HURWITZ' THEOREM

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1. Introduction

In this article we describe several results based on the paper [Hur98] and which we will refer to as Hurwitz' theorem. There are several related results: the classification of real normed division algebras, the classification of complex composition algebras and the classification of real composition algebras. The classification of real division algebras is an open problem. Our interest is in composition algebras.

This article was originally conceived of as an account of the result of Rost in [Ros96] and the intention was to promote the use of diagrams. This result can also be found in [Cvi08, Chapter 16] although it is less explicit. This is still the core of this article. However I then added a prequel giving some background on composition algebras and vector product algebras. In particular I give brief accounts of two other proofs of Hurwitz' theorem. Both proofs can be found with more detail in [Bae02, §2]. I then added a sequel based on [DEF$^+$99, Volume 1]. The relevant Chapters are Chapter 6 in Notes on Spinors and Chapters 1 & 2 in Supersolutions. The aim here is to explain an algebraic property of the Poincare supergroup and Minkowski superspace and to show that this is satisfied only for Minkowski space of dimension 3, 4, 6 and 10. As a stepping stone to this I have also included a section with an unconventional view on triality.

In these notes we work over a field F whose characteristic is not two and for (25) we require that the characterisric is not two or three.

2. Algebras

In this section we discuss composition algebras and vector product algebras. The first result is that these have the same classification. Then we also give a classification of composition algebras based on Cayley-Dickson doubling and a classification of vector product algebras based on the structure theory of Clifford algebras. For these classifications we work over the field of complex numbers.

An *algebra* is a vector space A with a bilinear multiplication or equivalently a linear map $\mu\colon A \otimes A \to A$. An inner product on the vector space A is *associative* if

$$\langle x, y \times z\rangle = \langle x \times y, z\rangle$$

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for all $x, y, z \in A$.

If A has a unit 1 then the *imaginary part* is

$$\{a \in A | \langle a, 1\rangle = 0\}$$

The imaginary part has a multiplication given by

$$u \times v = uv - \langle u, v\rangle .1$$

Let A be an algebra. A *derivation* of A is a linear map $\partial\colon A \to A$ that satisfies the Leibniz rule

$$\partial(ab) = \partial(a)b + a\partial(b)$$

for all $a, b \in A$. Then $\mathfrak{Der}(A)$ is the Lie algebra of derivations of A.

If A has an associative inner product then we define $D\colon A \otimes A \to \mathfrak{Der}(A)$ by

$$\langle \partial, D(a, b)\rangle = \langle \partial(a), b\rangle$$

for all $a, b \in A$, $\partial \in \mathfrak{Der}(A)$.

2.1. **Composition algebras.** A *composition algebra* has a bilinear multiplication, a unit, a symmetric inner product and an anti-involution called conjugation. The first condition is

$$\overline{a} = -a + 2\langle a, 1\rangle .1$$

In particular the conjugation is determined by the functional $a \mapsto \langle a, 1\rangle$. The second condition is

$$a\overline{b} + b\overline{a} = \langle a, b\rangle .2$$

In particular the inner product is determined by the conjugation. This condition is usually written as $a\overline{a} = |a|.1$ whereas we have preferred the polarised version. The final condition is the key condition. This is usually written as

$$|ab| = |a|.|b|$$

The polarisation of this condition is

$$2\langle a, b\rangle \langle c, d\rangle = \langle ac, bd\rangle + \langle ad, bc\rangle$$

Let A be a composition algebra. Then we define the Cayley-Dickson double of A, $D(A)$. The underlying vector space is $A \oplus A$. The multiplication is given by

$$(a, b)(c, d) = (ac - d\overline{b}, \overline{a}d + cb)$$

The unit is $(1, 0)$. The conjugation is given by

$$\overline{(a, b)} = (\overline{a}, -b)$$

The inner product is given by

$$\langle (a, b), (c, d)\rangle = \langle a, c\rangle + \langle b, d\rangle$$

This is the structure needed for a composition algebra but the conditions may not be met. The precise result is that $D(A)$ is a composition algebra if and only if multiplication in A is associative.

The basic example of a composition algebra over a field K is the field K itself. The multiplication is the field multiplication and this has a unit. The conjugation map is the identity map. The inner product is then determined and is given by $\langle a, b\rangle = ab$.

Take the field to be the field of real numbers, $\mathbb{R}$. Then starting with $\mathbb{R}$ and applying the Cayley-Dickson doubling, the algebras that are constructed are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. The multiplication in $\mathbb{O}$ is not associative and the Cayley-Dickson double of $\mathbb{O}$ is not a composition algebra.

Taking the field to be the field of complex numbers, $\mathbb{C}$, gives the complex versions of these algebras. The complex version of $\mathbb{R}$ is $\mathbb{C}$, the complex version of $\mathbb{C}$ is $\mathbb{C} \oplus \mathbb{C}$ and the complex version of $\mathbb{H}$ is the algebra of 2×2 matrices with entries in $\mathbb{C}$.

Next we give an outline of the classification of composition algebras over the complex numbers. The result is that the four examples we have constructed using Cayley-Dickson doubling are the only possibilities.

The proof of this result is based on the observation that if A is a composition algebra and A_0 is a proper subalgebra then $D(A_0)$ is a subalgebra of A. To see this choose $\imath \in A$ such that $\langle a, \imath\rangle = 0$ for all $a \in A_0$ and such that $\langle \imath, \imath\rangle = 1$. Then we have an inclusion $D(A_0) \to A$ given by $(a, b) \mapsto a + \imath b$ for all $a, b \in A_0$.

Then the result follows from this. Let A be a composition algebra. Then by the previous observation $\mathbb{O}$ is not a proper subalgebra of A. Also $\mathbb{C}$ is a subalgebra of A by taking scalar mutliples of the unit. Hence it follows from the previous observation that A is isomorphic to one of the four composition algebras constructed by Cayley-Dickson doubling.

2.2. **Vector product algebras.** A *vector product algebra* has a symmetric inner product and an alternating bilinear multiplication. The inner product is required to be associative. These are the conditions

$$\langle x, y\rangle = \langle y, x\rangle \tag{1}$$

$$x \times y = -y \times x \tag{2}$$

$$\langle x, y \times z\rangle = \langle x \times y, z\rangle \tag{3}$$

The final condition for a vector product algebra is

$$(x \times y) \times x = \langle x, x\rangle\, y - \langle x, y\rangle\, x$$

The polarisation of this condition is

$$(x \times y) \times z + (z \times y) \times x = 2\,\langle x, z\rangle\, y - \langle x, y\rangle\, z - \langle z, y\rangle\, x \tag{4}$$

Let V be a vector product algebra. Then $L(\mathbf{u})$ for $\mathbf{u} \in V$ is defined by

$$L(\mathbf{u})\colon (a, \mathbf{w}) \mapsto (-\mathbf{u}.\mathbf{w}, a\mathbf{u} + \mathbf{u} \times \mathbf{w})$$

Then a direct calculation shows that

$$L(\mathbf{u})^2\colon (a,\mathbf{w}) \mapsto -(\mathbf{u}.\mathbf{u})(a,\mathbf{w})$$

or equivalently

$$L(\mathbf{u})L(\mathbf{v}) + L(\mathbf{v})L(\mathbf{u})\colon (a,\mathbf{w}) \mapsto -\langle \mathbf{u},\mathbf{v}\rangle (a,\mathbf{w})$$

These are the defining relations for the Clifford algebra of V, $\mathrm{Cliff}(V)$. This shows that $\mathrm{Cliff}(V)$ acts on $F \oplus V$. This is a strong constraint and the classification of vector product algebras follows from the structure theory of Clifford algebras.

2.3. **Motivation.** First we discuss the relationship between composition algebras and vector product algebras. A composition algebra determines a vector algebra by taking the imaginary part. A vector algebra determines a composition algebra by formally adjoining a unit. Let V be a vector product algebra over a field F. Then we define the structure maps of a composition algebra on the vector space $F \oplus V$ as follows.

The multiplication map is given by

$$(s,\mathbf{u})(t,\mathbf{v}) = (st - \mathbf{u}.\mathbf{v}, s\mathbf{v} + t\mathbf{u} + \mathbf{u}\times\mathbf{v})$$

The unit is $(1,\mathbf{0})$. The conjugation is given by

$$\overline{(s,\mathbf{v})} = (s,-\mathbf{v})$$

The inner product is given by

$$\langle (s,\mathbf{u}) | (t,\mathbf{v})\rangle = st + \mathbf{u}.\mathbf{v}$$

Then this structure satisfies the conditions for a composition algebra. Furthermore these are inverse operations. Taking the imaginary part of this composition algebra recovers the vector product algebra. Also applying this construction to the imaginary part of a composition algebra recovers the composition algebra.

Also the composition algebra and the vector product algebra have the same derivation Lie algebra.

Next we discuss two situations in the study of the exceptional simple Lie algebras in which vector product algebras arise. Although it may not be apparent these are closely related. One situation is taken from [Cvi08, (16.11)]. Let A be an algebra with an associative inner product whose multiplication is anti-symmetric and whose inner product is symmetric. Assume that A considered as a representation of the derivation Lie algebra is irreducible. Consider the endomorphism algebra of $A \otimes A$ considered as a representation of the derivation Lie algebra of A. Assume further that the dimension of this endomorphism algebra is at most five. Then there are just two possibilities; either A is a Lie algebra or A is a vector product algebra.

The second situation is the Freudenthal-Tits construction of the exceptional simple Lie algebras given in [Tit66]. A careful account of this is given in [BS03].

Let H^+ be the algebra of 3×3 Hermitian matrices with entries in the octonions. This has a symmetric inner product and a symmetric multiplication. These are given by

$$\langle A, B\rangle = \operatorname{tr}(AB) \qquad A \circ B = \frac{1}{2}(AB + BA)$$

Here the product AB is given by matrix multiplication and is not an element of H^+. This is an associative inner product.

Let H_{26} be the imaginary part of H^+. The derivation algebra of H_{26} is the exceptional simple Lie algebra $\mathfrak{f}_4$, [CS50].

Now let V be an algebra and denote the derivation algebra by $\mathfrak{Der}(V)$. Then we start with the vector space

$$\mathfrak{g} = \mathfrak{f}_4 \oplus \mathfrak{Der}(V) \oplus H_{26} \otimes V$$

Then the aim is to construct a Lie bracket so that $\mathfrak{g}$ is a simple Lie algebra. Here $\mathfrak{f}_4 \oplus \mathfrak{Der}(V)$ is already a Lie algebra and this Lie algebra acts on $H_{26} \otimes V$. Hence to define the Lie bracket on $\mathfrak{g}$ it remains to construct an anti-symmetric map of representations of $\mathfrak{f}_4 \oplus \mathfrak{Der}(V)$

$$(H_{26} \otimes V) \otimes (H_{26} \otimes V) \to \mathfrak{g}$$

and to check the Jacobi identity. There is also a homomorphism of representations of $\mathfrak{Der}(V)$, $D\colon V \otimes V \to \mathfrak{Der}(V)$ and this is anti-symmetric. Then we extend the Lie bracket in the only way possible

$$[A \otimes \mathbf{u}, B \otimes \mathbf{v}] = \langle \mathbf{u}, \mathbf{v}\rangle\, D(A, B) + \langle A, B\rangle\, D(\mathbf{u}, \mathbf{v}) + (A * B) \otimes (\mathbf{u} \times \mathbf{v})$$

where $A * B$ is the multiplication on H. This multiplication is anti-symmetric and the Jacobi identity is satisfied if and only if V is a vector product algebra.

The examples are

$$\begin{aligned}
\mathfrak{f}_4 &= \mathfrak{f}_4 \oplus 0 \oplus H_{26} \otimes V_0 \\
\mathfrak{e}_6 &= \mathfrak{f}_4 \oplus 0 \oplus H_{26} \otimes V_1 \\
\mathfrak{e}_7 &= \mathfrak{f}_4 \oplus \mathfrak{sl}_2 \oplus H_{26} \otimes V_3 \\
\mathfrak{e}_8 &= \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus H_{26} \otimes V_7
\end{aligned}$$

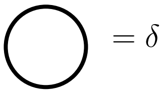
It follows from the classification of vector product algebras that these are the only examples.

3. Hurwitz' theorem

First we write the tensor equations in diagram notation. The bilinear multiplication is represented by a trivalent vertex. The conditions that the symmetric bilinear form is non-degenerate and that the multiplication is associative imply that the tensor only depends on the isotopy class of the planar diagram.

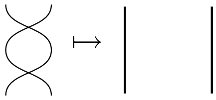
Then we write δ for the dimension of the vector algebra. This is written in diagram notation as

(5)

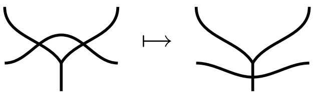


Our preference is to work with planar diagrams and to regard the crossing as part of the data. It is more usual to work with abstract graphs (with a cyclic ordering of the edges at a vertex) and so to regard crossing as part of the formalism. From this point of view we have the following relations.

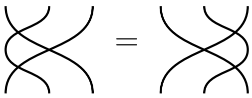
(6)



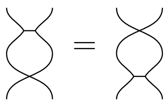
(7)



(8)

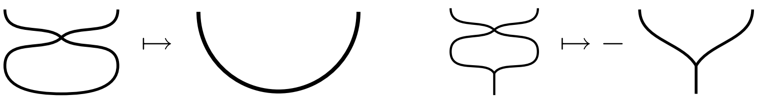


Note these have respectively four, five and six boundary points. The first two are taken as reduction rules. In proof of Hurwitz' theorem we only consider diagrams with four boundary points and so this argument only uses the first reduction rule. When we study the case $\delta = 7$ then we consider diagrams with five boundary points and we use the second rule. We will not consider diagrams with six boundary points and so we will not make use of the third relation. There is one other relation with four boundary points which we will use. This is the relation



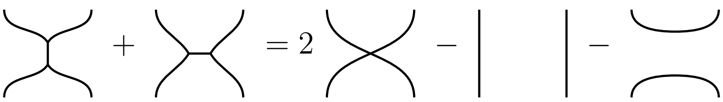
Then the symmetries are written in diagram notation as follows. These are taken to be reduction rules.

(9)

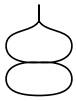


Then the fundamental identity is written in diagram notation as

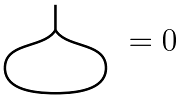
(10)



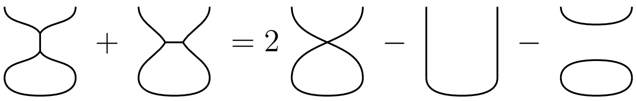
Then we want to deduce some consequences of these relations. Firstly we have the overlap



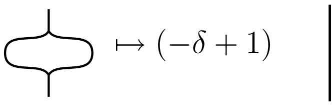
This can be simplified using either of the reduction rules (9). Since the characteristic is not two this gives the reduction rule

(11) 

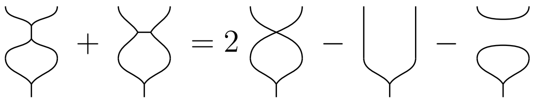
Next we find some consequences of the fundamental identity (10). The first consequence is



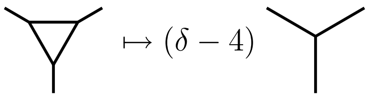
Simplifying this using (5), (9), (11) gives the reduction rule

(12) 

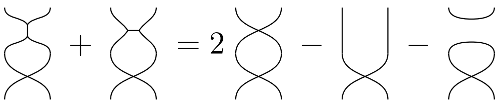
The second consequence is



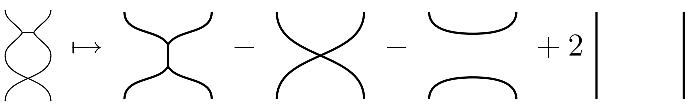
Simplifying this using (9), (11), (12) gives the reduction rule

(13) 

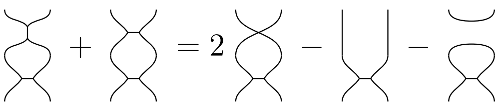
The third consequence is



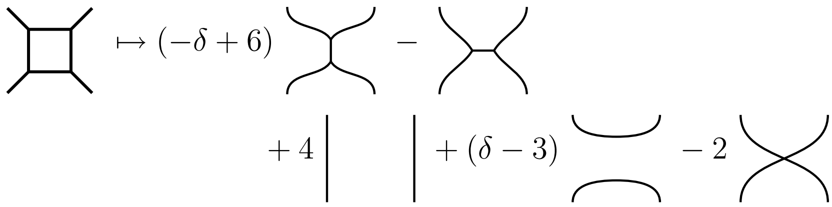
Simplifying this using (9), (13) gives the reduction rule

(14) 

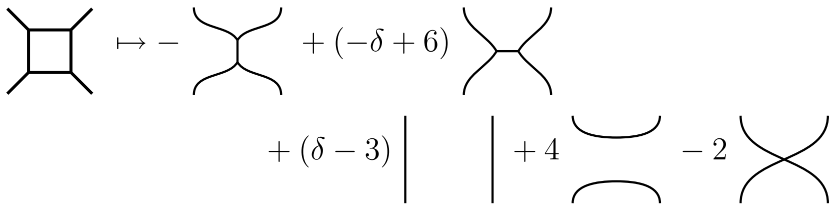
The fourth consequence is



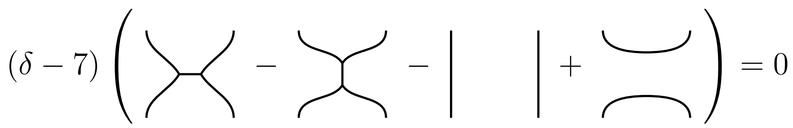
Simplifying this using (12), (13), (14) gives the reduction rule

(15) 

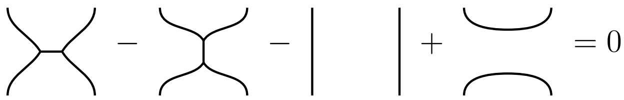
Now comes the punchline. The relation (15) can be rotated through a quarter of a revolution to give the reduction rule

(16) 

Then taking the difference between (15) and (16) gives the relation

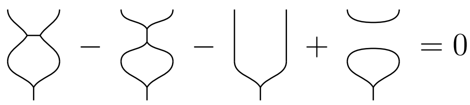
(17) 

This gives two possibilities either $\delta = 7$ or we have the relation

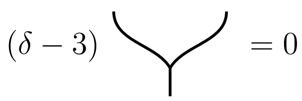
(18) 

The relation (18) is equivalent to the condition that the composition algebra associated to the vector algebra is associative. Therefore if we state this in terms of composition algebras we have that either the dimension is 8 or the algebra is associative.

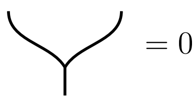
The relation (18) has the following consequence



Simplifying this gives a relation which can be compared with [Cvi08, (16.17)]

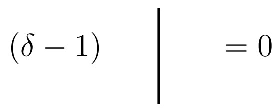


This gives two possibilities either $\delta = 3$ or we have the relation

(19) 

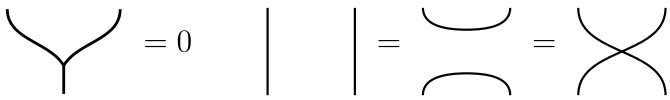
The relation (19) is equivalent to the condition that the composition algebra associated to the vector algebra is commutative.

This relation has the following consequence



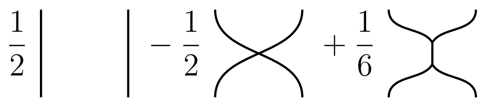
This gives two possibilities either $\delta = 1$ or all diagrams are 0.

In conclusion this gives four possibilities. The completely degenerate case is $\delta = 0$ and all diagrams are 0. The second case which is still degenerate is that $\delta = 1$ and the following relations hold

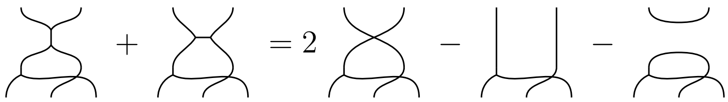


The third case is $\delta = 3$ and the relation (18) holds. The fourth case is $\delta = 7$.

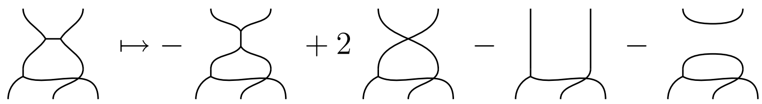
3.1. **Reduction rules.** Finally we derive some further reduction rules for the case $\delta = 7$. Our aim is to deduce the finite confluent set of reduction rules in [Kup94] and [Kup96]. An alternative approach is given in [Cvi08, Chapter 16]. The starting point for this approach is that the projection onto the derivation Lie algebra is the following idempotent in $\mathrm{End}(V \otimes V)$.



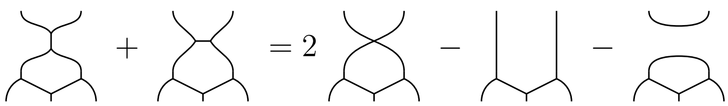
The relation (10) has the following consequence



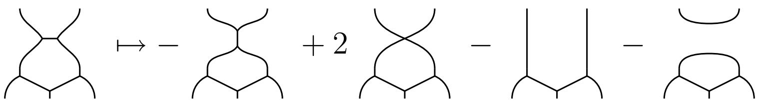
This simplifies to give the reduction rule

(20) 

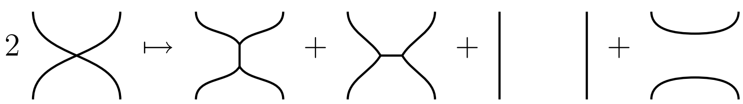
The relation (10) has the following further consequence



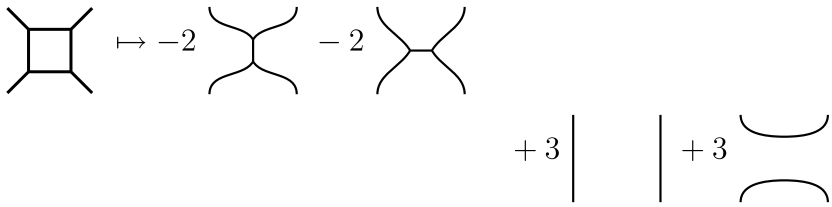
This simplifies to give the reduction rule



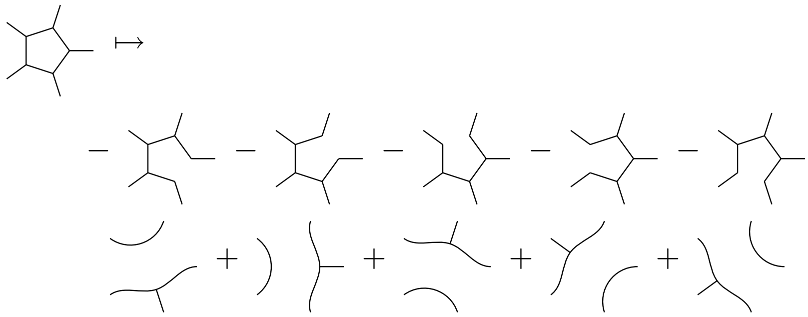
The fundamental relation (10) gives the reduction rule

(21) 

Then we substitute $\delta = 7$ in the reduction rules (5), (12), (13) The reduction rules (15) and (16) give the reduction rule

(22) 

Now simplify the reduction rule (20) to obtain the reduction rule



Then we arrive at the reduction rules (with $q = 1$) in [Kup94] and [Kup96]. In particular this set of rewrite rules is confluent. Furthermore any closed diagram reduces to a scalar by a curvature argument. Also given any k the set of irreducible diagrams with k boundary points is finite; this is an application of the isoperimetric inequality. A bijection between these finite sets and certain lattice walks is given in [Wes07].

4. Triality

This is a further application of Hurwitz' theorem. We have seen that if V is a vector product algebra then $\mathrm{Cliff}(V)$ acts on $F \oplus V$. In this section and the next we promote this to higher dimensions.

Let $\mathbb{K}$ be the composition algebra associated to V. Then we can identify $F \oplus V$ with $\mathbb{K}$ and $\mathrm{Cliff}(V)$ with $\mathrm{Cliff}^{\mathrm{even}}(\mathbb{K})$. Then from the action of $\mathrm{Cliff}(V)$ on $F \oplus V$ we can form an action of $\mathrm{Cliff}^{\mathrm{even}}(\mathbb{K})$ on $\mathbb{K}$. This is usually expressed as saying that $\mathbb{K}$ is a spin representation of $\mathrm{Spin}(\mathbb{K})$. Next we promote this to an action of $\mathrm{Cliff}(\mathbb{K})$ on $\mathbb{K} \oplus \mathbb{K}$ which makes this explicit.

For $a \in \mathbb{K}$ define $\rho(a) \in \mathrm{End}(\mathbb{K} \oplus \mathbb{K})$ by

$$\rho(a)\colon (x, y) \mapsto (\overline{ay}, \overline{xa})$$

for all $(x, y) \in \mathbb{K} \oplus \mathbb{K}$. Then to check the defining relations of the Clifford algebra we use the fact that if $x, y \in \mathbb{K}$ then the subalgebra generated by $\{1, x, y, \overline{x}, \overline{y}\}$ is associative. This is equivalent to the statement that a composition algebra is alternative. Then we calculate

$$\begin{aligned}\rho(a)^2\colon (x,y) &\mapsto (\overline{a}(ax), (ya)\overline{a}) \\ &= ((\overline{a}a)x, y(a\overline{a})) \\ &= |a|(x,y)\end{aligned}$$

This example has more structure. There are three vector spaces with non-degenerate inner products, V_1, V_2, V_3 and $V_1 \otimes V_2 \otimes V_3 \to F$. If (i,j,k) is a permutation of $(1,2,3)$ then this gives a map $V_i \otimes V_j \to V_k$. The condition is that for any permutation (i,j,k) the two maps

$$V_i \otimes V_j \to V_k \qquad V_i \otimes V_k \to V_j$$

define an action of $\mathrm{Cliff}(V_i)$ on $V_j \oplus V_k$. We will refer to this structure as a *triality*.

In the previous example all three vector spaces are identified with $\mathbb{K}$. The fundamental tensor is the trilinear form given by

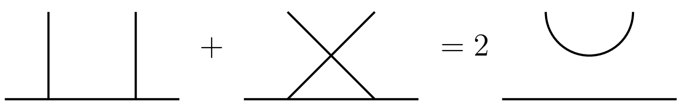
$$x \otimes y \otimes z \mapsto \langle \overline{x}\,\overline{y}, \overline{z}\rangle = \langle xy, z\rangle$$

This is invariant under cyclic permutations of (x,y,z).

Then we can show that this constructions gives all examples. Choose $e_1 \in V_1$ and $e_2 \in V_2$. Then all three spaces are identified. Take $e_3 = e_1e_2$. This gives $\mathbb{K}$ with the inner product, $1 \in \mathbb{K}$ and the map $x \otimes y \mapsto \overline{xy}$. This structure is equivalent to the structure of a composition algebra.

The diagrams for a triality are straightforward. There are three possible colours each edge. We consider trivalent graphs coloured so that for each vertex all three edges at the vertex have different colours. For the relations we take the one relation

(23)



with all possible colourings. There are three different ways to colour this relation.

All three representations have the same dimension which we denote by δ. Then the challenge is to deduce directly from these relations that there is a finite set of possible values for δ which includes $\{0,1,2,4,8\}$. The intention is not just to deduce this result but also to derive the additional relations that hold in each case.

5. Super gravity

In this section we stay with the same theme and move up another dimension. Let H be a two dimensional hyperbolic space. Then for any V we have that $\mathrm{Cliff}(V \oplus H) \cong M_2(\mathrm{Cliff}(V))$. Then this gives that $\mathrm{Cliff}(\mathbb{K} \oplus H)$ acts on the direct sum of four copies of $\mathbb{K}$ and $\mathrm{Cliff}^{\mathrm{even}}(\mathbb{K} \oplus$

H) acts on $\mathbb{K} \oplus \mathbb{K}$. Equivalently $\mathbb{K} \oplus \mathbb{K}$ is a spin representation of $\mathrm{Spin}(\mathbb{K} \oplus H)$. This is sometimes expressed as

$$\mathrm{Spin}(\mathbb{K} \oplus H) \cong \mathrm{SL}_2(\mathbb{K})$$

but this needs some interpretation for $\mathbb{K} = \mathbb{O}$.

Fix a vector space V with a non-degenerate inner product. Then a *system* consists of a representation of $\mathrm{Spin}(V)$, S and linear maps

$$\Gamma\colon S^* \otimes S^* \to V \qquad \Gamma\colon S \otimes S \to V$$

These maps are both required to be symmetric and to be maps of representations of $\mathrm{Spin}(V)$. There is a further condition which in index notation is written

$$\Gamma^{\mu}_{ab}\Gamma^{\nu bc} + \Gamma^{\nu}_{ab}\Gamma^{\mu bc} = 2g^{\mu\nu}\delta^c_a$$

Alternatively, there are linear maps

$$\rho\colon V \otimes S^* \to S \qquad \rho\colon V \otimes S \to S^*$$

such that $\rho \oplus \rho\colon V \to \mathrm{End}(S^* \oplus S)$ satisfies the Clifford relations. The equivalence between these definitions is given by

$$\langle \rho(v \otimes s), t\rangle = (\Gamma(s \otimes t), v) \tag{24}$$

for all $s, t \in S^*$ and all $v \in V$.

However there is an additional condition which is needed to construct supersymmetric Lagrangians. This is the condition that the following quartic invariant vanishes identically

$$s \mapsto |\Gamma(s, s)|$$

For further interpretations of this see [DEF$^+$99, (6.14),(6.69),(6.72)]. Polarising and taking into account the symmetries this is equivalent to

$$(\Gamma(s,t), \Gamma(u,v)) + (\Gamma(s,u), \Gamma(t,v)) + (\Gamma(s,v), \Gamma(t,u)) = 0 \tag{25}$$

A system that satisfies this condition is *special*.

Next we construct an example from a composition algebra. Let H be hyperbolic space with basis $\{\alpha, \beta\}$ and quadratic form $\lambda\alpha + \mu\beta \mapsto \lambda\mu$. Then the Clifford algebra is the algebra of 2×2 matrices. An isomorphism is given by

$$\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \beta \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then by the theory of Clifford algebras there is a representation of $\mathrm{Cliff}(\mathbb{K} \oplus H)$ on $\mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$. This is given explicitly by

$$\begin{aligned}
\rho(a)&\colon (u, v, x, y) \mapsto (\overline{a}\,\overline{v}, \overline{u}\,\overline{a}, -\overline{a}\,\overline{y}, -\overline{x}\,\overline{a}) \\
\rho(\alpha)&\colon (u, v, x, y) \mapsto (x, y, 0, 0) \\
\rho(\beta)&\colon (u, v, x, y) \mapsto (0, 0, u, v)
\end{aligned}$$

The decomposition as a representation of $\mathrm{Cliff}^{\mathrm{even}}(V)$ is

$$(u, v, x, y) = (u, y) + (v, x)$$

Then the definition of a system requires that these are dual representations. A non-degenerate pairing is given by

$$(u, y) \otimes (v, x) \mapsto -\langle u, \overline{x}\rangle + \langle v, \overline{y}\rangle$$

The proof that this pairing is a map of representations is left to the reader.

This shows that we have constructed a system. In the alternative formulation we have a symmetric morphism of representations of $\mathrm{Spin}(\mathbb{K} \oplus H)$,

$$\Gamma\colon (\mathbb{K} \oplus \mathbb{K}) \otimes (\mathbb{K} \oplus \mathbb{K}) \to \mathbb{K} \oplus H$$

This is given by

$$\Gamma\colon (u, y) \otimes (u', y') \mapsto (u'y + y'u) - \langle u, u'\rangle\, \alpha + \langle y, y'\rangle\, \beta$$

Then we check condition (24) which determines Γ uniquely. This follows from the calculations

$$\begin{aligned}
\langle \rho(a \otimes (u, y)), (u', y')\rangle &= \langle \overline{u}\,\overline{a}, \overline{y}'\rangle + \langle u', ya\rangle \\
\langle \rho(\alpha \otimes (u, y)), (u', y')\rangle &= \langle y, \overline{y}'\rangle \\
\langle \rho(\beta \otimes (u, y)), (u', y')\rangle &= -\langle u', \overline{u}\rangle
\end{aligned}$$

Then we have $(u, y) \otimes (u, y) \mapsto uy + yu - |u|\alpha + |y|\beta$ and this has norm $|uy| - |u|.|y| = 0$. This shows that this system is special.

Next we show that there is a correspondence between trialities for W and special systems for $V = W \oplus H$. Assume we have a special system for $V = W \oplus H$. Then we have a representation of $\mathrm{Cliff}(V)$. There is a decomposition of 1 into two orthogonal idempotents,

$$1 = \rho(\alpha)\rho(\beta) + \rho(\beta)\rho(\alpha)$$

These idempotents commute with $\mathrm{Cliff}(W)$ and so as a representation of $\mathrm{Cliff}(W)$ this decomposes. Each of these can be decomposed as a representation of $\mathrm{Cliff}^{\mathrm{even}}(W)$. This means the representation can be written

$$\begin{aligned}
\rho(a)&\colon (u, v, x, y) \mapsto (\rho(a \otimes v), \overline{\rho}(a \otimes u), -\rho(a \otimes y), -\overline{\rho}(a \otimes x)) \\
\rho(\alpha)&\colon (u, v, x, y) \mapsto (x, y, 0, 0) \\
\rho(\beta)&\colon (u, v, x, y) \mapsto (0, 0, u, v)
\end{aligned}$$

The decomposition as a representation of $\mathrm{Cliff}^{\mathrm{even}}(V)$ is

$$(u, v, x, y) = (u, y) + (v, x)$$

Then the definition of a system requires that these are dual representations. A non-degenerate pairing is given by

$$(u, y) \otimes (v, x) \mapsto -\langle u, x\rangle + \langle v, y\rangle$$

This is required to be a morphism of representations. At this stage we have three vector spaces (W, S^+, S^-) each with a non-degenerate inner product and $S^+ \oplus S^-$ is a representation of $\mathrm{Cliff}(W)$.

Next we discuss Γ. This is given by

$$\Gamma\colon (u,y)\otimes(u',y') \mapsto \gamma(u',y)+\gamma(y',u)-\langle u,u'\rangle\,\alpha+\langle y,y'\rangle\,\beta$$

Then the condition (24) determines Γ. This is equivalent to the following conditions which determine γ, $\overline{\gamma}$.

$$\langle\rho(a\otimes u),v\rangle=(a,\gamma(u\otimes v)) \qquad \langle\overline{\rho}(a\otimes v),u\rangle=(a,\overline{\gamma}(v\otimes u)) \tag{26}$$

This follows from the calculations

$$\begin{aligned}\langle\rho(a\otimes(u,y)),(u',y')\rangle&=\langle\overline{u}\,\overline{a},\overline{y}'\rangle+\langle u',ya\rangle\\ \langle\rho(\alpha\otimes(u,y)),(u',y')\rangle&=\langle y,\overline{y}'\rangle\\ \langle\rho(\beta\otimes(u,y)),(u',y')\rangle&=-\langle u',\overline{u}\rangle\end{aligned}$$

Then it remains to show that the condition (25) for Γ is equivalent to the Clifford relation for $\gamma\oplus\overline{\gamma}$. From the definition

$$\Gamma((u,y)\otimes(u,y))=\gamma(u,y)+\overline{\gamma}(y,u)-|u|\alpha+|y|\beta$$

Then the condition that this has norm zero is equivalent to

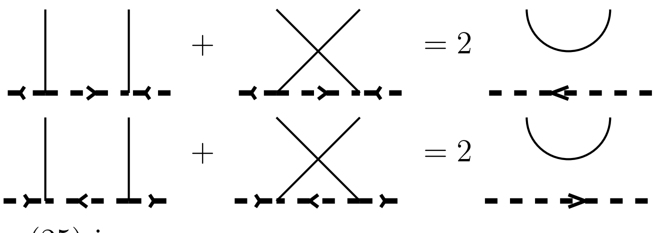
$$|\gamma(u,y)+\overline{\gamma}(y,u)|=|u|.|v|$$

Now consider $\gamma(u,y)\overline{\gamma}(y,u)+\overline{\gamma}(y,u)\gamma(u,y)$. This is a morphism of representations $V\to V$ and V is ireducible. So by Schur's lemma this is a scalar matrix. Hence (huh!) this is equivalent to condition (25).

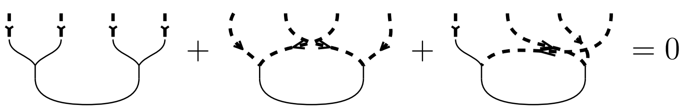
Again we can express this in terms of diagrams. There are two types of edges. An undirected edge for V and a directed edge for S. Then there are two trivalent vertices

(27) 

Then the relations these satisfy are the Clifford relations and the relation (25). The Clifford relations are



The relation (25) is

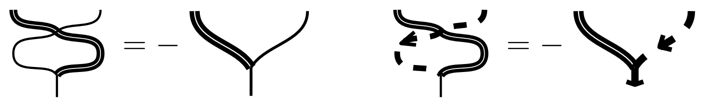
(28) 

However these are not all the relations. There is also the requirement that S is a representation of $\mathrm{Spin}(V)$ and that these are morphisms of representations. The way to understand these relations is to introduce a third type of edge for the adjoint representation, $\mathfrak{g}$. Next we write the relations using this edge and then we eliminate this edge using the fact that the adjoint representation is the exterior square of V. Here we will not draw the diagrams that say that $\mathfrak{g}$ is a Lie algebra and that V is a representation as these relations are satisfied.

There are three trivalent vertices involving the new edge. These give the action of $\mathfrak{g}$ on V, S, S^*, $\mathfrak{g}$. The four vertices are

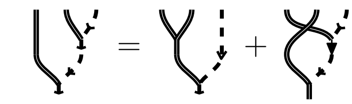
(29) 

The first relations are the antisymmetry relations

(30) 

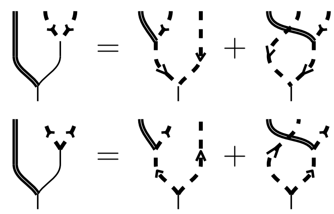
This implies that V is a self-dual representation and that the pairing between S and S^* is a morphism of representations.

Next we have the relations that say that V, S, S^*, $\mathfrak{g}$ are representations. This relation for S is

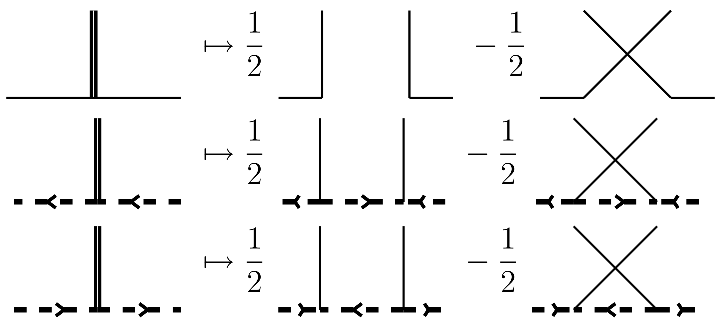
(31) 

The relation for S^* is obtained by reversing the directions. This is a consequence of (30) and (31).

Finally we have the relation that the maps (27) are morphisms of representations. These are the relations



Now we have obtained the relations the edge for the adjoint representation can be eliminated. Three of the four vertices in (29) are eliminated using



Again the challenge is to understand the consequences of these relations. It is reasonable to expect that there is a fit set of possible values for δ which includes $\{0, 2, 3, 4, 6, 10\}$. As before the intention is not just to prove this result but also to find additional relations which hold in each case.

If we omit the special relation (28) then this can be realised for an infinite set of values for δ. Therefore this makes sense for δ an indeterminate. Then from these relations we can formally construct a Lorentz supergroup for a spacetime of dimension δ. This is analogous to the way the Brauer category formally constructs an orthogonal group for space of dimension δ.

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