

# STRUCTURED COSPANS

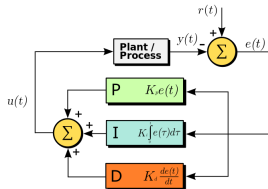
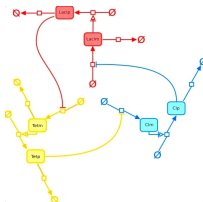
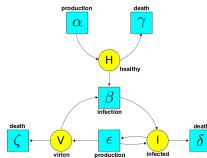
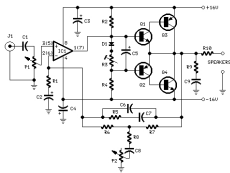


**John Baez, Kenny Courser, Christina Vasilakopoulou**

CT2019

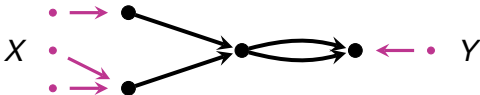
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Throughout science and engineering, people use *networks*, drawn as boxes connected by wires:



So, they're using categories! Which categories are these?

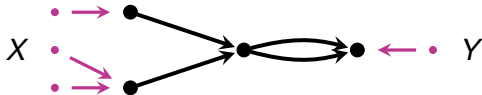
Networks of some particular kind, with specified inputs and outputs, can be seen as morphisms in some symmetric monoidal category:



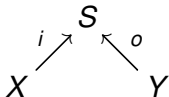
Such networks let us describe “open systems”, meaning systems where:

- ▶ stuff can flow in or out;
- ▶ we can combine systems to form larger systems by composition and tensoring.

We can describe networks with inputs and outputs using cospans with extra structure. For example, this:



is really a cospan of finite sets:



where  $S$  is decorated with extra structure: edges making  $S$  into the vertices of a graph.

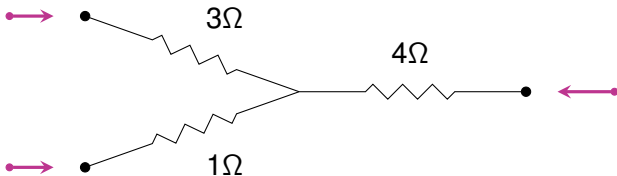
Fong invented 'decorated cospans' to make this precise:

- ▶ Brendan Fong, [Decorated cospans](#), arXiv:1502.00872.

We've used them to study many kinds of networks.

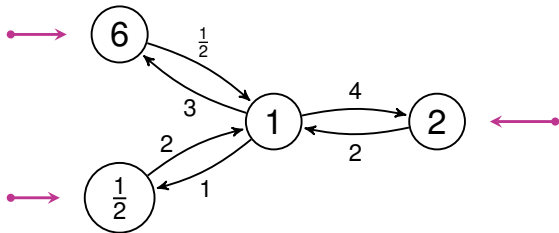
## Electrical circuits:

- ▶ Brendan Fong, JB, [A compositional framework for passive linear networks](#), arXiv:1504.05625.



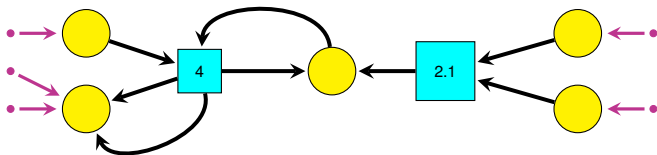
Markov processes:

- ▶ Brendan Fong, Blake Pollard, JB, [A compositional framework for Markov processes](#), arXiv:1508.06448.



Petri nets with rates:

- ▶ Blake Pollard, JB, *A compositional framework for reaction networks*, arXiv:1704.02051.





Now Kenny Courser has developed a simpler formalism — ‘structured cospans’ — that avoids certain problems with decorated cospans.

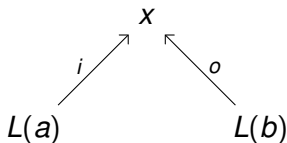
Kenny has redone most of the previous work using structured cospans:

- ▶ Kenny Courser, *Open Systems: A Double Categorical Perspective*, <https://tinyurl.com/courser-thesis>.

Given a functor

$$L: A \rightarrow X$$

a **structured cospan** is a diagram

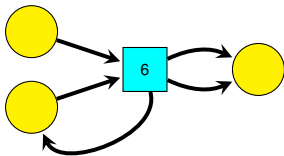




Think of  $A$  as a category of objects with 'less structure', and  $X$  as a category of objects with 'more structure'.  $L$  is often a left adjoint.

For example, a **Petri net with rates** is a diagram like this:

$$(0, \infty) \xleftarrow{r} T \xrightleftharpoons[t]{s} \mathbb{N}[S]$$

where  $S$  and  $T$  are finite sets, and  $\mathbb{N}[S]$  is the underlying set of the free commutative monoid on  $S$ .



We call elements of  $S$  **species** ,  
elements of  $T$  **transitions** ,  
and  $r(t)$  the **rate constant** of the transition  $t \in T$ .

There is a category Petri where morphisms are the obvious things:

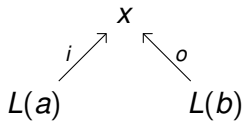
$$\begin{array}{ccccc}
 & & T & \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} & \mathbb{N}[S] \\
 & \swarrow r & \downarrow f & & \downarrow \mathbb{N}[g] \\
 (0, \infty) & & & & \mathbb{N}[S'] \\
 & \swarrow r' & & \begin{array}{c} \xrightarrow{s'} \\ \xrightarrow{t'} \end{array} & \\
 & & T' & & 
 \end{array}$$

where the square involving  $s$  and  $s'$  commutes, as does the square involving  $t$  and  $t'$ .

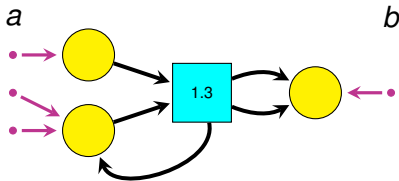
There is a functor  $R: \text{Petri} \rightarrow \text{FinSet}$  sending any Petri net with rates to its underlying set of species.

This has a left adjoint  $L: \text{FinSet} \rightarrow \text{Petri}$ .

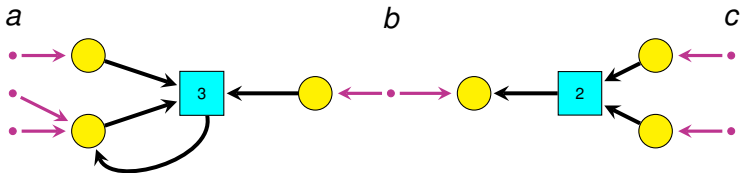
In this example, a structured cospan



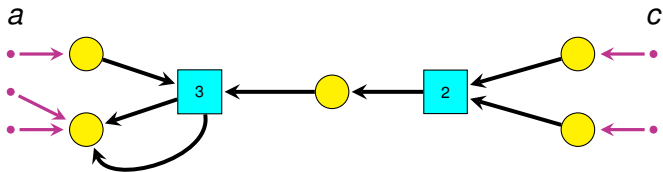
is called an **open Petri net with rates**:



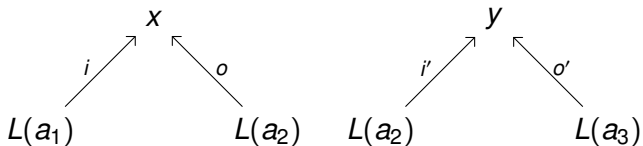
We can compose open Petri nets with rates:



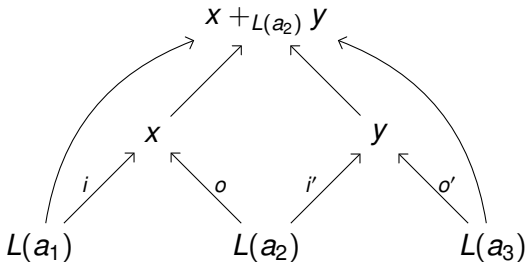
by identifying the outputs of the first with the inputs of the second:



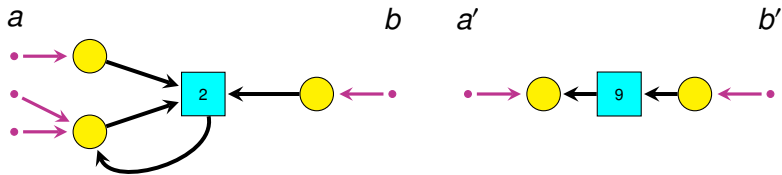
In other words, given open Petri nets with rates:



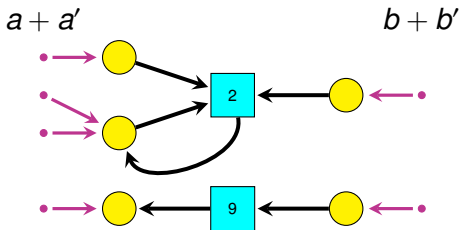
we compose them by taking a pushout in the category Petri:



To tensor open Petri nets with rates:

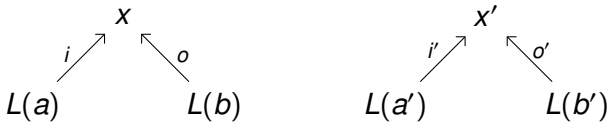


we set them side by side:

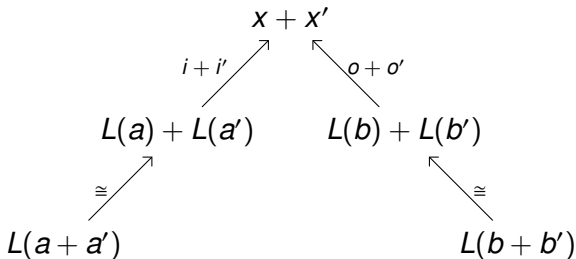




In other words, to tensor open Petri nets with rates:



we use coproducts in Set and Petri:



and the fact that  $L: \text{FinSet} \rightarrow \text{Petri}$  preserves coproducts.

In general:

### Theorem (Kenny Courser, JB)

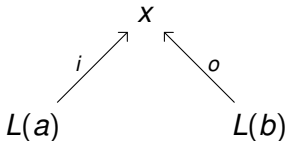
*Let  $A$  be a category with finite coproducts,*

*$X$  a category with finite colimits, and*

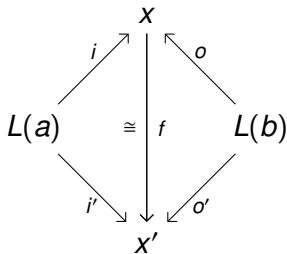
*$L: A \rightarrow X$  a functor preserving finite coproducts.*

*Then there is a symmetric monoidal category  ${}_L\text{Csp}(X)$  where:*

- ▶ *an object is an object of  $A$*
- ▶ *a morphism is an isomorphism class of structured cospans:*



Here two structured cospans are **isomorphic** if there is a commuting diagram of this form:



This theorem applies to many examples, giving structured cospan categories whose morphisms are:

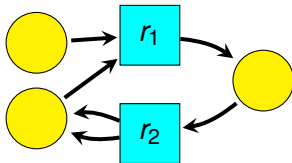
- ▶ open electrical circuits
- ▶ open Markov processes
- ▶ open Petri nets
- ▶ open Petri nets with rates

etcetera.

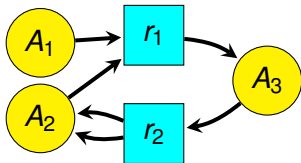
In all these examples  $A$  and  $X$  have finite colimits and  $L: A \rightarrow X$  is a left adjoint, so all the conditions of the theorems hold.

What can we do with structured cospan categories?

Given a Petri net with rates, we can write down a **rate equation** describing dynamics. For example, this Petri net with rates:



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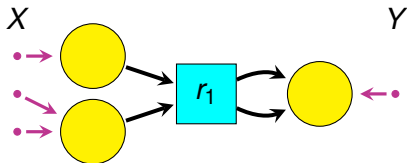
gives this rate equation:

$$\frac{dA_1}{dt} = -r_1 A_1 A_2$$

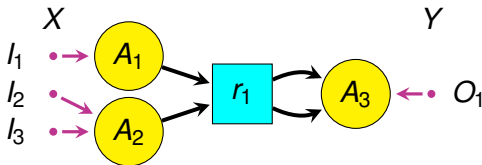
$$\frac{dA_2}{dt} = -r_1 A_1 A_2 + 2r_2 A_3$$

$$\frac{dA_3}{dt} = r_1 A_1 A_2 - r_2 A_3$$

An open Petri net with rates  $f: X \rightarrow Y$  gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:



An open Petri net with rates  $f: X \rightarrow Y$  gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:



gives:

$$\frac{dA_1}{dt} = -r_1 A_1 A_2 + l_1(t)$$

$$\frac{dA_2}{dt} = -r_1 A_1 A_2 + l_2(t) + l_3(t)$$

$$\frac{dA_3}{dt} = 2r_1 A_1 A_2 - O_1(t)$$



Let  $\text{Open}(\text{Petri})$  be the category with open Petri nets with rates as morphisms. The map sending open Petri nets to their open rate equations gives a symmetric monoidal functor

$$\square: \text{Open}(\text{Petri}) \rightarrow \text{Dynam}$$

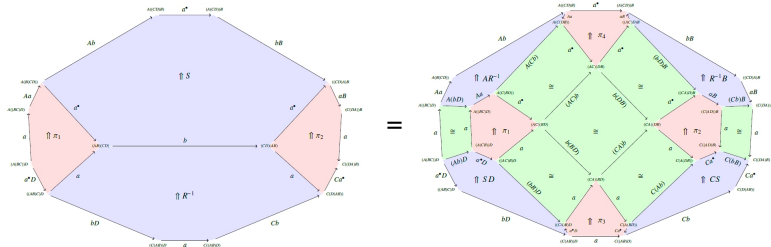
where  $\text{Dynam}$  is a category of ‘open dynamical systems’.

So, we can describe dynamical systems *compositionally*, a piece at a time, using open Petri nets with rates.

Jonathan Lorand and I are using this to study questions from biochemistry.

What if we want to use actual structured cospans, rather than isomorphism classes?

You might be thinking we should use a symmetric monoidal bicategory... and we *could*.



But Mike Shulman noticed that it's easier to use a symmetric monoidal double category!

For us a **double category** is a weak category object in the 2-category **Cat**. It has a category of objects  $\text{Ob}$  and a category of morphisms  $\text{Mor}$ . Composition

$$\circ : \text{Mor} \times_{\text{Ob}} \text{Mor} \rightarrow \text{Mor}$$

is associative and unital up to 2-isomorphisms obeying the usual equations.

There is a 2-category **DbI** of double categories, double functors, and transformations. **DbI** has finite products.

In any 2-category with finite products we can define symmetric pseudomonoids. In **Cat** these are symmetric monoidal categories. In **DbI** we call them **symmetric monoidal double categories**.

More concretely, a double category has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ C & \xrightarrow{N} & D \end{array}$$

So, it has:

- ▶ **objects** such as  $A, B, C, D$ ,
- ▶ **vertical 1-morphisms** such as  $f$  and  $g$ ,
- ▶ **horizontal 1-cells** such as  $M$  and  $N$ ,
- ▶ **2-morphisms** such as  $\alpha$ .

2-morphisms can be composed vertically and horizontally, and the interchange law holds:

$$\begin{array}{ccc}
 A & \xrightarrow{M} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 D & \xrightarrow{N} & E
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{M'} & C \\
 g \downarrow & \Downarrow \beta & \downarrow h \\
 E & \xrightarrow{N'} & F
 \end{array}$$
  

$$\begin{array}{ccc}
 D & \xrightarrow{N} & E \\
 f' \downarrow & \Downarrow \alpha' & \downarrow g' \\
 G & \xrightarrow{O} & H
 \end{array}
 \quad
 \begin{array}{ccc}
 E & \xrightarrow{N'} & F \\
 g' \downarrow & \Downarrow \beta' & \downarrow h' \\
 H & \xrightarrow{P} & I
 \end{array}$$

Vertical composition is strictly associative and unital, but horizontal composition is not.

## Theorem (Kenny Courser, JB)

Let  $A$  be a category with finite coproducts,

$X$  a category with finite colimits, and

$L: A \rightarrow X$  a functor preserving finite coproducts.

Then there is a symmetric monoidal double category  ${}_{L}\mathbf{Csp}(X)$  where:

- ▶ an object is an object of  $A$
- ▶ a vertical 1-morphism is a morphism of  $A$
- ▶ a horizontal 1-cell is a structured cospan  $L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$
- ▶ a 2-morphism is a commutative diagram

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(b) \\ L(f) \downarrow & & h \downarrow & & \downarrow L(g) \\ L(a') & \xrightarrow{i'} & x' & \xleftarrow{o'} & L(b') \end{array}$$

Horizontal composition is defined using pushouts in  $X$ ;  
 composing these:

$$\begin{array}{ccccc}
 L(a) & \longrightarrow & x & \longleftarrow & L(b) \\
 \downarrow & & \downarrow & & \downarrow \\
 L(a') & \longrightarrow & x' & \longleftarrow & L(b')
 \end{array}
 \qquad
 \begin{array}{ccccc}
 L(b) & \longrightarrow & y & \longleftarrow & L(c) \\
 \downarrow & & \downarrow & & \downarrow \\
 L(b') & \longrightarrow & y' & \longleftarrow & L(c')
 \end{array}$$

gives this:

$$\begin{array}{ccccc}
 L(a) & \longrightarrow & x +_{L(b)} y & \longleftarrow & L(c) \\
 \downarrow & & \downarrow & & \downarrow \\
 L(a') & \longrightarrow & x' +_{L(b')} y' & \longleftarrow & L(c')
 \end{array}$$

Vertical composition is straightforward.

Tensoring uses binary coproducts in both  $A$  and  $X$ , and the fact that  $L: A \rightarrow X$  preserves these:

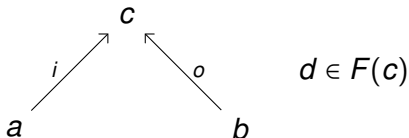
$$\begin{array}{ccc}
 L(a_1) \longrightarrow x_1 \longleftarrow L(b_1) & & L(a'_1) \longrightarrow x'_1 \longleftarrow L(b'_1) \\
 \downarrow & & \downarrow & & \downarrow \\
 L(a_2) \longrightarrow x_2 \longleftarrow L(b_2) & \otimes & L(a'_2) \longrightarrow x'_2 \longleftarrow L(b'_2)
 \end{array}$$

$$\begin{array}{ccc}
 L(a_1 + a'_1) \longrightarrow x_1 + x'_1 \longleftarrow L(b_1 + b'_1) \\
 = \quad \downarrow & & \downarrow & & \downarrow \\
 L(a_2 + a'_2) \longrightarrow x_2 + x'_2 \longleftarrow L(b_2 + b'_2)
 \end{array}$$

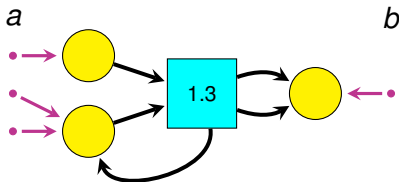


How do structured cospans compare to *decorated* cospans?

Given a suitable functor  $F: A \rightarrow \text{Set}$ , Fong defined an  **$F$ -decorated cospan** to be a pair



For example,  $F(c)$  could be the set of Petri nets with rates having  $c$  as their set of species.



The problem is that a functor  $F: A \rightarrow \text{Set}$  corresponds to a *discrete* opfibration  $R: X \rightarrow A$ . These are not general enough!

For example: the functor  $R: \text{Petri} \rightarrow \text{FinSet}$  sending any Petri net with rates to its underlying set of species is an opfibration, but not a discrete one.

The solution: use pseudofunctors  $F: A \rightarrow \mathbf{Cat}$ .

## Theorem (Kenny Courser, Christina Vasilakopoulou, JB)

Given a finitely cocomplete category  $A$  and a symmetric lax monoidal pseudofunctor  $F: A \rightarrow \mathbf{Cat}$ , there is a symmetric monoidal double category  $F\mathbf{Csp}$  where:

- ▶ an object is an object of  $A$
- ▶ a vertical 1-morphism is a morphism of  $A$
- ▶ a horizontal 1-cell is an  $F$ -decorated cospan:

$$a \xrightarrow{i} c \xleftarrow{o} b \quad d \in F(c)$$

- ▶ a 2-morphism is a commutative diagram and a triangle:

$$\begin{array}{ccccc}
 a & \xrightarrow{i} & c & \xleftarrow{o} & b \\
 f \downarrow & & h \downarrow & & \downarrow g \\
 a' & \xrightarrow{i'} & c' & \xleftarrow{o'} & b'
 \end{array}
 \qquad
 \begin{array}{ccc}
 & F(c) & \\
 1 & \xrightarrow{d} & \\
 & \Downarrow \iota & \\
 & F(c') & \\
 & \xleftarrow{d'} & 1
 \end{array}$$

$F(h)$  is the vertical arrow from  $F(c)$  to  $F(c')$ .

## Theorem (Kenny Courser, Christina Vasilakopoulou, JB)

Suppose  $A$  is finitely cocomplete,  $F: A \rightarrow \mathbf{Cat}$  is a symmetric lax monoidal pseudofunctor, and  $F$  factors through the 2-category  $\mathbf{Rex}$  of finitely cocomplete categories. Then the opfibration

$$R: \int F \rightarrow A$$

has a left adjoint

$$L: A \rightarrow \int F$$

and there is an isomorphism of symmetric monoidal double categories

$${}_L\mathbf{Csp}(\int F) \cong F\mathbf{Csp}.$$

So in this situation, which is common, structured cospans agree with the 'new improved' decorated cospans!