Open Petri Nets

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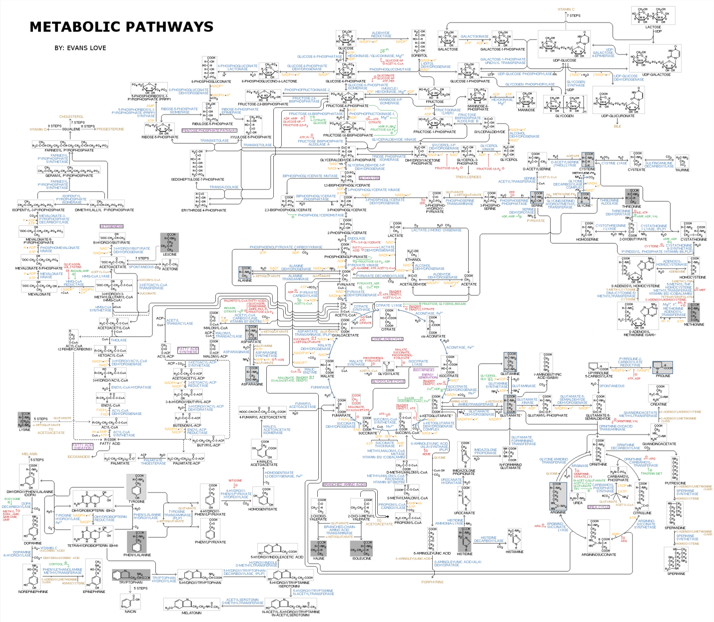
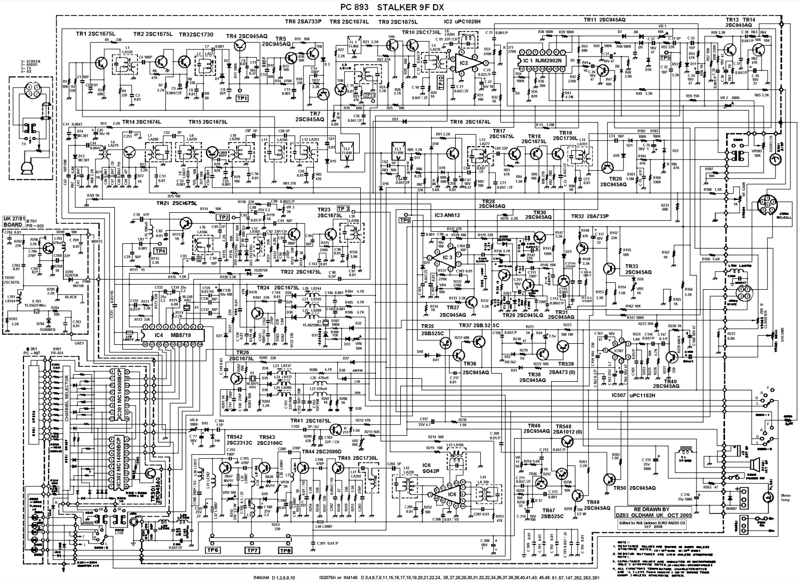
Complex networks are everywhere.

The Metabolic System



An Electrical Circuit



Divide and Conquer

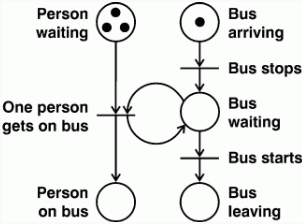
Lawvere's *Functorial Semantics of Algebraic Theories* describes notation which people use for systems and quantitative meaning which people attach to this notation. The former is called syntax and the latter is called semantics.

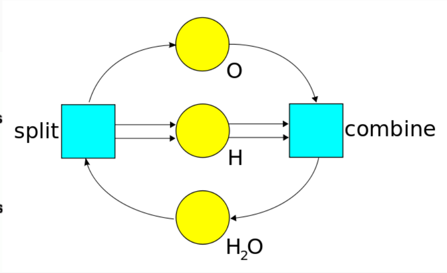
$$\text{Syntax} \xrightarrow{\text{Functor}} \text{Semantics}$$

Definition: A Petri net is a pair of functions of the following form

$$T \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{N}[S]$$

where $\mathbb{N}\colon \mathsf{Set} \to \mathsf{Set}$ is the free commutative monoid monad which sends a set X to $\mathbb{N}[X]$ the free commutative monoid on X.





Petri nets are useful because they are a general language for representing processes which can be performed in sequence and in parallel. This can be summarized with following slogan:

Petri nets present free symmetric monoidal categories

Commutative monoidal categories

The devil is in the details.

Because Petri nets have a free commutative monoid of species, they more naturally present commutative monoidal categories. These are commutative monoid objects in Cat.

$$\mathrm{Mor}\, C \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}\, C$$

Maclane's coherence theorem doesn't apply.

In *Petri Nets are Monoids*, Messeguer and Montanari introduced the idea. We use a variant of this: starting with a functor

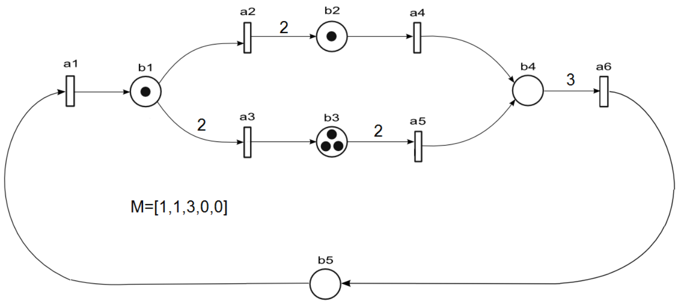
$$\hat{F} \colon \mathsf{Petri} \to \mathsf{CMC}$$

we restrict to the essential image of $\hat{F}$ to get an adjoint equivalence

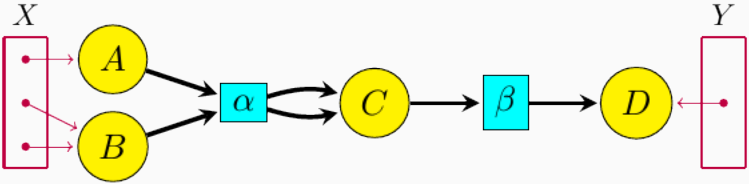
$$\begin{tikzcd} \mathsf{Petri} \arrow[r, bend left, "F"] & \mathsf{CatPetri} \arrow[l, bend left, "U"] \end{tikzcd}$$

For a Petri net P, the commutative monoidal category FP has

- objects given by possible markings of P with tokens
- morphisms given by ways that these markings can be shuffled around using sequences of transitions



Petri nets generate free symmetric monoidal categories but also they *are* morphisms in a symmetric monoidal category.
is a Petri net equipped with inputs and outputs.



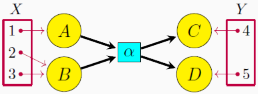
We can think of this as a morphism between two sets.

Definition: An open Petri net $P\colon X \to Y$ is a cospan in Petri of the form

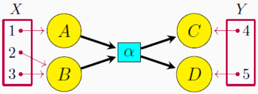
$$
\begin{tikzcd}
& P & \\
LX \arrow[ur] & & LY \arrow[ul]
\end{tikzcd}
$$

Where LX and LY are the Petri nets with no transitions and X or Y as their set of places.

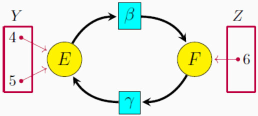
Given an open Petri net from X to Y



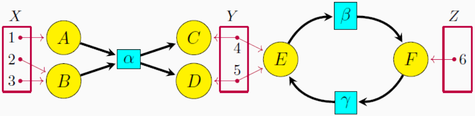
Given an open Petri net from X to Y



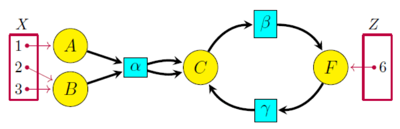
and an open Petri net from Y to Z



To compose them first you place them end to end



and identify the places which come from the same element of Y



This is formalized using pushouts

$$
\begin{tikzcd}
 & & P +_{LY} Q & & \\
 & P \arrow[ur] & & Q \arrow[ul] & \\
LX \arrow[ur] & & LY \arrow[ul] \arrow[ur] & & LZ \arrow[ul]
\end{tikzcd}
$$

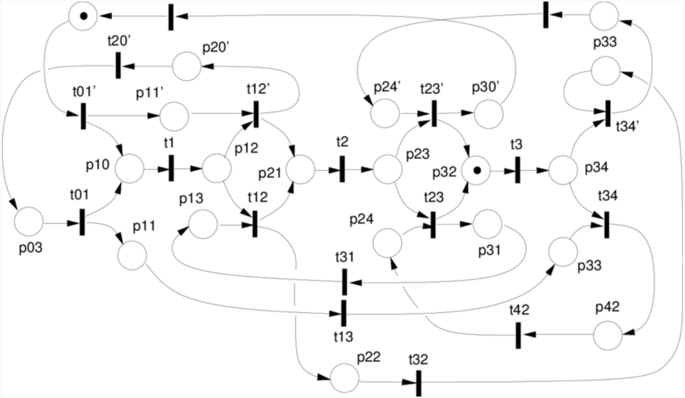
which takes the disjoint union and mods out by the equivalence relation described above.

Theorem. (John Baez, JM) There is a symmetric monoidal category Open(Petri) where

- objects are sets $X, Y, \ldots$
- morphisms are (equivalence classes of) open Petri nets $P \colon X \to Y$,
- composition is given by pushout and,
- monoidal product is given by coproduct on sets and pointwise coproduct on morphisms.

Open(Petri) is more naturally a bicategory or double category because composition using pushout is not strictly associative. To make this into a category we need to define open Petri nets *up to isomorphism*.

The coherence laws of a symmetric monoidal category ensure that complex networks can be built in a coherent way using open Petri nets.



Reachability Semantics

Some Motivation

The reachability problem asks: given two markings m and n, is there a sequence of transitions which can fire starting at m and ending in n. Reachability is good for formal verification but also and EXSPACE algorithm can be reduced to Petri net reachability in polynomial time.

- In 1984 Mayr showed that the reachability problem was decidable but ...

- In 1984 Mayr showed that the reachability problem was decidable but ...
- In 2018 the complexity was shown to be greater than *any primitive recursive function*

The analogue of reachability for Turing machines is the halting problem so Petri nets are right on the edge of being Turing complete. This puts them in the sweet spot of expressiveness.

Open Petri nets are a natural setting to discuss reachability.

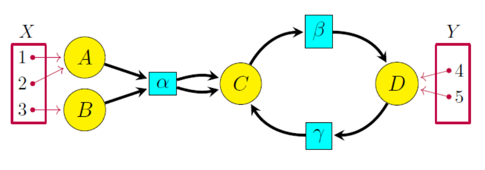
Definition: For an open Petri net $P \colon X \to Y$ it's reachability relation

$$\blacksquare(P) \subseteq \mathbb{N}[X] \times \mathbb{N}[Y]$$

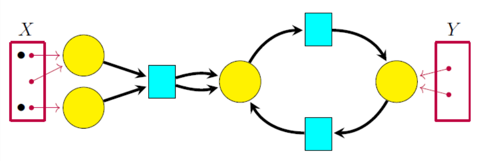
contains an element (x, y) if y is reachable from x.

Example

Let $P\colon X \to Y$ be the following open Petri net:

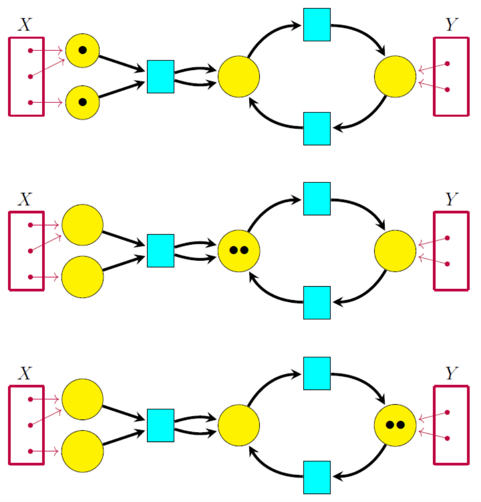


then we can equip X with an initial marking,



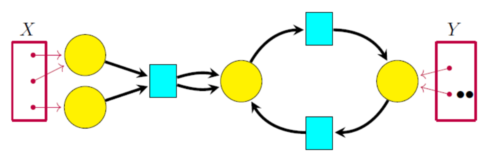
Example

shuffle this marking around using the transitions,



Example

and pop the tokens back into Y leaving no tokens behind.



This can all be made categorical.

Proposition: (folklore) For a Petri net P, a marking n is reachable from m if and only if there is a morphism $f\colon m \to n$ in the free commutative monoidal category FP.

Definition: For a cospan of categories

$$\begin{tikzcd} & C & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "j"'] \end{tikzcd}$$

its reachability relation

$$\pi_0(C) \subseteq \mathrm{Ob}\, X \times \mathrm{Ob}\, Y$$

contains an element (x, y) if there is a morphism $f\colon i(x) \to j(y)$ in C.

So to get the reachability relation of an open Petri net

$$\begin{tikzcd} & P & \\ LX \arrow[ur] & & LY \arrow[ul] \end{tikzcd}$$

we apply the semantics functor $F\colon \mathsf{Petri} \to \mathsf{CatPetri}$

$$\begin{tikzcd} & FP & \\ FLX \arrow[ur] & & FLY \arrow[ul] \end{tikzcd}$$

and take the reachability of this.

The Upshot

This process is laxly functorial!
Let Rel be the 2-category where

- objects are sets X, Y, ...
- morphisms are relations $R \subseteq X \times Y$ and
- a 2-morphism from $R \subseteq X \times Y$ to $R' \subseteq X \times Y$ is an inclusion

$$R \subseteq R'$$

And Open(Petri) can be upgraded to a 2-category where the 2-morphisms can only be the identity.

We get the following diagram

$$\mathrm{Open}(\mathrm{Petri}) \xrightarrow{\mathrm{Open}(F)} \mathrm{Open}(\mathrm{CatPetri}) \xrightarrow{\pi_0} \mathbf{Rel}$$

where the application of F gives the first arrow and the reachability of categories gives the second arrow.

Theorem: (Baez, JM) There is a lax symmetric monoidal 2-functor

$$\blacksquare \colon \mathsf{Open}(\mathsf{Petri}) \to \mathsf{Rel}$$

which makes the following assignment on morphisms

$$\begin{tikzcd} LX \arrow[d, "i"] \\ P \\ LY \arrow[u, "j"'] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathbb{N}[X] \arrow[dd, "\blacksquare(P)"] \\ \\ \mathbb{N}[Y] \end{tikzcd}$$

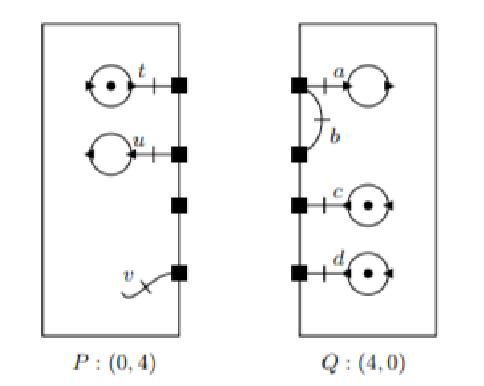
This result describes the extent to which we can reason about reachability in compositional way.

Laxness means that we have an inclusion

$$\blacksquare(P) \circ \blacksquare(Q) \subseteq \blacksquare(P \circ Q)$$

which allows us to break up reachability problems into smaller subproblems.

Penrose, Statebox, and formal verification.



Conclusion

Petri nets are inherently categorical. Grothendieck said

> *The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!*

References

- José Meseguer and Ugo Montanari (1990) Petri nets are Monoids
- Ernst Mayr (1981) Persistence of vector replacement systems is decidable
- Wojciech Czerwinski, Slawomir Lasota, Ranko Lazic, Jérôme Leroux and Filip Mazowiecki (2018) The reachability problem for Petri nets is not elementary. Available at arXiv:1809.07115.
- John Baez, JM (2018) Open Petri Nets. Available at arXiv:1808.05415