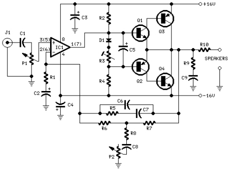
STRUCTURED COSPANS

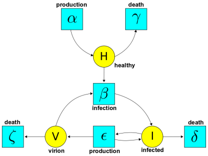
John Baez, Kenny Courser, Christina Vasilakopoulou

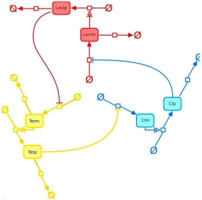
CT2019

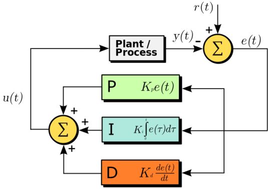
11 July 2019

Throughout science and engineering, people use *networks*, drawn as boxes connected by wires:



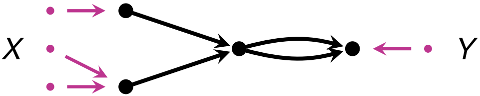






So, they're using categories! Which categories are these?

Networks of some particular kind, with specified inputs and outputs, can be seen as morphisms in some symmetric monoidal category:



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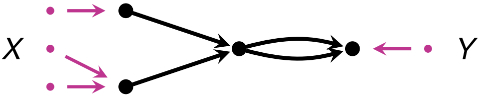
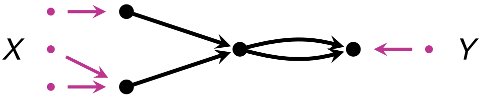


Such networks let us describe “open systems”, meaning systems where:

- stuff can flow in or out;
- we can combine systems to form larger systems by composition and tensoring.

We can describe networks with inputs and outputs using cospans with extra structure. For example, this:



is really a cospan of finite sets:

$$X \xrightarrow{\;i\;} S \xleftarrow{\;o\;} Y$$

where S is decorated with extra structure: edges making S into the vertices of a graph.

Fong invented 'decorated cospans' to make this precise:

- Brendan Fong, Decorated cospans, arXiv:1502.00872.

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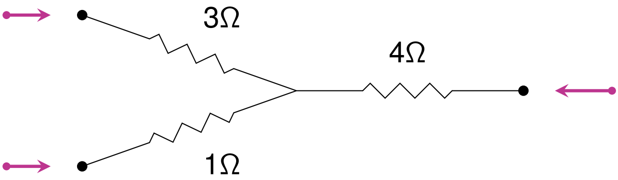
We’ve used them to study many kinds of networks.

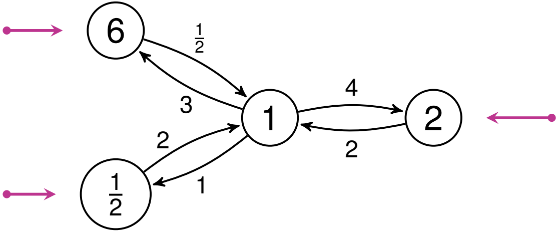
Electrical circuits:

- Brendan Fong, JB, A compositional framework for passive linear networks, arXiv:1504.05625.



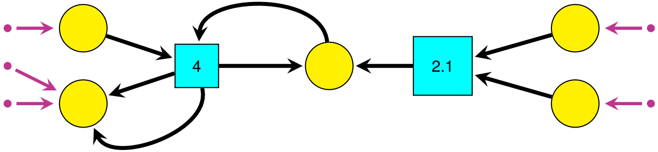
Markov processes:

- Brendan Fong, Blake Pollard, JB, A compositional framework for Markov processes, arXiv:1508.06448.



Petri nets with rates:

- Blake Pollard, JB, A compositional framework for reaction networks, arXiv:1704.02051.



Now Kenny Courser has developed a simpler formalism — ‘structured cospans’ — that avoids certain problems with decorated cospans.

Kenny has redone most of the previous work using structured cospans:

- Kenny Courser, *Open Systems: A Double Categorical Perspective*, https://tinyurl.com/courser-thesis.

Given a functor

$$L\colon \mathsf{A} \to \mathsf{X}$$

a **structured cospan** is a diagram

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

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Think of A as a category of objects with 'less structure', and X as a category of objects with 'more structure'. L is often a left adjoint.

For example, a **Petri net with rates** is a diagram like this:

$$(0,\infty) \xleftarrow{\;r\;} T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

where S and T are finite sets, and $\mathbb{N}[S]$ is the underlying set of the free commutative monoid on S.

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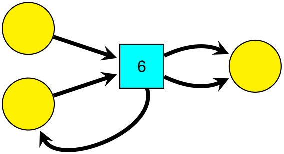
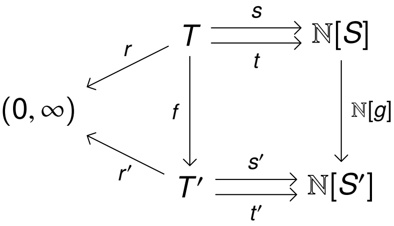


We call elements of S **species** ,
elements of T **transitions** ,
and $r(t)$ the **rate constant** of the transition $t \in T$.

There is a category Petri where morphisms are the obvious things:



where the square involving s and s' commutes, as does the square involving t and t'.

There is a functor $R\colon \mathrm{Petri} \to \mathrm{FinSet}$ sending any Petri net with rates to its underlying set of species.

This has a left adjoint $L\colon \mathrm{FinSet} \to \mathrm{Petri}$.

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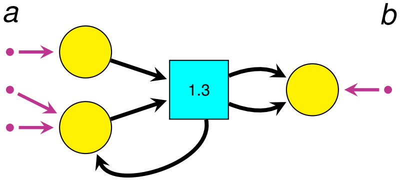
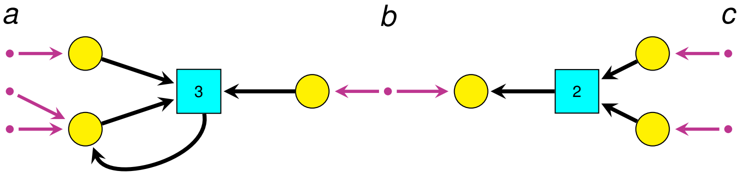
In this example, a structured cospan

$$L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$$

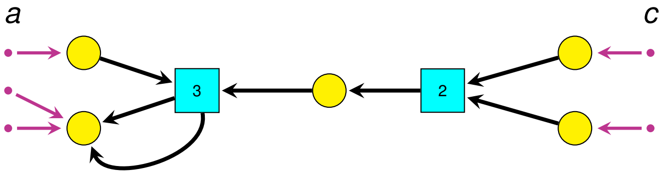
is called an **open Petri net with rates**:



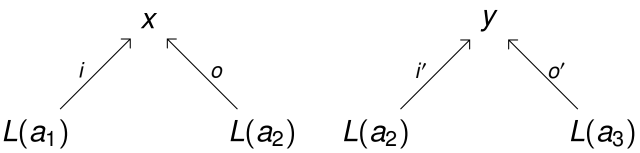
We can compose open Petri nets with rates:



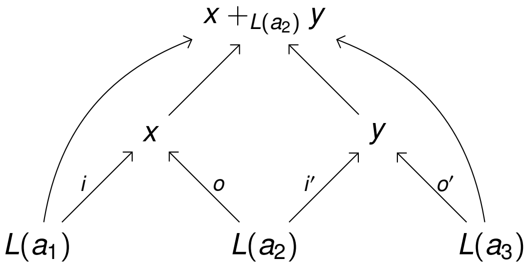
by identifying the outputs of the first with the inputs of the second:



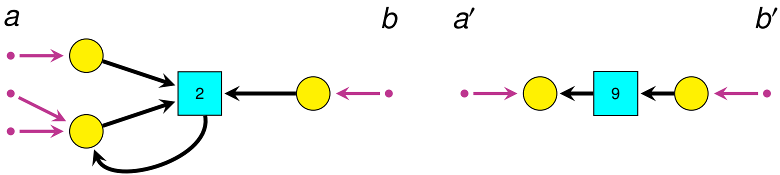
In other words, given open Petri nets with rates:



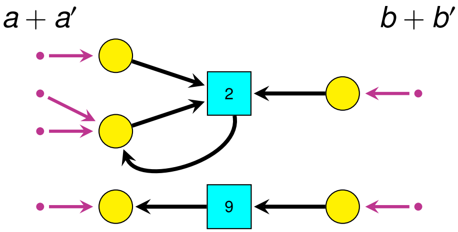
we compose them by taking a pushout in the category Petri:



To tensor open Petri nets with rates:



we set them side by side:



In other words, to tensor open Petri nets with rates:

$$
\begin{array}{ccc}
 & x & \\
{\scriptstyle i}\nearrow & & \nwarrow{\scriptstyle o} \\
L(a) & & L(b)
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & x' & \\
{\scriptstyle i'}\nearrow & & \nwarrow{\scriptstyle o'} \\
L(a') & & L(b')
\end{array}
$$

we use coproducts in Set and Petri:

$$
\begin{array}{ccccc}
 & & x + x' & & \\
 & {\scriptstyle i+i'}\nearrow & & \nwarrow{\scriptstyle o+o'} & \\
 & L(a) + L(a') & & L(b) + L(b') & \\
{\scriptstyle \cong}\nearrow & & & & \nwarrow{\scriptstyle \cong} \\
L(a + a') & & & & L(b + b')
\end{array}
$$

and the fact that $L\colon \mathrm{FinSet} \to \mathrm{Petri}$ preserves coproducts.

In general:

Theorem (Kenny Courser, JB)

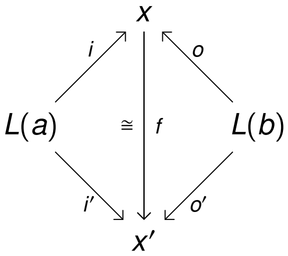
Let A *be a category with finite coproducts,*
X *a category with finite colimits, and*
$L \colon \mathsf{A} \to \mathsf{X}$ *a functor preserving finite coproducts.*

Then there is a symmetric monoidal category ${}_L\mathrm{Csp}(\mathsf{X})$ *where:*

- *an object is an object of* A
- *a morphism is an isomorphism class of structured cospans:*

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

Here two structured cospans are **isomorphic** if there is a commuting diagram of this form:



This theorem applies to many examples, giving structured cospan categories whose morphisms are:

- open electrical circuits
- open Markov processes
- open Petri nets
- open Petri nets with rates

etcetera.

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In all these examples A and X have finite colimits and $L\colon \mathsf{A} \to X$ is a left adjoint, so all the conditions of the theorems hold.

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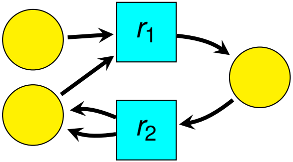
What can we do with structured cospan categories?

Given a Petri net with rates, we can write down a **rate equation** describing dynamics. For example, this Petri net with rates:



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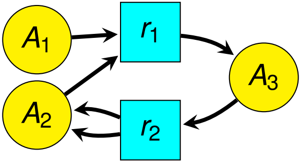


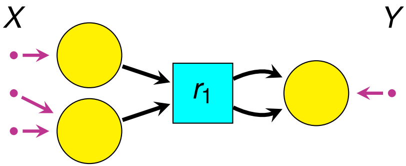
gives this rate equation:

$$\frac{dA_1}{dt} = -r_1 A_1 A_2$$

$$\frac{dA_2}{dt} = -r_1 A_1 A_2 + 2r_2 A_3$$

$$\frac{dA_3}{dt} = r_1 A_1 A_2 - r_2 A_3$$

An open Petri net with rates $f\colon X \to Y$ gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:



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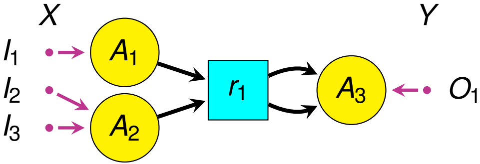

gives:

$$\frac{dA_1}{dt} = -r_1 A_1 A_2 + I_1(t)$$

$$\frac{dA_2}{dt} = -r_1 A_1 A_2 + I_2(t) + I_3(t)$$

$$\frac{dA_3}{dt} = 2r_1 A_1 A_2 - O_1(t)$$

Let Open(Petri) be the category with open Petri nets with rates as morphisms. The map sending open Petri nets to their open rate equations gives a symmetric monoidal functor

$$\square \colon \mathrm{Open}(\mathrm{Petri}) \to \mathrm{Dynam}$$

where Dynam is a category of ‘open dynamical systems’.

So, we can describe dynamical systems *compositionally*, a piece at a time, using open Petri nets with rates.

Jonathan Lorand and I are using this to study questions from biochemistry.

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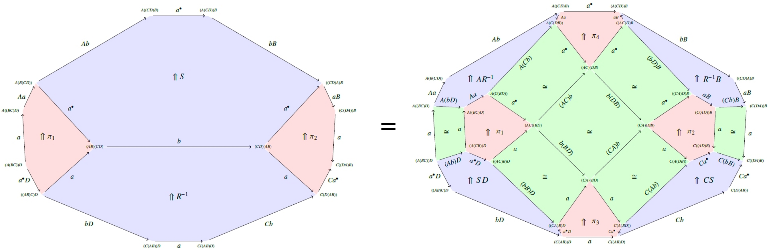
You might be thinking we should use a symmetric monoidal bicategory... and we *could*.



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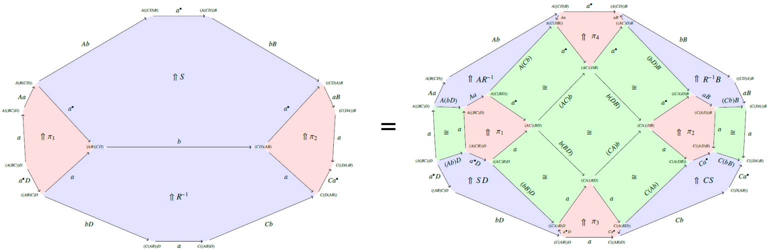

But Mike Shulman noticed that it’s easier to use a symmetric monoidal double category!

For us a **double category** is a weak category object in the 2-category **Cat**. It has a category of objects Ob and a category of morphisms Mor. Composition

$$\circ \colon \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor} \to \mathrm{Mor}$$

is associative and unital up to 2-isomorphisms obeying the usual equations.

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There is a 2-category **Dbl** of double categories, double functors, and transformations. **Dbl** has finite products.

In any 2-category with finite products we can define symmetric pseudomonoids. In **Cat** these are symmetric monoidal categories. In **Dbl** we call them **symmetric monoidal double categories**.

More concretely, a double category has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

So, it has:

- **objects** such as A, B, C, D,

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So, it has:

- **objects** such as A, B, C, D,
- **vertical 1-morphisms** such as f and g,
- **horizontal 1-cells** such as M and N,
- **2-morphisms** such as α.

2-morphisms can be composed vertically and horizontally, and the interchange law holds:

$$
\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle g} \\
D & \xrightarrow[N]{} & E
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{M'} & C \\
{\scriptstyle g}\downarrow & \Downarrow\beta & \downarrow{\scriptstyle h} \\
E & \xrightarrow[N']{} & F
\end{array}
$$

$$
\begin{array}{ccc}
D & \xrightarrow{N} & E \\
{\scriptstyle f'}\downarrow & \Downarrow\alpha' & \downarrow{\scriptstyle g'} \\
G & \xrightarrow[O]{} & H
\end{array}
\qquad
\begin{array}{ccc}
E & \xrightarrow{N'} & F \\
{\scriptstyle g'}\downarrow & \Downarrow\beta' & \downarrow{\scriptstyle h'} \\
H & \xrightarrow[P]{} & I
\end{array}
$$

Vertical composition is strictly associative and unital, but horizontal composition is not.

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Then there is a symmetric monoidal double category ${}_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a structured cospan* $L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$
- *a 2-morphism is a commutative diagram*

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(b) \\ {\scriptstyle L(f)}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle L(g)} \\ L(a') & \xrightarrow[i']{} & x' & \xleftarrow[o']{} & L(b') \end{array}$$

Horizontal composition is defined using pushouts in X; composing these:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x & \longleftarrow & L(b) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' & \longleftarrow & L(b') \end{array} \qquad \begin{array}{ccccc} L(b) & \longrightarrow & y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(b') & \longrightarrow & y' & \longleftarrow & L(c') \end{array}$$

gives this:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x +_{L(b)} y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' +_{L(b')} y' & \longleftarrow & L(c') \end{array}$$

Vertical composition is straightforward.

Tensoring uses binary coproducts in both A and X, and the fact that $L\colon \mathsf{A} \to \mathsf{X}$ preserves these:

$$
\begin{array}{ccccc}
L(a_1) & \longrightarrow & x_1 & \longleftarrow & L(b_1) \\
\downarrow & & \downarrow & & \downarrow \\
L(a_2) & \longrightarrow & x_2 & \longleftarrow & L(b_2)
\end{array}
\quad \otimes \quad
\begin{array}{ccccc}
L(a'_1) & \longrightarrow & x'_1 & \longleftarrow & L(b'_1) \\
\downarrow & & \downarrow & & \downarrow \\
L(a'_2) & \longrightarrow & x'_2 & \longleftarrow & L(b'_2)
\end{array}
$$

$$
= \quad
\begin{array}{ccccc}
L(a_1 + a'_1) & \longrightarrow & x_1 + x'_1 & \longleftarrow & L(b_1 + b'_1) \\
\downarrow & & \downarrow & & \downarrow \\
L(a_2 + a'_2) & \longrightarrow & x_2 + x'_2 & \longleftarrow & L(b_2 + b'_2)
\end{array}
$$

How do structured cospans compare to *decorated* cospans?

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Given a suitable functor $F\colon \mathsf{A} \to \mathsf{Set}$, Fong defined an **$F$-decorated cospan** to be a pair

$$a \xrightarrow{i} c \xleftarrow{o} b \qquad d \in F(c)$$

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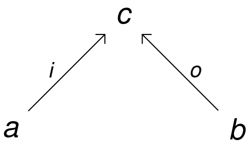
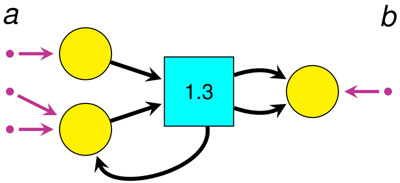


$$d \in F(c)$$

For example, $F(c)$ could be the set of Petri nets with rates having c as their set of species.



The problem is that a functor $F\colon \mathsf{A} \to \mathsf{Set}$ corresponds to a *discrete* opfibration $R\colon \mathsf{X} \to \mathsf{A}$. These are not general enough!

For example: the functor $R\colon \mathsf{Petri} \to \mathsf{FinSet}$ sending any Petri net with rates to its underlying set of species is an opfibration, but not a discrete one.

The solution: use pseudofunctors $F\colon \mathsf{A} \to \mathbf{Cat}$.

Theorem (Kenny Courser, Christina Vasilakopoulou, JB)

Given a finitely cocomplete category A *and a symmetric lax monoidal pseudofunctor* $F\colon \mathsf{A} \to \mathbf{Cat}$, *there is a symmetric monoidal double category* $F\mathbb{C}\mathbf{sp}$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is an* F*-decorated cospan:*

$$a \xrightarrow{\;i\;} c \xleftarrow{\;o\;} b \qquad d \in F(c)$$

- *a 2-morphism is a commutative diagram and a triangle:*

$$\begin{array}{ccccc} a & \xrightarrow{\;i\;} & c & \xleftarrow{\;o\;} & b \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow[\;i'\;]{} & c' & \xleftarrow[\;o'\;]{} & b' \end{array} \qquad \begin{array}{ccc} & & F(c) \\ & \overset{d}{\nearrow} & \\ 1 & \Swarrow \iota & \downarrow{\scriptstyle F(h)} \\ & \underset{d'}{\searrow} & \\ & & F(c') \end{array}$$

Theorem (Kenny Courser, Christina Vasilakopoulou, JB)

Suppose A *is finitely cocomplete,* $F\colon \mathsf{A} \to \mathbf{Cat}$ *is a symmetric lax monoidal pseudofunctor, and* F *factors through the 2-category* **Rex** *of finitely cocomplete categories. Then the opfibration*

$$R\colon \int F \to \mathsf{A}$$

has a left adjoint

$$L\colon \mathsf{A} \to \int F$$

and there is an isomorphism of symmetric monoidal double categories

$${}_L\mathbb{C}\mathbf{sp}(\textstyle\int F) \cong F\mathbb{C}\mathbf{sp}.$$

So in this situation, which is common, structured cospans agree with the 'new improved' decorated cospans!