploration of physics from a very general language: A Lagrangian for a universe, the particle spectra, the local and global topological properties,…
5. Along the lines of exploring physics using very general language, I strongly recommend "Geometry, Topology, and Physics" by Nakahara. It's readable and shows how gauge field theories can be expressed in terms of connections and fiber bundles. This motivates the use of differential forms, a far more general theory than vector and tensor calculus. (For a quick, but to-the-point introduction on differential forms see, "Introduction to Differential Forms," by Donu Arapura; I honestly can't recommend any good physics or math book for a good introduction into forms.)
6. To the physicist, "Quantum Field Theory," by Lewis Ryder would make the efforts in 1 through 5 above worth the pain, especially the material that relates differential geometry (General Relativity) with Lie Groups/Algebras—taking an infinitesimal loop in the underlying space (connection in GR, commutator in Lie Groups/Algebras. I also recommend "A First Course in String Theory," by Barton Zweibach, 1st or 2nd eds. A great tease full of history and ideas for further study is "Knots, Mathematics With a Twist," by Alexei Sossinsky—you'll see that the knot theory built up by Vortex atom physicists in the 19th century resembles today's string theory work.

I strongly recommend algebra to the physicist, not so much the engineer. The physicist will appreciate aspects of quantum mechanics far better with a solid foundation in Abstract Algebra. Since this is not recommended for all majors, I'll elaborate on its importance here in Part 1. I took a junior and senior level course in algebra, and years later a full year of graduate classes in a mathematics department. The incredible dryness and ivory 'towers' of detachment modern mathematicians have brought to this powerful field is what made me quit pursuing a doctoral program in mathematics and switch to physics. One of the founding fathers of this field, Evariste Galois, died at twenty after a duel. He spent the night before his duel penning his mathematical thoughts. Galois theory solved an old problem, but this first requires some history. Babylonians and Egyptians, and probably earlier peoples, knew how to solve quadratic equations. The derivation of a formula to solve cubic equations had to wait until the early 16^{th} century. Working at this, it may have been the Italian Girolamo Cardano who first ran into what today we call the complex numbers. Cardano's assistant, or student, Ferrari found a formula for the general quartic, which was published in Cardano's *Ars Magna* in 1545. Naturally, mathematicians then went on to search for a formula for the general quintic <u>and</u> higher order polynomials. No dice. Galois put an end to this pursuit in 1832 before a bullet put an end to him. I should say that by the time Galois penned his notes, another ill-fated young man, Abel of Norway had already proved the impossibility of solving the quintic. Abel died at twenty-seven. Is pursuing modern algebra ill-fated?

Methods of modern algebra were used to classify all 230 possible three-dimensional types of crystals (symmetry groups). In the plane, there are 17 symmetry groups. You have seen them all on stained glass windows in ancient cathedrals, in Roman mosaics, and beneath your feet on bathroom floor tessellations (tiling patterns). Today, circa 2011, the first hard x-ray free electron laser has started making images of the crystallographic structures of whole viruses unperturbed by any preparation techniques required by earlier methods. The crystal structure of table salt was the first resolved by x-ray crystallography in 1914. The first organic crystal structure was solved in 1923, cholesterol in 1937, and by the end of 2010, just over 70,000 protein structures have been worked out by this method. We're about to cross 75,000 in August of 2011.

The modest Jewish physicist Eugene Wigner (who's brief autobiography is a delight to read) was one of the earlier promoters of group theory to physics early in the 20^{th} century. Many physicists reviled him for bringing this incomprehensible "*gruppen-pest*" to quantum physics, a mathematical tool which now underlies one of the most basic paradigms through which we describe existence. Group theory underpins our most advanced description of all that we see in the universe, the so-called Standard Model, which we know is likely not a complete theory, as it is too rife with parameters we must put in by hand from experimental results, and it does not include gravity. Algebra also underpins our work to move beyond the Standard Model in the exploration of Grand Unified Theories (GUTs) and Theories of Everything (TOEs) which do include gravity. An honest-to-goodness surfer dude, Garrett Lisi, stirred up the world of physics in 2007 with his TOE based on the group E8. This work is now defunct, but it may serve as a model for further investigations.

My breakout from mathematics to physics lays with the book on Abstract Algebra by Lang. The moon has more atmosphere. Thankfully, I eventually ran into "*Groups, Representations and Physics*," 2nd ed. by H. F. Jones, Institute of Physics Publishing, Bristol and Philadelphia, 1998. There is an old, high school book which can serve as a great introduction: "*Groups and Their Graphs*" by Israel Grossman and Wilhelm Magnus, The Mathematical Association of America, 9th printing, 1964. A sophomore level book that can also serve as introduction is "*Modern Algebra, An Introduction*," 2nd ed. by John R. Durbin, John Wiley & Sons, 1985. A harder read, to follow the Jones text is by physicist Howard Georgi: "*Lie Algebras in Particle Physics, From Isospin to Unified Theories,"* Frontiers in Physics, 1999; it's worth the time if you are willing to fill in the steps. The Georgi text is one of several books in mathematics and physics from which I've extracted cleaned up notes. Read "*Lie Groups, Lie Algebras, and some of their applications*" by Robert Gilmore, Dover Publications, Inc., 1974, 2002. Many of the results of mathematical physics (this subject discussed below) are tied together by Lie groups and Lie algebra. Get "***Symmetry Methods for Differential Equations, A Beginner's*** **Guide**" by Peter E. Hydon, Cambridge Texts in Applied Mathematics, 2000. I found it last. It's now my core book tying differential equations (ordinary, partial, linear and nonlinear) to algebra and topology. Start with Hydon and work backwards!

A historical aside: After leaving Germany, Eugene Wigner found his way to Los Alamos to work on atomic weapons. Another Jew who escaped then Nazi Germany, and was instrumental to the development of nuclear physics was Lise Meitner (*Lise Meitner: A Life in Physics*" by Ruth Lewin Sime, 1997). Only smelly politics kept her from earning a Nobel Prize in Physics. One is shocked by the challenges women in science faced in the early 20th century. Her modesty was not unlike that of Eugene Wigner.

Mathematical Methods of Physics: Since this is only mandatory for physicists, I'll elaborate here on why this subject matter is so important far beyond physics. Books for this subject are a dime a dozen. The best, most thorough one I have run into is "*A Course of Modern Analysis*" by E. T. Whitaker and G. N. Watson first published in 1902. It was a standard reference for giants such as G. H. Hardy, the great mathematician who brought the world the number theory works of Ramanujan, an enigmatic genius so ahead of his time that mathematicians are still pouring over his notebooks to date. See "*Number Theory in the Spirit of Ramanujan*" by Bruce C. Berndt, American Mathematical Society, 1996. You'll need a good background in complex variables to pursue Ramanujan. As for Whitaker and Watson, their book was reprinted (more like poorly photocopied) by Cambridge Press in November of 2009. Allow me to quote an anonymous Amazon.com reviewer. "It is certainly the most useful book of mathematics I ever put my hands on. If you read its page of contents, you'll call it prophetic! Every kind of function he studied became important in theoretical physics some time. String theory was started with an amplitude containing only Gamma functions. Renormalization, reborn from the ashes, discovered the Zeta-function (in Whittaker-Watson, for sure), Legendre's less familiar functions were prominent in Regge pole theory (again, the source was Whittaker), and even the Theta functions became important for some field theory skirmishes. You could travel light: Whittaker, Watson, tooth brush, etc. It's not only what there is in it. It's also the fact that it's done better! Consider this: I had once an ugly series to sum up. These were the days before Maple! I couldn't find it anywhere, having looked into immense mathematical tables. I came back to old Whittaker and there it was: in an exercise, asking you to prove that the sum of MY series was some function he wrote in all detail! This is Whittaker-Watson. God bless them." Whittaker and Watson filled their book with citations to the literature dating back hundreds of years, *e.g.*, as far back as 1655.

As for Ramanujan, I'm not sure how applicable his work is to physics. Though I suspect that his work on partition number theory might have deep connections to statistical physics. In number theory, a partition of a positive integer *n* is a way of writing *n* as a sum of positive integers, *e.g.*, 4 = 3 +1 = 2 + 2 = 1 + 1 + 1 + 1. The number of partitions grows very rapidly, and recently in January of 2011, mathematician Ken Ono made a major advance in the study of partition functions: they behave like fractals.
A subset of mathematical methods of physics is the calculus of variations. One of the best introductory books I have ever found is "*Calculus of Variations*" by Elsgolc, now in reprint by Dover Publications. In ordinary calculus, we often search for where a function takes its minimum or its maximum. In the calculus of variations, we look for a function which extremizes a functional. One can imagine many ways for a telephone pole wire to dangle from two poles of different heights, one taller than the other. That is, one can imagine many functions whose graphs look like dangling wire, but there is only one curve (function) which is realized in physics. This curve is what the calculus of variations provides us. Actually it provides us differential equations, the solutions of which are extremizing curves we're after. As an example, think of the surface defined by a soapy water solution forming a curved sheet around an arbitrary loop of wire, typically found in a children's science museum. The Standard Model of physics itself, and its extensions, are written in the language of the calculus of variations, connected to group theoretic symmetry by Emmy Noether's theorem and our belief in symmetries such as super symmetry (SUSY). As a last example of the calculus of variations, consider Fermat's theorem in the following way. A lifeguard has to get to a drowning person off to the side from her station. Naturally she wants to find the path of minimal time. She can ran faster on sand than she can swim in water. The calculus of variations gives her just how much distance to run on the sand, and just how much distance to swim in water to minimize her time. This is exactly what light beams do when they refract from air to water, where the speed of light is different in air and water. Lastly, the calculus of variations plays a role outside of physics in optimization theory with applications ranging from finance to engineering.

Perturbation and stability is another staple of mathematical physics. In the limit I discussed based on bifurcation, it is instability which lies beneath complexity. Perturbation theory helps us define the line of when and/or where a system is no longer stable, *e.g.*, imagine a star passing our solar system. At a certain point, if it gets near enough, its gravitational perturbation might make our planetary orbits unstable.

Why is fractional calculus not taught? Ever wonder about a 1/3-order derivative, or a $\sqrt{\pi}/3$-order integral? Or even a derivative or integral of complex number order? The fractional calculus is very old, a generalization of integer order differentiation or integration. It ties together many of the results of mathematical physics. I've enjoyed the Dover publication "*The Fractional Calculus, Theory and Applications of Differentiation and Integration to Arbitrary Order*" by Oldham and Spanier.

Analysis: At the undergraduate level the books "*Analysis, An Introduction to Proof*" by Steven R. Lay, and "*Elementary Analysis: The Theory of Calculus*" by Kenneth A. Ross serve as very readable and enjoyable analysis texts building careful proofs to results in limit, derivative and integral theory. Is it important to master this stuff? Calculus texts do a pretty good job of presenting sketches to the results of calculus, but Ross and Lay give one a feeling of comfort about one's understanding. Moreover, the tools built in Ross and Lay help people who use applied mathematics obtain valuable information about the boundaries, pathologies, uniqueness, existence, and convergence properties of solutions. "*Real Analysis*" 2nd ed. by Royden, and "*Real and Complex Analysis*" 3rd ed. by Rudin are first year graduate texts in mathematics that proceed into (Lebesgue) measure theory and more advanced methods of analysis. However, these latter books begin to lose contact with applications, and start to trail off into abstraction that seems to be there just for the sake of abstraction. It was not until I had finished my doctoral degree in physics, and was working in the energy trading pits in Houston, Texas that I began to realize that a great model and motivation for the machinery in Rudin and Royden was the theory of probability with applications to stochastic processes. One of the best books not only for your mind, but for your retirement portfolio, is "*Options, Futures, and Other Derivatives*" 5th ed. (or beyond) by John C. Hull. As mentioned earlier, Hull's only shortfall is failing to give a decent understanding of what underlies quadratic variation. As for Rudin and Royden, you cannot proceed to a deeper understanding of stochastic processes without mastering what they cover. It was finance again, that led me to appreciate Rudin and Royden in "*Risk-Neutral Valuation, Pricing and Hedging of Financial Derivatives*" by N. H. Bingham and Rüdiger Kiesel, Springer, 1998. At the level of Hull is "*Options Pricing*" by Wilmott, Dewynne, and Howison. Wilmott sponsors a great website in finance, a nexus for mathematicians, physicists and other financial professionals to discuss everything in advanced mathematical finance.

Physics class list

A warning to American students of graduate physics. It is my understanding that the competition from China is better prepared. In the United States, it is typical to progress from a bachelor's degree in physics directly into a graduate program in physics. In China, students heading to American graduate physics programs go through a preparatory program. For example, The Physics coaching Class, University of Science and Technology of China, has put out a series of books literally taken from problems and solutions of qualifying Ph.D. physics examination questions taken from top American universities. *Major American University Ph.D. Qualifying Questions and Solutions* is a series of seven volumes covering:

Mechanics,
Electromagnetism,
Optics,
Atomic, Nuclear and Particle Physics,

Thermodynamics,
Quantum Mechanics, and
Solid State Physics, Relativity, and Miscellaneous Topics.

I am very grateful that a professor of mine pointed out these books to me. They are readily available in the US. I did not need to buy the one on optics, nor the one on solid state physics to prepare for my particular qualifying exams. My department also kept a forty year list of the qualifying exams problems it had given out. Ask for, and start solving these kinds of problems as soon as possible.

Classical Mechanics: I've previously mentioned the importance of mastering the calculus of variations (the Elsgolc Dover book). The Baierlein text, also previously mentioned, is a fantastic undergraduate introduction into classical mechanics, including mechanics formulated in variational calculus. If you want to understand quantum mechanics, read the graduate text "*Classical Mechanics*" 3rd ed. by Herbert Goldstein, Addison Wesley, 2001. I love my 2nd ed. The 3rd ed. better reflects the use of computers and nonlinearities in mechanics. First edition dates back to 1950. I strongly recommend "Variational Principles In Dynamics and Quantum Theory (Dover books on physics)", Wolfgang Yourgrau and Stanley Mandelstam, Dover Publications 1979 to tie the calculus of variations to mechanics, quantum mechanics and beyond.

Continuum Mechanics: Physicist do not typically study continuum mechanics. Physicists who do study a semester of general relativity at the graduate level, will however, run into tensor calculus, and tensor equations, namely, the Einstein Field Equations in a four dimensional spacetime manifold. The typical engineer does not work in curved spacetime, but the theory of continuum mechanics is best expressed in tensor form, and the tensor mathematics employed in continuum mechanics is far deeper than a one semester graduate course in general relativity. I think a physicist should understand continuum mechanics and all of its mathematics, not just those who go on to pursue advanced topics in general relativity and create numerical models of extreme stellar physics phenomenology. I therefore recommend "*Continuum Mechanics*" by A. J. M. Spencer, Dover Publications Inc., 1980.

Electricity and Magnetism: Probably the best book out there at the undergraduate level is "*Introduction to Electrodynamics*" 3rd ed. by David J. Griffiths. It "thinks" like a physicist, and I keep my 1st ed. as a reference. At the graduate school level comes one year of "*Classical Electrodynamics*" 3rd ed. by John David Jackson, Wiley, 1998. I had 2nd ed., and the first edition dates back to 1962. I hated this book initially, but have come to greatly appreciate it later in life, as my skills in mathematical methods of physics grew to match it, and I faced down the barrel of two powerful x-ray accelerators at Los Alamos National Laboratory built to support of Stockpile Stewardship. One of the accelerators is an induction accelerator, the other a resonant cavity accelerator. They make you eat and breath Jackson. Jackson is essentially self contained, although research through the literature can significantly add to your understanding of classical electrodynamics. 3rd ed. Jackson is in MKS SI units. It is missing a great chapter from 2nd ed. covering plasma physics, a dumb omission.

Thermodynamics and Statistical Physics: Start with "thermodynamics" by Enrico Fermi, Dover Publications Inc., 1956. This will begin your path into one of the most fundamental areas of physics. My undergraduate book was "*Heat and Thermodynamics*" 6th ed. by Zemansky and Dittman, McGraw Hill, 1981. It has a very accessible introduction to statistical physics using the method of maximum likelihood (which can have mathematical pathologies at times). The best graduate level book I've ever run across was the one I used in graduate school, "*A Modern Course in Statistical Physics*" by Linda E. Reichl, University of Texas Press, 1980. It covers a lot of ground: thermodynamics, complexity and entropy, phase transitions, equilibrium statistical physics, fluctuation and dissipation, hydrodynamics, transport coefficients, and nonequilibrium phase transitions. Also good is "*Statistical Mechanics, An Advanced Course with Problems and Solutions*" 2nd ed. by R. Kubo, North-Holland, 1988. It has hundreds of problems solved in great detail, and makes for a perfect study guide in probability methods in statistical physics. "*Statistical Mechanics*" 2nd ed. by R. K. Pathria, Butterworth Heinemann, 1996 is worth keeping for a perspective different than Reichl. "*An Introduction to equations of state: theory and applications*" by S. Eliezer, A. G. Ghatak and H. Hora (1986) gives a pretty good treatment of where our knowledge in thermodynamics and statistical physics abuts with our ignorance, as well as shows how quickly mathematical models and methods become complex, difficult and approximate. This compact book has applications literally far outside of weapons work, like astronomy and cosmology.

General Relativity (optional): This material is optional if you are not interested in big picture stuff at a very accessible level, *e.g.*, cosmology. A very short read that gets to essentials of general relativity is from the master himself. "*The Meaning of Relativity*" 3rd. ed. by Albert Einstein, Princeton Science Library, was first published 1921, but has been reprinted. My favorite short course is "*A short course in General Relativity*" by J. Foster and J. D. Nightingale, Springer. It is now in third edition, and very carefully builds up the required mathematics in tensor calculus and differential geometry required to use Einstein's field equations to calculate the precession of orbits, the bending of light by gravity, and basic cosmological models already discussed in the Dover book by Einstein. Only the linearized rudiments of gravitational radiation are developed in Foster and Nightingale.

Quantum Mechanics: "*Quantum Mechanics*" Vols. I and II by Cohen-Tannoudji, Diu, and Loloë, Wiley Interscience, 1977. Tannoudji shared the 1997 Nobel Prize in Physics for his work on laser cooling and trapped atoms. The book is great for self education, presenting theory, then applications chapter by chapter. Another great book to supplement your studies is "*Modern Quantum Mechanics*" by Sakurai now in a 2nd edition.

Relativistic Quantum Mechanics: Note—This material comes with better presentations in newer books treating Quantum Electrodynamics (QED) and Quantum Field Theory (QFT). However, for historical connectivity, and still a very good introduction, I like "*Advanced Quantum Mechanics"* by Sakurai, Addison Wesley, 1967, especially his comment on the EPR paradox. The book is currently in its 11th printing. This should tell you something. In a similar vein, I like "*Quantum Mechanics*" 2nd ed. by Schiff, McGraw Hill, 1955. It transitions well into Relativistic Quantum Mechanics.

Solid State Physics (optional): The standard book is "*Introduction to Solid State Physics*" 7th ed. by C. Kittel, Wiley, 1996. However, my take on Solid State Physics is that it is a nonrelativistic quantum field theory that led Wilson to the Renormalization Group Method, a very powerful tool in field theories. This can be seen by reading the well written book, "*A Guide to Feynman Diagrams in the Many-Body Problem*" 2nd ed. by R. D. Mattuck, Dover Publications Inc., 1967; 2nd ed. 1992.

Quantum electrodynamics (QED): QED is the relativistic quantum theory of electrodynamics. QED describes how light and matter interact, and produces extremely accurate predictions, *e.g*., the Lamb shift and the energy levels of hydrogen. QED is a renormalizable perturbation theory of the electromagnetic quantum vacuum. It is the prototype theory of all of our perturbative quantum field theories (QFTs), e.*g*., electroweak and Quantum Chromodynamics (QCD) covering the strong interaction. I do not believe that one can fully understand modern QFTs without first understanding QED and how it was originally practiced. This is masterfully done in "*Quantum Electrodynamics*" 4th ed. by Greiner and Reinhardt, Springer, 2008. I'm happy with my older edition.

Note: A departure is beginning to happen at this level of physics. The texts are beginning to be less self contained, requiring the reader to track down cited literature to fully understand the material.

Under the abstract algebras section, I described how fundamental this subject is to the physics of constructing models of the universe. We have physics expressed in the language of the calculus of variations and symmetry (algebra). The best book I have run into for QFTs up to the simplest SUSY work is "*Quantum Field Theory*" 2nd ed. by Lewis H. Ryder, Cambridge University Press, 1996. I also strongly recommend "*An Introduction to Quantum Field Theory"* by Peskin and Schroeder, Westview Press, 1995. There are other QFT texts much beloved by other physicists which also cover the renormalization group method, spontaneous symmetry breaking, etc.

From the requirements of how the Dirac equation transforms under the restrictions of relativity, one can see the logical necessity of the (massless photon) electromagnetic gauge field. QED, an Abelian gauge theory with the symmetry group U(1), thus shows us a way to build more general, non-Abelian gauge theories such as QCD for strong interactions, with the SU(3) symmetry group. To sum up, we may consider electromagnetism as an extra fifth dimension. The extra dimension can be understood to be the circle group U(1), and electromagnetism can be formulated as a gauge group on a fiber bundle, namely the circle bundle with gauge group U(1). Once this geometrical interpretation was understood, physicists started to replace U(1) with general Lie groups often under the name of Yang-Mills theories. These kinds of theories have predictable, therefore testable consequences of whether they correspond to nature, or are exercises in mathematics. Of course, all of these field theories are the fruit of particle physics. It stands to reason one can only better appreciate fields theories with a good foundation in particles, and Griffiths comes through again. Read "*Introduction to Elementary Particles*" 2^{nd} ed. by David Griffiths, Wiley-VCH, 2008. The first edition is also good.

Aside from the algebraic/symmetry and the calculus of variation (extremal-action principles) framework of theoretical models of the universe, there is also input from the language of topology and algebraic topology as I showed above in connecting U(1) with general Lie groups. A great way to see what I mean for yourselves, and not just applicable to 'big idea' physics, is through reading, "*Geometry, Topology, and Physics*" by N. Nakahara, Graduate Student Series in Physics, Institute of Physics Publishing, Bristol and Philadelphia, 1990. One can see that gauge fields can be framed in algebraic topology terms. If you want to work on GUTs and TOEs, you will have to master extremal-action principles, group theory, and algebraic topology. Nakahara is a good springboard for the discussion that follows on GUTs and TOEs.

It was Einstein's dream to find a final theory of existence: a theory of everything that fully explains and links together all known physical phenomena, and predicts the outcome of any experiment that could be carried out in principle—that is conceiving idealized 'thought experiments' with imperfect laboratory measurements minimized as much as we want. Not long after Einstein published his general theory of relativity, which combined space and time into a four-dimensional manifold, others produced models of the universe with higher dimensions as well, *e.g.*, Kaluza-Klein theory in 1921, a five-dimensional theory that sought to unify gravity and electromagnetism using the machinery of compactification much like today's string theories do, which instead try to unify gravity with quantum mechanics (quantum fields including QED). In fact, this kind of theorizing involving other geometries had already happened long before. At the outset of the development of non-Euclidean geometry, for example, Bolyai, born in 1802. prepared a treatise on hyperbolic geometry between 1820 and 1823. He ended this work by mentioning that it is not possible through mathematical reasoning alone to decide in the geometry of the universe as Euclidean or non-Euclidean. In the 1840s, Hermann Grassmann wrote a Ph.D. dissertation on abstract algebra and exterior algebra wherein he argued that the dimensionality of the universe was not necessarily three. However the first solid physical evidence of the non-Euclidean nature of the universe would have to await the conundrum of the constancy of the speed of light as derived from Maxwell's equations. Another geometrically based attempt to unify gravity and electromagnetism can be reviewed in "*The Weyl-Dirac Theory and Our Universe*", M. Israelit, Nova Science Publishers, Inc., 1999. String and M theory are some of the latest manifestations of extremal-action/symmetry/geometry/topology based TOEs.

Here is where I have given up my search for, and my belief in a final theory. Our understanding of the Standard Model has come from rapid progress in accelerator physics and astrophysics. Accelerator physics, however, is grinding down to a screeching halt as accelerators have grown so large and so expensive, only collections of nations can afford to build them to the next order-of-magnitude in energy every twenty or more years apart. Theoretical physicists, on the other hand, create GUTs and TOEs at a pace far faster than experimental physics can get data. It stands to reason that most of these theoreticians are creating mathematics, and only some of these theories will pass the test of experimental discovery within the error bars. Then it's likely that the problem of finding a final theory starts anew, only shifted a little from the past. Richard Feynman was always impressed by how many handcuffs experimental physics imposed on theory, but now it seems to me that there are too many degrees of freedom in mathematics to create high energy theories of physics (energies outside of our technological range), that mathematically match what we observe at low energy (our accessible energy levels). Thankfully the rapid advances in astrophysics technology can help pare down fantasy from reality, but I suppose there will always be uncertainty about higher dimensions in small, meso, and large scales much in the same way Bolyai was uncertain about the geometry of the universe without guidance from experimental physics. Some of the latest ideas of physics concerning cosmology are actually a collection of theoretical perspectives dubbed Digital physics, the physicists' versions of The Matrix. Wikipedia will get you started, including links to works in this field by some of the biggest names in physics like John A. Wheeler, Feynman's dissertation advisor. I know that Wheeler's paper "*Law without Law*" pages 182-213 in Quantum Theory and Measurement, Princeton University Press, 193 will blow your mind.

On the light side, there are some very pleasant, general audience books that give the reader a great appreciation of what we do understand. I recommend "*The Magic Furnace*" by Marcus Chown, Oxford University Press, 2001. It gives the human history, as well as the scientific history, of the evidence we have for the Big Bang Theory. At a more advanced level, including far more detailed history is "*The Anthropic Cosmological Principle*" by J. D. Barrow and F. J. Tipler, Oxford Paperbacks, 1988 covering physics from the beginning of the Big Bang to dinosaurs to religion to extraterrestrial life. I also recommend some biographies. "*Adventures of a Mathematician*" by S. M. Ulam who played a pivotal role in figuring out how to make a hydrogen bomb. Ulam was very creative, and it's insightful seeing how he thought. "*Boltzmann's Atom: The Great Debate That Launched a Revolution in Physics*" by D. Lindley is the biography of the physicist who pushed the development of statistical physics very far. Boltzmann was Lise Meitner's advisor. Another book very accessible for peering into the minds of classical mathematical geniuses is "*Journey through Genius: The Great Theorems of Mathematics*" by W. Dunham. A beautiful book on what mathematics is, is "*What is Mathematics?*" by Richard Courant and Herbert Robbins, Oxford University Press, 1957. It appears to have been recently reprinted. One of the best biographers of physicists is the physicist himself, Abraham Pais, who managed to survive German occupation. Pais's doctoral dissertation attracted the attention of Niels Bohr. Pais worked on SU(6) symmetry breaking. In the late 1970s Pais became a historian of modern physics. He wrote, "*Subtle is the Lord—": The science and life of Albert Einstein*", Oxford University Press, 1982. I learned much of the history of QED from "*Inward Bound: of matter and forces in the physical world*" Clarendon Press/Oxford University Press, 1988. Pais provides some of the best books for getting into the technical mind of physicists.

Part 2—Not yet a full elaboration of the courses recommended in Part 1.

Calculus III. At the end of this three part calculus series the student will have mastered operational skills in limiting processes, differentiation including partial differentiation, the computation of integrals, the testing the convergence of infinite series, and working with vector algebra and vector calculus, as in working with several key differential vector operators, computing line, surface and volume integrals, understanding how to apply Gauss' law, Green's theorem on the plane, and Stokes' Law. Clearly this listing is not all inclusive, but it corresponds to a set of tools I definitely had to use to understand the principles of nuclear weapons design and accelerator physics at Los Alamos National Laboratory (LANL) over a span of eight years.

Let us, as they say in the Air Force, take a vector check of where a student sits after having completed a one year course in calculus based physics. A one year course in calculus based physics with a laboratory portion will involve the use of limits, derivatives and integrals, and will build that portion of vector algebra and vector calculus that is required to study electricity and magnetism. I 'understood' calculus I, II, and III better from this physics training than from the mathematics courses. Students of mathematics and engineering would thus do themselves good to study a one year conventional text in calculus based physics, possibly along with the "*Feynman Lectures On Physics*". The laboratory section was also critical to my understanding of physical systems. Thankfully, the emergence of the world wide web (WWW) has at least made videos of laboratory classes available to all.

So where does the anxious student interested in modern Theories Of Everything (TOEs), or interested in what possible meaning complex numbers might have, fit relative to the Renaissance and today's latest headlines from the edges of theoretical physics?
Having studied one year of calculus based physics and calculus I, II, and III, he or she has a good foundation in classical mechanics, electricity and magnetism, and special relativity, but only the basic rudiments of thermodynamics and statistical physics, and quite possibly only an awe of the mysteries of quantum mechanics. As for myself, I did not feel comfortable, or honest with my understanding of thermodynamics and statistical physics until I wanted to understand the processes that occur in nuclear weapons from equations of state for gases, crystals, high explosives, high pressure shock waves, ions and plasmas, to issues in non-equilibrium statistical physics. I say 'wanted' to learn because a modern nuclear weapons designer need not learn any more than how to create pretty pictures, slideshows, and three dimensional movies from the output of nuclear weapons codes built over generations by true physicists, mathematicians, engineers and computer scientists. "*An Introduction to equations of state: theory and applications*" by S. Eliezer, A. G. Ghatak and H. Hora (1986) gives a pretty good treatment of where our knowledge in thermodynamics and statistical physics abuts with our ignorance, as well as shows how quickly mathematical models and methods become complex, difficult and approximate. This compact book has applications literally far outside of weapons work, like astronomy and cosmology. As a physicist, you will not be ready to work at this level until you get past at least your first year in graduate school in physics. Ditto goes for engineers who go into the broad field of pulse power and a few other related fields. The

applied mathematician without any physics or engineering training will never fully understand Eliezer, Ghatak, and Hora.

Up until students reach doctoral dissertation work, they typically see mathematics and physics in their highly refined forms. This may give students the impression of more human genius than is justified from the exceptional individual cases because they do not see the personalities, politics, defeats, reversals, and ugly dead ends of scientific history. It is eye opening to see the fits, starts, and serpentine paths individuals and governments have taken over the centuries in developing mathematics and physics. I therefore make the following suggestion to help you develop an understanding of the intertwined development of mathematics, physics, and engineering. When you have the time, read a copy of "*Mathematical Thought from Ancient to Modern Times*" by Morris Kline, a thick tome of comprehensive history tracing the development of mathematical ideas, most often motivated by investigations in physics and engineering, as well as the careers of the prominent mathematicians, physicists and engineers responsible for these developments. Also if possible, read a copy of, "*A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*" by Michael J. Crowe. This is a particularly relevant history after a student has finished the vector related material in calculus III. Two key ideas that I recall are that: 1) It was difficult for mathematicians to discard with the notion of multiplicative commutativity, *e.g.*, for vectors, $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$, and 2) there is a far more general theory of "space analysis" than vector calculus, namely the method of exterior algebra which is fundamental in differential forms, differential geometry, topology and algebraic topology, all staples of graduate and post-graduate texts in theoretical physics, and to a degree, in some advanced engineering texts in continuum mechanics.

Subjects using calculus III (vector calculus): Electricity and magnetism from the junior level and upwards. Ditto classical mechanics, quantum mechanics and thermodynamics. Note—Classical mechanics as studied by physicists from the junior level and upwards is different than classical mechanics as understood by mechanical engineers. For physicists, classical mechanics is presented in various variational guises, *e.g.*, Lagrangian, Hamiltonian, Hamilton-Jacobi, Poisson brackets, etc., each serving to a greater or lesser degree as paths to quantum physics and quantum field theories at least up to the level of quantum electrodynamics (QED); this is required knowledge in Ph.D. programs in physics. Some of the main topics in classical mechanics studied by physicists include orbital mechanics (think electrons orbiting atomic nuclei) and coupled systems of springs and masses (think of atomic lattices in crystals). Mechanical engineers learn some of the same topics covered in a physicists' course in classical mechanics, but they are not as interested in progressing into quantum physics as much as they are interested in continuum mechanics (think of a fluid flowing over a submarine, or an automobile's frame deforming as it crashes into a bus as computed by numerical methods). Continuum mechanics proceeds into deep mathematics involving tensor calculus and differential forms, but to be useful, it is forced into making contact with the laboratory. Experimental work, for example, drives our mechanical and thermodynamical understanding of metals experiencing, for example, strain rate hardening while undergoing an amalgamation of various shock physics processes driven by conventional or nuclear explosives.

Note—The differences we observe between experiment, perhaps what we think of as reality, and our simulations of reality hindered by finite computer memory, hence limited numerical discretization of space and time, limited processor speed, various kinds of limited I/O speeds and bandwidths, together with our choices in numerical algorithms, their various convergence properties and instabilities, and what physics we exclude from our numerical models, *e.g*., an inviscid hydrocode, among other considerations, makes me believe that this approach to creating a virtual world in the sense of The Matrix move would quickly lead intelligent beings within the simulations to conclude that the nature of their universe is inconsistent with the basic laws of physics they experience even at the classical level. For a different reason, this, if we can believe in what current physics tells us about the deep future, will also be true for intelligent beings born eons from now, for there will come a time when galaxies will become isolated islands, and though the inhabitants of these islands will be able to discover and develop physics pretty much as we know it, they will not be able to reconcile it with a sensible cosmology consistent with the abundance of elements in their galaxy the way we can reconcile the abundance of elements we observe with a Big Bang cosmology. Has this already happened to us? Has it happened to our antecedents? Where are the SUSY sparticles? Or the next missing particles?

Introductory Linear Algebra concurrent with Ordinary Differential Equations. The laws of nature, and many human made models, such as economic models, can be expressed as differential equations. However, except for a small number of exceptions, most differential equations cannot be solved analytically, instead being subjected to approximate numerical methods. That is, differential equations are converted into difference equations to be solved by approximate numerical methods. These numerical methods invariably involve solving systems of linear equations, best expressed in matrix form. An introductory level linear algebra course will give the student a basic understanding of what numerical methods do to solve systems of linear equations. Note—By ordinary differential equation, it is meant differential equations involving only one independent variable, and one or more their derivatives. Ordinary differential equations involve equations with derivatives in the sense of calculus I. More generally, systems of coupled ordinary differential equations may be solved, sometimes exactly, using methods from linear algebra as well; also see eigenvalues and eigenvectors and the method of eigenfunction expansion. Partial differential equations are a type of differential equation involving an unknown function (or functions) of several independent variables and their partial derivatives with respect to those variables. Partial differential equations involved derivatives in the sense of calculus III, often involving differential vector operators used by physicists and engineers.

Some future motivation for at least introductory level mastery of linear algebra. Electricity and magnetism at the junior level and above is expressed in the language of partial differential equations, as are the subjects of heat transfer, diffusion theory, and many other fields including mathematical finance. We have already stated that numerical methods involving partial differential equations ultimately involve solving systems of linear equations. Some problems in classical mechanics are problems involving metric preserving orthogonal transformations. That is, in classical mechanics the length of an object (a metrical concept) is independent of the reference frame (coordinate system) we choose. Linear algebra provides the matrix algebra of such transformations. In non-relativistic quantum mechanics, unitary transformations connect a wave function solving Schrodinger's wave equation between two points in time. In special relativity, it is a four-vector length that is preserved as expressed by Lorentz transformations. In Dirac's equation, a (special) relativistic equation for the electron, we deal with matrix transformations which preserve probability currents. The bottom line: having some knowledge of elementary linear algebra is essential to an eventual understanding of many physical systems and concepts independent of, but in conjunction with corresponding numerical methods.

Advanced undergraduate course in linear algebra (optional, but highly recommended): The need to solve systems of linear equations derives from many sources. The approximate numerical solution of partial and ordinary differential equations has already been mentioned. Multiple linear least square regression to fit a line through data requires the solution of systems of linear equations. Problems in linear optimization, *e.g.*, the simplex method involve solving systems of linear inequalities. Advanced courses in classical, continuum, quantum, special and general relativistic mechanics are bested treated in a coordinate free, linear algebraic language proceeding to tensor algebra and calculus. This list is far from complete, but the reader should get the idea as to the deep importance of linear algebra in both applied and pure mathematics.

Ordinary Differential Equations. When a student is first introduced to partial differential equations, it is shown how some very important partial differential equations can be reduced (by separation of variables) into a system of ordinary differential equations. Hence the reason a first course in differential equations is a course in ordinary differential equations. Advanced courses in mathematical physics at the advanced undergraduate or beginning graduate level introduce the student to other methods of treating partial differential equations such as integral transform methods, Green's functions and the use of the Dirac delta function, which opens a path to the theory of distributions and probability that a graduate student or post doctoral fellow in physics or engineering may wish to pursue on their own in greater depth.

Complex Variables concurrent with Partial Differential Equations. When recently an undergraduate engineering student asked me if complex numbers have meaning, I took his question to mean if complex numbers have a correspondence to physical 'reality'. I explained to him that complex numbers are useful in describing the motion of a damped harmonic oscillator, modeled by an ordinary differential equation. I then thought of the importance of the phases of quantum mechanical wave functions in the Bohm-Ahronov experiment, but I kept quiet for the moment. This free flow of ideas and questions further led me to think about Hamilton's quarternionic extension of the complex number system and the connection of this number system to spinors and Dirac's relativistic quantum mechanical partial differential equation for the electron and positron. The Lie algebra of the Lorentz group then came to mind, an algebra shared with the direct product group $SU(2) \times SU(2)$, which is compact while the Lorentz group is not. I was struck by the magnitude of the difference in experience between my current self and the young man I had once been three decades ago pondering the same type of questions as the young man standing before me. I had often wished for a mentor.

More mundanely, in certain idealized problems of aerodynamics and hydrodynamics, methods from complex variables lead to solutions to flow line fields. Particularly important is the generalization of integration as understood from a calculus II level. Integration in complex variables is deeply connected with infinite series expansions and much more. Incidentally, we can construct 'larger' sets of numbers from 'smaller' sets of numbers: from natural numbers we can construct the integers, from the integers the rationals, from the rationals the real numbers, from the real numbers the complex numbers, from the complex numbers the quaternions, from the quaternions the Cayley numbers, and so on, but we progressively abandon more structure so that we get progressively less axiomatic richness. Does this limit how wild physics theories can be? Probably not. One can create illimitably rich number systems by defining direct products.

Numerical methods. A student running through a calculus sequence is not likely to escape some homework involving numerical methods. Computational fluid dynamics (CFD), hydrodynamics, radiation hydrodynamics and many other computationally intense fields including statistics drive students deep into numerical methods. A great reference, heavy on linear algebra, is "*Numerical Recipes*" in Fortran 77, or C, or C++, etc. One need not take a course in numerical methods. They can be mastered with self study and experimentation with computers. Having a good, general understanding in numerical methods, however is essential to understanding how we model physical systems with computers. Creating approximate numerical models of physical systems aids in our understanding of 'real' systems. One of the best books I've ever run into is the book on numerical methods by Richtmyer and Morton, "*Difference Methods for Initial Value Problems*." Richtmyer worked at Los Alamos during WW II, and became the leader of its theoretical division after the war. This old book of limited availability covers many areas from shock physics to diffusion, heat flow, particle transport, and fluid (hydrodynamics) in one space variable including the method of artificial viscosity which was pioneered in "*A method for the Numerical Calculation of Hydrodynamic Shocks*" by J. Von Neumann and R. D. Richtmyer, Journal of Applied Physics, Vol. 21, March, 1950. Also useful in this vein of reference books I've used is, "*Numerical Methods in Applied Physics and Astrophysics*" by R. L. Bowers, and J. R. Wilson, Jones and Bartlett, 1991. I recommended "*Computational Plasma Physics, with Applications to Fusion and Astrophysics*" by Toshiki Tajima, Kansai Research Establishment, Japan Atomic Energy Research Institute, Kyoto, Westview Press, 2004. I enjoyed working through "*Nuclear Reactor Analysis*" by James J. Duderstadt, and Louis J. Hamilton, Wiley Interscience, 1976. I also like, "*Computational Methods of Neutron Transport*" by E. E. Lewis, and W. F. Miller, Jr., American Nuclear Society, 1993. A friend of mine from Los Alamos recommended to me the three volume series "*Computational Fluid Dynamics*" 4th ed. Vols. I, II, and III by K. A. Hoffmann and S. T. Chiang; they are very good books. Working these three books out will give you a strong foundation in the numerical treatment of partial differential equations in Computational Fluid Dynamics and Computation Fluid Turbulence.

Underlying the applied numerical methods above are the following texts. "*Explosives Engineering"* by Paul. W. Cooper, Wiley-VCH, 1996, and "*Explosive Effects and Applications*" edited by J. A. Zukas and W. P. Walters. An essential text and reference book is "*Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*" by Ya. B. Zeldovich, and Yu. P. Razier, edited by W. D. Hayes, and R. F. Probstein, Dover Publications Inc., first translated from Russian and published in English in Volume I in 1966 and in Volume II in 1967. You will deal with much applied thermodynamics in Zeldovich and Razier. "*Shock Compression of Condensed Materials*" by R. F. Trunin is another book translated from the All Russian Research Institute of Experimental Physics, Sarov, which was translated into English by Cambridge University Press in 1998. The book covers condensed matter physics phenomenology and data tables of materials experiencing up to 10 TPa in large scale underground nuclear tests. Lastly, I recommend "*Foundations of Radiation Hydrodynamics*" by Dimitri Mihalas and Barbara Weibel-Mihalas Dover Publications Inc., 1984. You would be well on your way to readings in cosmology and astrophysics with the book by S. Eliezer, A. G. Ghatak and H. Hora cited above treating equations of state.

An old but great reference book is "*Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*" edited by Milton Abramowitz and Irene A. Stegun, Dover Publications Inc., 9th printing, November 1970. Another essential handbook is "*Tables of Integrals, Series, and Products*" 2nd ed. by I. S. Gratsheyn and I. M. Ryzhik, American Press Inc, 1980. Lastly, there is "*Handbook of Differential Equations*" 3rd ed. by Zwillinger, Academic Press, 1997. I'm happy with the 2nd edition.
Probability and Statistics for Physicists, Mathematicians, and Engineers. Here is a list of topics/ideas that I see as critical: The Central Limit Theorem for the sample mean. Propagation of errors. Least Square Estimation. Maximum Likelihood Estimation (for statistical physics through logistic regression). Singular Value Decomposition and other Principle Component (PC) methods with applications ranging from Molecular Dynamics to pricing a portfolio of stocks and options, or pricing long term weather derivatives for North America based on El Niño/La Niña climatology. A doctoral candidate in physics will have had to have passed at least a one semester course in statistical physics. Problems in econophysics require a deeper foundation in graduate level (measure theoretic) analysis and probability theory, together with courses in stochastic processes modeled by stochastic differential equations. Note—Monte Carlo methods (numerical methods first developed at LANL during WWII to solve neutron transport problems in early nuclear weapons) are good enough, but a good scientist should always pine for a deeper understanding.

Optimization. Outside of the Calculus of Variations, the areas of linear and nonlinear programming, linear and nonlinear optimization are good things to be familiar with. I like "*Optimization Theory with Applications*" by D. A. Pierre, Dover Publications Inc., 1969, 1986.

Design of Experiments. One of the best, most useful applications of Probability and Statistics, with Optimization Theory is the field of Designed Experiments, or Design of Experiments (DOE). I like "*Design and Analysis of Experiments*" 5th ed. or above by Douglas C. Montgomery, Wiley, 1997.

Shannon information theory. No thinker of mathematics and physics, statistical physics and communication theory, from streams of bits to art, to deep theoretical models of possible universes, can be complete without reading "*The Mathematical Theory of Communication*" by Claude E. Shannon and Warren Weaver. The guts of this writing were written by Shannon while working at the Bell Labs. Weaver added a few chapters at the beginning to increase the accessibility of this work, which is mostly accessible to anyone with at least one year of calculus based physics. Today, information theory wends its way from efficient transmission and storage of information all the way to the holographic principle and black holes. An even more accessible book which bends your mind on matters of perception at levels from physics to psychology to human factors, is "*An Introduction to Information Theory, Symbols, Signals and Noise*" by John R. Pierce (also from Bell Labs), Dover Publications Inc., 1980.

Part 2 is minimally elaborated for now.

A few more words on pitfalls

Perhaps one of the best ways to illuminate the pitfalls and sand traps of becoming a self trained man or woman is best illustrated by considering the uncanny parallels between the leading edges of physics during the days of James Clerk Maxwell as a triumphant theoretical physicist and today's fine mess. You can teach yourself like he did, but the fruit today, however, is not as low hanging as it was for Maxwell, and the edge of theory expands much faster. So unless you start with a good advisor from a good school, you're halfway screwed. No matter how good you are, without a good advisor you will forever be chasing an ever more rapidly receding edge (like me), and should you pull off a miracle (a super surfer dude), you likely won't get attention.

Leading physicists. I'm not ashamed to say that I would have liked becoming a leading physicist like an Einstein, Fermi, or Wigner as an ancillary result of my having figured out a final theory. This deeper passion to understand existence is what motivated me to study the way I have, pretty much exclusively on my own from the days when I began teaching myself calculus the summer after I learned algebra in the seventh grade. I would take soundings of the leading ideas in theoretical physics, figure out what I needed to learn, *e.g.*, abstract algebra, and hit the books, often at the cost of very little sleep. It has been a haphazard process with much unnecessary wastage of time, and I have yet not gotten to the bleeding edges—I have reached the edges, but I'm getting old. I wish I could have had a mentor, or at the very least, access to these words I am penning at the age of forty-five via time travel to my past. I could have acquired the same education in at least a decade less. Sometimes when I see a young physicist on television, in his or her thirties or forties, I wonder if they are smarter than me. Probably. But the thought has recently occurred to me, that not all of them can have acquired the depth of background that I have. I suspect that for many of them, great mentoring has probably played a great role in their success in getting to where the bleeding edges of physics are. It has to be the case, and a great argument for going to a school with a great reputation with great professors. The fruit today is not as low hanging as it was when someone like James Clerk Maxwell was training himself in physics and mathematics. On the other hand, I suppose that physics was already accelerating rapidly in wild directions even back then. Professor Tait spent years studying the abstract mathematics of classifying knots of string in an effort to explain atomic spectra, and if I had been a kid back then, I suppose I would have tried to understand his work, which turned out to be, for a long time, an exercise in pure mathematics—knot theory has had a resurgence in physics for other reasons. Then I suppose that I would have stumbled into Cantor's work on transfinites fresh off the presses, and perhaps I would have gone insane. Maybe being a leading physicist is about having an instinct for what is pure mathematics and what is physics, and a little luck. But I'm not sure this is true today. When non-Euclidean geometries were discovered in the early 1800s, there were experimental attempts to determine if our universe was Euclidean or not, but this branch of mathematics remained pure mathematics until Einstein produced general relativity. Of course this mathematical theory allows one to grope for more general post-Einstein theories, like Kaluza-Klein theory early last century. Today we have many more theories in physics from which to springboard into even more mathematical theorization. At some point mathematical models must be brought against "reality" as we struggle to build ever more powerful and expensive accelerators and space telescopes. Until then, models remain models. Of course, when the laboratory pares down the number of models, it's likely that mathematics and theoreticians will find the freedom to start growing new mathematics and new theories ad infinitum from the new experimental benchmark.

Some comfort: I don't worry about reality anymore, the way I used to when I was so into physics. I don't think the process of discovering layers of reality ever stops. But I do appreciate that I am living in a time when we have figured out how to make mathematical models of existence beyond what we can measure, and that I have some mastery of this set of skills. Up until as recently as the 1970s, we were still trying to figure out mathematical models for what we were observing in accelerators.

Take noise, good old Gaussian noise. It is a collection of random hiss across the frequency spectrum, indistinguishable and uninteresting. I suppose I can think of the universe this way before life evolved on Earth. Slowly humans arrived and begin to figure out patterns in the noise. Things became interesting: the orbit of the moon, the seasons. There was a beat to things, a music of the spheres if you will. In time we fleshed out the first level of universal abstraction: Newtonian gravity, Newtonian physics, and the calculus (Newtonian and Leibnizian). We progressed to electricity and magnetism via Maxwell. Then things got stormy. We couldn't figure out black body radiation. The Earth was old according to geologists, but young according the physicists who could not imagine a greater source of energy than burning coal—like radioactivity. The constancy of the speed of light brought further confusion. More results came from the laboratory: x-rays, cathode rays, radioactivity, and yet more results came from theory driving experiment, *i.e.*, Brownian motion, osmosis, the photon of Planck. Then came a false breakthrough along with a correct breakthrough. That is, special relativity came alongside old, Bohr orbital quantum mechanics. Then we got our QED and our QFTs. And now, as with Professor Tait's investigating of knots of strings to explain atomic spectra, we're playing with knots of strings in compactified dimensions to explain the spectra of fundamental particles, many yet undetected, like the SUSY spectrum, the Higgs boson or bosons. Within this pattern of history, I am reminded of fractals and their self-similarity across scales. We magnify in, or zoom out of a fractal and we see (at least approximately) self similar geometric structure. Now fractals do have mathematical, iterative origins, and if the universe's structures are truly fractal, I suppose some might see a god in this. To me, this fractal idea is just an analogy. Perhaps the universe is more like a musical composition. As children we can appreciate the simple structure of a fugue: row, row, row, your boat. As older music listeners we can appreciate the more complex structure of Johan Sebastian Bach, arguably several orders of complexity removed from a children's song. As a Johan Sebastian Bach (or an Einstein) we probably appreciate even greater orders of structure, but I'm not in this class. As we evolve and merge ourselves with strong artificial intelligences and beyond, I'm sure the appreciation for more complex structures will continue to increases until some limits are hit, if any. I think it similar to flying in a passenger jet over first an ocean, then a coastal mountain range, then plains, then another mountain range, and so on. Some of us see patterns and recognize a loose theme of design in the flowing landscape, a part of a continuum of patterns down below us at smaller scales, extending up above us into the voids of space—a mindless universe doing its thing. Most others, however, see an intelligent, causal purpose and design in the same semi-haphazard flow of landscapes. Finding patterns is what we're built to do. Is that a tiger? Run!

Open Questions: This 'syllabus' has been, absent lots of personal history in dealing with dead ends, a representation of my path to understanding how we theorize about existence in terms of generalizations of universes/multiverses from the Standard Model, to good training in physics, mathematics, and engineering. Could this syllabus be a basis for the knowledge base of an AI model of a fairly mature physicist, mathematician, and statistician? (Lawyers and doctors are having some of their baser work taken from them by artificial intelligence for the better good of the customer and patient.) By this question, I am asking if engineers, mathematicians, and scientists shouldn't map out the world as they know it in terms of ideas, books, and research literature as the basis to train strong AI representations of themselves. Networking these AIs follows tacitly. Do good, useful, accelerated results follow?

It boggles my mind to think that such AIs might soon grasp all of our literature on cancer, and easily grasp hidden secrets to cures the way a child might grasp a simple fugue. Just a few months ago, around mid February 2011, an IBM computer, by no means a supercomputer, wiped the floor with two human Jeopardy! champions. Computer scientists, evidently, have been developing a dizzying number of formal systems: spatial calculus, event calculus, and so on, and the art of amalgamating these formal systems into artificial intelligence is what is finally leading to machine intelligence of sufficient power to begin to deal with natural human language. The formal systems I've been 'raised' in are those of mathematics: the formal system of arithmetic, the formal system of plane geometry, the formal system of the Zermelo-Fraenkel axioms, group theory, etc., and by the way. I'm out of my element in the formal systems of AI. I strongly suspect things will get interesting now at a much faster rate, not that they haven't been for a few hundred years already.

Guides are good. T the best teacher is oneself. Just because I've mapped out some limits, doesn't mean anything. Be optimistic. Go out there, trust nothing, and learn.

Additional notes and history for Parts 1 and 2

Circular situations. From observation we have been inspired to develop (create or discover) mathematics and physics. When we model things, we begin with (stochastic) partial differential equations, boundary and initial conditions, discretize them, creating a system of simultaneous linear equations, which are then reduced to 0s and 1s inside memory. We then let the computer evolve the solutions, dealing with whatever rounding errors. We can imagine the 0s and 1s being laid out on a one-dimensional tape, as in a Turing machine, or a two-dimensional lattice, as in a checkerboard. Conversely, taking the view that the universe is a computable, one-dimensional Turing machine, or some such type of discrete cellular automata, processing grid cells (or 0s and 1s) according to some simple rules, *e.g.*, the two-dimensional game of life by John Conway, by which complex, meta, self-organizing rules evolve, we can imagine the eventual evolution of our universe as we know it by differential equations. (Since black holes are two-dimensional surfaces, some people talk about the universe being an information theoretic holographic machine.) Either way, one will run into the limits and/or paradoxes of rejecting or keeping the Axiom of Choice. The Wikipedia article on digital physics is a fun read.

For two thousand years mathematicians sought to capture eternal, immutable truths in formalized, axiomatic constructions divorced of any taint from fuzzy, imperfect physical reality, despite the fact that their points, lines and planes were abstracted from imperfect experience. They were snobs. Today mathematicians are plagued by deep divisions between set theorists, formalists, intuitionists and other camps. They have failed miserably, and there is no end in sight from mathematical incompleteness, ambiguity, a welter of incompatible choices, paradoxes, and the threat of inconsistency. Physicists, drunk with many successes, have recently become the snobs. They seek, even promise us god particles and theories of everything. Notwithstanding that the language of physics is mathematics, and that it therefore inherits whatever ails mathematics, physics is on the same path to its own emergent incompleteness, ambiguity, a welter of incompatible choices, divided camps, paradoxes and the threat of inconsistency.

Richard Feynman, who was no snob, noted that observation pins down theorizing because theories have to agree with observation, and given how much we know about physics to so many decimal places, *e.g.*, the fundamental constants of nature, spectral emission and absorption lines, etcetera, we are severely limited to what kinds of mathematical models we can construct. I beg to differ. Mathematicians have learned that there are infinitely many nonisomorphic interpretations to first order logic systems such as arithmetic, upon which we construct our common number systems, mathematics in general, and the mathematics employed in physics in particular.

Let history be a guide. In 1862 Lord Kelvin severely misestimated the age of the Earth by billions of years as was being argued by geological evidence. Completely ignorant of nuclear processes, he calculated how much time it would have taken the Earth to cool to its present temperature if it had started off as a molten object. At best, physicists could have calculated an estimate of the missing energy to account for the geological dating of the Earth, but they would have had no solid basis to cook up a reasonable model to explain the discrepancy even though much of the mathematics of quantum mechanics and general relativity was already extant. In 1860 Lord Kelvin conjectured a knot theoretic model of atoms, while the enemy camp supported corpuscular theory, the view that atoms are rigid bodies occupying precise positions in space. Corpuscular theory had nothing in it to suggest radioactivity, a source of heat. The same could be said for Kelvin's vortex atoms. Vortex atoms were inspired by mechanical analogies—in this case, smoke rings. We are more experienced today. Current astronomical data indicates that galaxies rotate fast enough to fly apart if their mass is composed only of the matter that we can directly observe with our instruments. We therefore conjecture dark matter, and estimate how much of the matter in the observed universe is dark—most of it. This kind of reasoning was forced upon us by, ironically, radioactive processes. Energy seemed to be destroyed in certain types of radioactive decays. The physicist Pauli invented the neutrino, an as yet unobserved particle to account for the missing energy. The neutrino was eventually detected, and this successful experience has since been repeated many times over. We expect to detect dark matter soon, and to see how far off the mark today's incompatible theories of particle physics turn out to be against some as yet unrealized experimental breakthrough.

Given our history, we might wonder if it is possible to develop a relative measure of plausibility to our current collection of speciated theoretical models. First, let us consider two near misses by mathematicians as a warning to how difficult the question is. The mathematicians who developed non-Euclidean geometries speculated—they even tried to measure the curvature of space—that there was no *a priori* basis for any particular geometry to the universe. This was a near miss indeed. The precession of the perihelion of Mercury—a deviation from Newtonian gravity—was first recognized in 1859, and not resolved until the appearance of Einstein's general theory of relativity gave us curved spacetime. Another relatively near miss was the development of the Hamilton-Jacobi equation in which the motion of a particle can be represented as a wave. The equation was first realized by Hamilton in 1834, and shares the same conditions for separability as the Schrödinger wave equation of quantum mechanics. In his derivation, Hamilton had had no experimental justification other than to take the limit of the wavelength going to zero. Unlike the case for atoms, which had a growing body of evidence from chemistry, thermodynamics, kinetic theory and other scientific sources, there was no observational evidence to support one spacetime geometry over another, or to tell us that we needed a probabilistic, quantum mechanical wave equation approach to mechanics at small scales. The mathematical works that turned out closest to observational reality, *e.g.*, Hamilton-Jacobi theory, had no plausible connection to the observed facts, and the mathematical works that turned out furthest from observational reality, *e.g.*, vortex atoms, had been motivated by experimental observation indicating the existence of indivisible atoms.

In the case of estimating the age of the Earth, the technological and scientific maturity wasn't there to support any notion of radioactivity. The first solid evidence for fundamental particles involved in radioactive processes awaited Röntgen's discovery of x-rays in 1895, the discovery that uranium emitted "strange rays" by Becquerel in 1896, and the discovery of the electron in cathode ray tubes by J. J. Thompson in 1897. Thus the knot theoretical approach to building a model of atoms by Kelvin's contemporary, Professor Tait, turned out to be nothing more than an exercise in mathematics. No less than James Clerk Maxwell had lent ideological support to vortex atoms based on his understanding of kinetic theory and its experimental evidence. Maxwell was presciently convinced of a connection between spectral emission lines of light, light being an electromagnetic phenomenon, and the vibrations of atoms and molecules. It was not until 1913 that the halfway decent Rutherford-Bohr model was put together atop a foundation of a half-century's worth of additional scientific and technological progress. The prior collection of abstract mathematical models of atoms ended far off the mark. Theoretical progress didn't occur until the laboratory had spoken with sufficient information to bracket mathematical freedom to correspond to the observable facts in the sense that Feynman spoke of. There have been cases, however, when the laboratory hadn't yet spoken so definitively, and speculative mathematical models turned out to correspond closely to physical reality. Consider the case in which Planck conjectured that the distribution of the energy of the (molecular) oscillators in a blackbody is described by a quantized energy distribution. An unbridgeable gap had existed between models of the spectral distribution of light emitted at short wavelengths (Wien law) and long wavelengths (Raleigh-Jeans law) which Plank's ansatz resolved.

In the case of vortex atoms, physicists were attempting to match experimental observation with theoretical models. However vortex atom theory was fanciful and abstract, aiming at a deep and general foundation far beyond the reaches of extant experimental capability to verify, much like today's string theories or loop quantum gravity. Planck's ansatz, on the other hand, was simple and well within the reach of laboratory methods to verify. Is this a clue to developing a measure of relative plausibility? It has long been convention that between two theories, the simplest one is more likely closer to the truth—Occam's razor. Unfortunately, though there have been many successes drawn from abstruse, abstract theoretical models anticipating results in the laboratory by years, today there seems to be little low hanging fruit by way of "simple" theories. All of today's theories are mind bogglingly complicated. To be fair, today's theorists are grappling at unifying much more baggage than the spectrum of blackbody radiation at short and long wavelengths.

For a short while, discoveries in mathematics and physics were considered to give evidence of a watchmaker god. When obvious problems arose, the burden was placed on the limitations of humanity's genius to resolve things. For example, we understood from Maxwell's electricity and magnetism that accelerated charges radiate. In the Rutherford-Bohr atom, arrived at by experimentation and mathematical modeling, we envisioned electrons as orbiting around the nucleus, experiencing constant acceleration, yet they don't radiate away their energy. Between the two theories, something was amiss. It was up to humanity to fix this glaring collision between a pair of models each derived from observational reality. Indeed, humanity cooked up quantum mechanics and quantum electrodynamics not long afterwards.

Today we are in a similar situation. We cannot reconcile general relativity with quantum mechanics despite nearly a century of intense effort. Most of us feel that the failure to resolve this unification has, as we have suspected before, more to do with our limited genius and the limits of our current experimental art. Some of us, however, have recently wondered if the problem is not with us, but with our reality. Is our reality virtual? We ask this question, even though our own virtual realities are still very poor, because we anticipate the day when our art will be more accomplished. At that future point, we might wonder if our virtual beings might go on to develop a civilization of their own, eventually sufficiently advanced enough to detect how poorly and self-contradictory we set up the laws of their universe, but we are getting ahead of ourselves. We should probably first look at our current, narrowly scoped models of reality. Take high explosive hydrodynamic processes for example. Suffering limitations of memory and processing speed, our best hydrocodes are fairly coarse gridded, on the order of a tenth of a millimeter. They exclude much physics that is still too burdensome to compute, and numerical methods run away due to hydrodynamics instabilities. These codes are inviscid. They employ artificial numerical schemes to smooth out shock wave propagation physics. Burning thermonuclear explosive processes ignited by lasers or x-rays from pulse power machines run through temperature and pressure regimes where we have very limited first principle knowledge. Thermodynamic equations of state are contrived with knobs that allow us to match data only somewhat consistently from shot to shot. If the shock front moves too fast relative to the collected data, we might reduce the specific energy of the high explosive, and/or its mass, and so on with such a collection of knobs tuned (not necessarily uniquely) until the code matches the shot under investigation—forget about matching a plurality of differing shots. If we repeated a set a shots as consistently as we could, and we observed detectable, inconsistent variations in the lab, we wouldn't worry about matching computer codes to data, we'd worry about figuring out what was going on with reality, possibly including if we weren't part of some cosmic joke. In this sense, it seems reasonable to suppose that unless we figure out how to compute our own reality consistently, we're likely not to be able to create any consistent virtual realities. Our virtual offspring would eventually figure out that their nature is capriciously stochastic beyond reasonable sources of noise. We're not fooled by actual images from an actual hydro shot compared to hydro shot images made up from a computer simulation. We are not fooled by supercomputer weather forecasts versus real weather. I doubt that we will ever realistically model most real systems from weapons physics to stellar physics, from molecular dynamics to molecular biology, from economics models to actual stock market behavior by this approach of using tunable parameters to make up for what we cannot measure due to extreme pressures and temperatures or complexity and incompleteness. I think we should forget about making a self-consistent artificial reality good enough to fools its inhabitants into believing they can create a final theory of everything until we can make our own models more realistic.

Perhaps a way out of this problem is to suppose we create a virtual reality from primitive structures and simple rules like cellular automata. On this idea, more will follow further below.

To begin to understand the scope of these types of questions, we should first begin our investigation with the basics. Is mathematics complete, unambiguous, and self-consistent? The answer from history turns out to be no, no, and not necessarily. What are the limits of mathematics? Is physics, expressed in mathematical language, complete, unambiguous, and self-consistent? Certainly not at the mathematical foundations level. What are the limits of physics? Is it possible to build an information theoretic construct to weigh the relative likelihood of models of nature? Might the laws of nature be, as Wheeler postulated, an evolutionary process? Cellular automata develop hierarchies of emergent behavior after all, and wouldn't an evolutionary process be free of contradiction no matter how complex things might get?

Mathematics history of fundamental questions.

In 1923 Brouwer showed that the Bolzano-Weierstrass theorem (every bounded infinite set has a limit point) cannot be proved without assuming the law of the excluded middle—a proposition is true, or its negation is true. Nor is the existence of a maximum of a continuous function on a closed interval demonstrable without the law of the excluded middle. The Heine-Borel theorem (from any set of intervals that enclose or cover an interval, a finite number can be selected which cover the interval) also cannot be demonstrated.

There are issues with the axiom of choice and the continuum hypothesis. First, let's take a brief detour into history. The axiom of choice is the assertion that given any collection of sets, finite or infinite, one can select one object from each set and form a new set. To be able to order his transfinite numbers as to size, Zermelo showed in 1904 that Cantor needed the theorem that every set of real numbers can be ordered. Cantor had guessed this without proof in 1883. The axiom of choice is used in many proofs of topology, measure theory, algebra, and fundamental analysis, *e.g.*, in a bounded infinite set one can select a sequence of numbers that converge to a limit point of a set. The axiom of choice is used to construct the real numbers from Peano's axioms. It is also used to prove that the power set of a finite set is itself finite. By 1923, Hilbert described the axiom as a general principle that is necessary and indispensable for the first elements of mathematical inference. In 1884 Cantor conjectured the continuum hypothesis, namely, that there is no transfinite number between $\aleph_o$ and c (between the countable infinity of the natural numbers and uncountable infinity of the real numbers). This history is relevant because the Zermelo-Fraenkel axioms of set theory, which are used by some mathematicians as a desirable foundation upon which to build mathematics, contain the axiom of choice and the continuum hypothesis. This axiom system is, as I understand things, currently the most general and fundamental theory on which we build our analysis and geometry.

Not every mathematician accepted the axiom of choice or the continuum hypothesis. Much like some mathematicians insisted for two-thousand years that Euclid's fifth postulate was surely derivable from the other postulates, some mathematicians conjectured that the axiom of choice and the continuum hypothesis could be demonstrated from the other axioms in the Zermelo-Fraenkel axioms. To this point, Gödel showed in 1940 that the axiom of choice and the continuum hypothesis cannot be disproved if the Zermelo-Fraenkel axioms are consistent. However, in 1963 Cohen showed that both the axiom of choice and the continuum hypothesis are independent of the Zermelo-Fraenkel axioms. That is, the two assertions cannot be proved on the basis of the Zermelo-Fraenkel axioms. Moreover, Cohen showed that if the axiom of choice is included in the Zermelo-Fraenkel axioms, the continuum hypothesis (and the generalized continuum hypothesis) could not be proved, but the other way around (including the generalized continuum hypothesis) does imply the axiom of choice. In other words, the two independence results mean that in the Zermelo-Fraenkel system the axiom of choice and the continuum hypothesis are undecidable. It could be that there is a transfinite number between $\aleph_o$ and c, though no set of objects is currently known which might have such a transfinite number.

As already noted, this situation is similar to the effort mathematicians made to prove the necessity of Euclid's fifth postulate by mucking with it. They tried to show that absurd contradictions would arise if the parallel postulate was modified, *e.g.*, by supposing that parallel lines meet at infinity. Instead, mathematicians arrived at non-Euclidean geometries, *e.g.*, spherical and hyperbolic. It turns out that just as there are many non-Euclidean geometries, there are many set theoretic mathematics. In fact, the variety of choices is bewildering.

Other troubles arose. Hilbert's dream of finite, formal consistency—a finite, mechanizable formal procedure to ensure that no contradiction will arise in a given system—became a nightmare. Gödel gave us his incompleteness theorem. It tells us that if any formal system T adequate to include the theory of whole numbers is consistent, then T is incomplete. In other words, there is a meaningful statement of number theory, S, such that neither S nor not S is provable within the theory. Now either S or not S is true. There is then a true statement of number theory which is not provable, and so is undecidable. Godel incompleteness applies to the Russell-Whitehead (Principia Mathematica) system, the Zermelo-Fraenkel system, Hilbert's axiomatization of number theory, and to all widely used accepted axiom systems.

Perhaps the *coup d'état* to the formalization of set theory came in what is now known as Löwenheim-Skolem theory regarding first order theory. With it, one can show that axiom systems dealing with infinite cardinalities, *e.g.*, arithmetic and set theory, that are designed to characterize a unique class of mathematical objects do not do so. The most pertinent example is to the set of whole numbers. One intends that these axioms should completely characterize the positive whole numbers and only the whole numbers. Yet one can find interpretations—models—that are drastically different and yet satisfy the axioms. Thus, whereas the set of whole numbers is countable ($\aleph_o$), there are a plurality of interpretations that contain as many elements as the set of real numbers (c).

Mathematical reality cannot be unambiguously incorporated in first order axiomatic systems. For something as ‘simple’ as arithmetic, there is no innate consistency in mathematics, and no ultimate truth in the sense of Plato as far as we know. High order logics are not afflicted by Löwenheim-Skolem theory, but they have their own weaknesses. It is easy to digress.

Physics history of fundamental questions.

Let us press to physics. Physics is a codification of what we experience expressed in mathematical language. It goes without saying that by inheritance, therefore, it is innately incomplete, and any axiomatization, like axiomatic quantum mechanics drawn from operator algebras and functional analysis, is not unambiguous. Physics also inherits strange paradoxes (not contradictions) from mathematics. In one version of the Banach-Tarski paradox of the Zermelo-Fraenkel axioms, a large sphere may be divided into two parts and rearranged into a smaller sphere. In a special version of this paradox, a sphere’s surface may be decomposed to give two complete spherical surfaces, each of the same radius as the original sphere. This kind of existence proof is hard to swallow in a world seemingly made of atomistic matter. Existence proofs are problematic in perhaps an even more fundamental sense. Existence proofs that a solution to a differential equation exists, for example, are plentiful enough, but these proofs provide no guide to actually finding a solution. It goes without saying that there is active work on creating a set of axioms that are less paradoxical than the Zermelo-Fraenkel axioms.
What is perhaps more interesting than the inheritance of mathematical paradoxes and limitations is the emergence of paradoxes and limitations innate to physics. For a short while, there was a time when the universe (existence) was considered to be the product of a watchmaker god. Some of the limitations and paradoxes follow below. They help me bracket how much our problems in physics might stem from mathematical incompleteness, and how much these problems might be caused by our ignorance of nature. The famous Heisenberg uncertainty principle, for example, is an innate property of Fourier transform pairs, exemplifying a limitation in physics originating in mathematics. Assuming the Earth had started as a molten sphere, Lord Kelvin in the 1860s computed that it could not be more than four-hundred-million years old to cool to its present temperature. This exemplifies a limitation of physics due to our ignorance of yet undiscovered nuclear processes. During this same period, Professor Tait developed abstract, abstruse knot theoretical mathematics to explain atoms decades before Rutherford’s work gave us a much better picture of the gross structure of the atomic nucleus. Today we are not certain about the existence of more than four space-time dimensions at large, cosmological scales, or at subatomic scales. We invent mathematical string theories, loop quantum gravity, variable dimension cosmologies and other constructions in what seems to be infinitely pliable mathematical creativity just as divorced from reality as vortex atoms.

The coverage of physics limitations ends here. Cleary, it is not complete, but perhaps the reader can more fully appreciated the difficulties of faking a complete, consistent, virtual universe by following the brute force classical approach we have been following since the times of the Greeks all the way to today’s incomplete modeling of thin slices of physical phenomenology inside massively parallel supercomputers. It is in this sense that I better understand the proclamations of Stephen Wolfram, his predecessors and contemporaries that the universe is better modeled as a kind of evolving cellular automaton (or some other kind of digital physics machine) which can compute the evolution of the universe.

The subject of digital physics is vast and diverse. It is divided into philosophical camps, incompatible in their natures of course. Some of the approaches are easy to frame. If an electron switches from one quantum state to another, it may be viewed the same as if a bit changed from one value to the other, say from zero to one. As the universe appears to be composed of elementary particles whose behavior can be described by the quantum switches they undergo, the implication is that the universe as a whole can be described by bits. Every state is information, and every change of state is a change in information (requiring the manipulation of one or more bits).

In this newer approach, we are proceeding backwards. We are beginning with elementary members of the universe, assigning them simple (computational) rules, and letting them evolve (compute) the future. What we see today, the Standard Model, weather systems, geopolitical systems, are to be seen as emergent phenomena at different hierarchies and time scales. We are back to a kind of watchmaker god: elementary particles, elementary rules, spacetime evolution. However, some have pointed out that digital physics is based on the theory of finite state machines. It cannot therefore abide a physical theory whose mathematics requires the real numbers, which is the case for all physical theories having any credibility. Thus others have argued that the universe is not of continuous cardinality. "God made the integers; all else is the work of man," Kronecker. The arguments go back and forth. Pro: computers can manipulate and solve formulas describing real numbers using symbolic computation, therefore avoiding the need to approximate real numbers by using an infinite number of digits—a Greek trick of using geometrical quantities to describe numbers. Before symbolic computation, a real number with an infinite number of digits was said to be computable if a Turing machine would continue to spit out digits endlessly. There is no last digit, but—con—this sits uncomfortably with any proposal that the universe is the output of a virtual-reality exercise carried out in real time (or any plausible kind of time). The universe seems to be able decide on its values in real time, moment by moment. Richard Feynman put it this way, "It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space, and no matter how tiny a region of time. How can all that be going on in that tiny space? Why should it take an infinite amount of logic to figure out what one tiny piece of space/time is going to do?"

To a practical person, it would thus seem that we arrive at an impasse no matter which philosophy we begin with when we think about the universe, or existence. I am thus forced to care more about the relative likelihood of one theory over another to better explain the observational results, say cellular automata versus classical economics theory. As for physics, which is better in the absence of further evidence from the universe: Technicolor or string theory? String theory or loop quantum gravity? The notion of a light cone comes to mind. By analogy, at the intersection between past and present we have what we know from observation, the Standard Model and General Relativity. Above this, stretching from the present to the future, lay infinite possibilities of theoretical constructions, inconsistent among each other, all of which can be tuned to match present observation. There is minimal supersymmetry (SUSY) for example, and it goes with saying, more than minimal supersymmetry. As we advance our technology to probe existence further and further, we will evolve the vertex of the light cone of knowledge and theory forward until, if ever, we reach insuperable limitations, *e.g.*, we might begin to create miniature, very short-lived black holes in our accelerators, and this would spell the end of particle physics wouldn't it?

My introduction to graduate level topology came from studying the economic work of Arrow and Intriligator—Arrow was awarded a Nobel Prize. How distant this approach is from reality, I compare with the information theoretic work Shannon published on how much of the English language we can capture from the statistics of successive letters pulled from speech and written works. I will take examples from "*The Mathematical Theory of Communication"* by Shannon and Weaver. In the simplest case, we may choose a successive letter probabilistically depending only the preceding letter, and not the letters before that. We might get: TUCOOWE AT TEASONARE FUSO TIZIN… If the choice of word depends probabilistically on the previous two words, we might get: IN NO IST WHEY CRATICT… The difference between these simple approaches to capture English and real English is large. The information content of these statistical tables is very small. In "*Le ton beau de Marot*," Douglas Hofstadter argues that we cannot create good machine translators until the machine translators become as complex as we ourselves are. It is implicit that Hofstadter believes a given human language is very dense in information. On the other hand, some relatively simple systems like cellular automata, or agent-based modeling, might capture complex, emergent dynamics, such as herding behavior observed in volatile stock markets. Thus it seems how much fidelity a model captures a given slice of reality depends greatly how efficiently we encode information about a slice of reality. THE A IN CAME TO OF was produced from the statistics of proceeding words from only the previous word, and THE HEAD AND IN FRONTAL ATTACK OF AN ENGLISH WRITER came from the statistics of proceeding words from two preceding words. In absolute terms—which I cannot establish—the theory of vortex atoms seems far more removed from modeling physical systems than kinetic theory. Perhaps developing relative measures of goodness-of-fit between theories is a better way to proceed. The information content of how much more observational phenomenology (and emergent observational phenomenology) is captured by simple billiard ball kinetic theory than by deep knot theoretic vortex atoms might be easier to answer. This kind of thinking smacks of being a useless version of Occam's razor. Only time, and growing experimentally derived knowledge separated out kinetic theory as being an approximate description of atomic and molecular physics in certain regimes, and vortex atoms as being an exercise in mathematics.

Part 3. An approach for describing the relative likelihood of theories.

On setting apart a Nobel Prize winning physics theory, such as quantum electrodynamics (QED) from a Nobel Prize winning theory in econophysics, as in either econometrics or economics. To begin with, the statistical moments characterizing the various elementary players in QED are better pinned down. Take the mass and charge of the electron for example. The 2010 CODATA source gives the mass of the electron as $9.10938291(40)\times10^{-31}$ kg, with the figure in the parenthesis being the estimated standard deviation. The same source gives the electron charge (and standard deviation of the electron charge) as $1.602176565(35)\times10^{-19}$ C. On the other hand, take the so-called Greeks in finance. The Greeks represent sensitivities of the price of derivatives to changes in underlying parameters. In the Black-Scholes-Merton derivatives (options) pricing model, rho, for example, is the partial derivative of the option value with respect to the risk free interest rate. However, the risk free rate is given by a regularly evolving term structure, a meta level concept of the relationship between interest rates and time to maturity for various maturities. There are many ways to estimate the term structure from market data: least squares regression, Lagrange polynomials, splines, kernel methods, linear programming to name a few. The probability density functions in finance typically require higher order moments than just means and standard deviations, and are not as solidly pinned down as they are in QED.

Herein lays another clue as to what sets physics apart from econophysics. Deviations from Gaussian densities, as in densities with long tails typical of stochastic econophysics processes can give rise to extreme behavior like market crashes or wild oscillations. Long tails indicate richer network topologies than those in physics. The network topology in Ising model theory seldom extends beyond nearest neighbors interactions, or next nearest neighbors interactions. Ising model networks are sparse and rigid. This is in stark contrast to econophysics networks, which are crude, coarse, and dynamic, and being better characterized by power laws. Stated in terms of correlation functions, many physics models describe natural systems well with no more than pairwise correlation functions. Econophysics models, instead, require many, poorly defined, general (multidimensional) correlation functions. Consider energy market options pricing. Certain energy options depend heavily on long term weather forecasting, requiring tea leaf reading of El Niño and the Pacific Decadal Oscillation, as well as other progressively less well understood oscillator phenomenology. Energy options also depend on many intangibles, such as political risk: witness the market effects of the unforeseen, broad based uprisings that erupted in the Middle East in 2011. Potential EPA rulings also color energy market dynamics, as do propagating rumors, hidden insider player activity, shipping or refinery accidents, traders who base their trading decisions on horoscopes and biorhythms, and so on.

When financial markets overextend risk, and probe the tails of their underlying densities, several related physics concepts come into play. Market crashes have been related to (equilibrium statistical physics) phase transitions and order parameters, and analogies have been found between quantitative measures of fluctuations in an economic index and the (non-equilibrium statistical physics) fluctuations in the velocity of a fluid in a fully turbulent state. Yet we would never class the modeling capability of any current, or foreseeable econophysics theory as we class the success of QED theory to describe, unify and predict experimental observation. There is already too great a gap in applying physics to arguably more tractable modeling problems such as weather forecasting, despite the existence of well established conservation principles and well researched systems of partial differential equations. Weather systems are chaotic, suffer hydrodynamic instabilities, and are very sensitive to initial and boundary conditions, which we can only specify coarsely from gridded spacetime sensors in finite state machines solving systems of truncated, low order difference equations.

A little more light can be shed on the gap between descriptive theories and their ability to match observation by considering excerpts from Shannon information theory. Consider the concept of producing algorithm-based English text beginning with the English character usage statistics. As a first go, assigning equal probabilities to successive letters (and the space character) results in the production of gibberish, as in the obvious difference between meaningless machine output and English text written by a native English speaker. Assigning single English letter usage statistics from a basis of English sources produces greater fidelity. Assignment of double letter (digram) frequencies further improves things, as does the assignment of triple letter (trigram) frequencies, producing many common words such as: and, has, the, men. However, there are diminishing returns to this process of collecting higher order *n*-gram statistics. Not enough English has been written to determine the asymptotic Frequentist statistics of, say, rare, low probability 11-gram letter sequences of English, like ‘probability’ or ‘benignantly’. Even better mimicry of English text can be produced from word level *n*-grams language models built from (*n*-1)-order Markov models, with the probabilities being culled from, say, the Google *n*-gram corpus. Still, we would never expect the reproduction of Hamlet in any reasonable amount of time. Underlying meaningful, human produced text are human brains, minds, and evolving social relationships, each of these being as yet indefinably complex networks within networks. It was probably ignorance of the complexity of the underlying networks and their topologies that made it easy for Renaissance thinkers to believe that the Universe followed mechanically from a small set of laws (equations) together with the appropriate initial and boundary values. In principle it might, but based on current studies of so-called digital physics, there is great questioning of whether discrete, Turing machine or cellular automata principles correspond to our possibly continuum based observable universe. Early workers in Artificial Intelligence (AI) failed to appreciate the richness of the underlying network topologies of the human mind, and early econometricians were fooled into believing simplistic physics-based probability/network topology models could tame markets.

Moving away from considering the gap between physics and econophysics relative to their observation spaces, this approach of using “relative Bayesian information, and corresponding relative network topology, “the model”, may be useful as a starting point for comparing the relative goodness between competing theories of physics, or even to say something about how we would assign likelihoods to whether we inhabit a virtual reality, or have souls. That is,

$$Bayes\ Factor = \frac{\int p(Data \mid \boldsymbol{\theta}_2, Model_2) \times p(\boldsymbol{\theta}_2 \mid Model_2) d\boldsymbol{\theta}_2}{\int p(Data \mid \boldsymbol{\theta}_1, Model_1) \times p(\boldsymbol{\theta}_1 \mid Model_1) d\boldsymbol{\theta}_1}.$$

In the nineteenth century, a great debate arose between a camp of physicists that advocated for corpuscular theory and another camp that advocated for vortex atoms, a debate much the same as more recent debates between string theories, loop quantum gravity, and other theories of everything (TOEs). Perhaps, between currently competing theories, one might compare the relative likelihood of producing the next level of emerging complexity, *e.g.*, going from more fundamental competing theory A or B to the next level of observed emergence, *e.g.*, the relative likelihood of going from corpuscular atoms versus vortex atoms to observed atomic and molecular spectral lines. In this sense, thanks to experimental science and technology, we have a large training set. We observe a wide spanning chain of emergences. From 'fundamental' particles (and possibly detectable higher dimensions at the small scale) we observe neutrons and protons, atoms, molecules, life, and civilizations within an observable universe that may itself have more detectable structure in terms of higher dimensions at the large scale, and we have clothed each of these levels of emergence with theories: quantum field theory, chemistry, molecular biology, economics, and general relativity; and though we don't practice organic (carbon based) chemistry with QED, we can certainly infer from basic QED that we should observe bound states (atoms), even atoms of carbon, if not even molecules containing carbon. The path from organic chemistry to life might be spanned by massive molecular dynamics simulations, and molecular dynamics simulations may predict the existence of cellular, in silico life. The computational resources to close the gap between in silico life and complex, intelligent multicellular life onwards to civilizations seems more daunting however, if not necessarily impossible. This union of the scales that we observe, and their theoretical connections from lower level to higher level emergent complexity, together with a relative likelihood method, might then give us a framework to estimate the likelihoods of souls or gods. If we presume that souls and gods must interact with the substrate of our minds, namely our electrochemical brains, and hence with the electrons in our brains, we have so far not observed such interactions as detectable terms in the Dirac equation for the electron. If these interactions are there, they are very, very weak. From this point of view, it thus seems more likely that souls and gods are emergent phenomenology in our heads, and that when we die, we truly die. This is not to say, however, that exterior hidden variables outside the purview of Bell's nonlocality inequalities are not at play. If we are virtual beings inside a virtual reality, then the coders of our existence can place all sorts of nonlocal correlations in our universe, like correlating my bowling score to always be ten minus your bowling score—magic to us. The appearance of such inexplicable nonlocal correlations might then give evidence to our being virtual entities. Conversely, however, we must realize that the amount of resources our coders have to expend to keep up the deceit should eventually grow very quickly and outrun all of their resources. In the 1800s, given the slower pace of advancement of astronomical instruments, the coders would have had to spend a lot less on generating an evolving map of the universe—only so many stars and smudges—compared to today, with billions of known galaxies beyond our solar system, and billions of minor objects orbiting our sun. After more than thirty years of thought, I have come up with no better way to frame deep questions in the language of mathematics and physics.

A. Alaniz