Structured cospans

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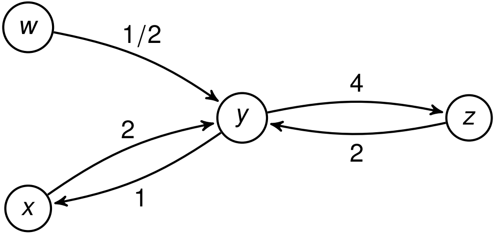
University of California, Riverside

May 22, 2019

Networks can very often be viewed as sets equipped or 'decorated' with extra structure...

For example,



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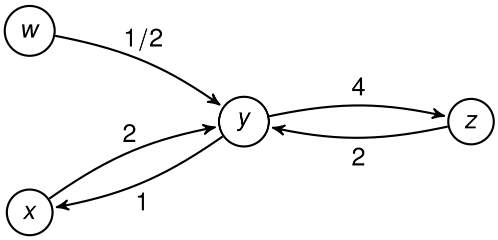

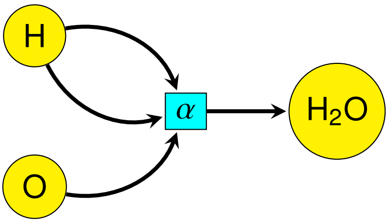

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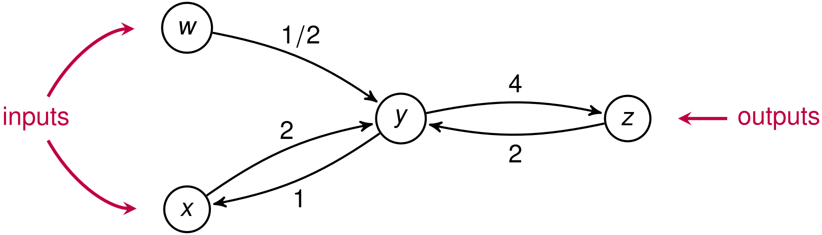

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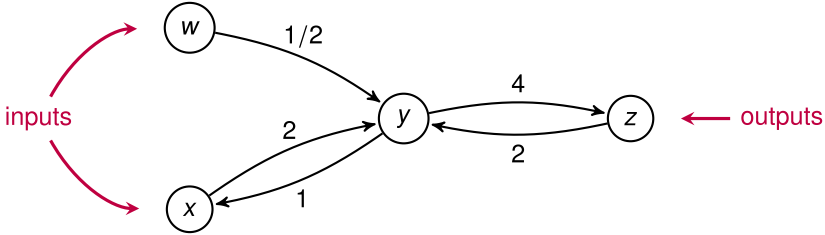
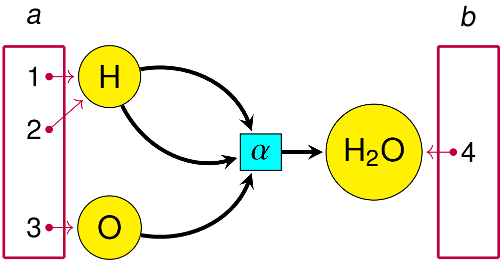
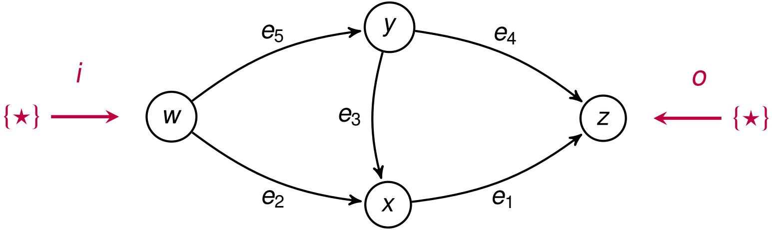




An easy example to have in mind is the example of open graphs:



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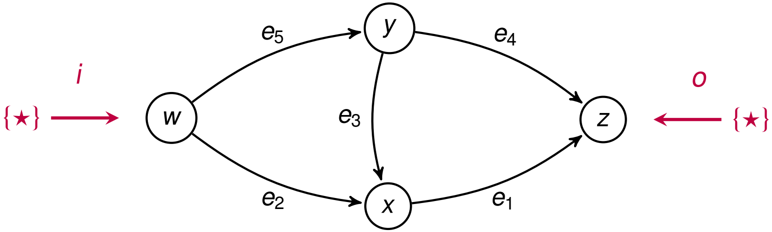
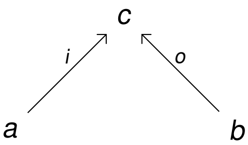


The overall shape of this diagram resembles that of a **cospan**:



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Theorem (B. Fong)

Let A *be a category with finite colimits and* $F\colon \mathsf{A} \to \mathrm{Set}$ *a symmetric lax monoidal functor. Then there exists a category* $F\mathrm{Cospan}$ *which has:*

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- *objects as those of* A *and*
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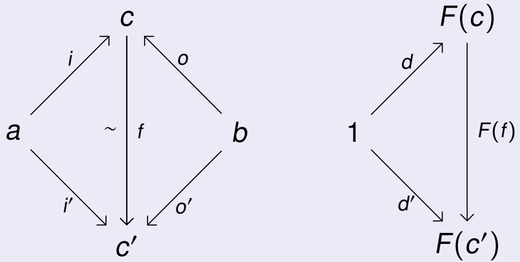
$$a \xrightarrow{i} c \xleftarrow{o} b \qquad\qquad d \in F(c)$$

Theorem (B. Fong continued)

Two F-decorated cospans are in the same isomorphism class if the following diagrams commute:



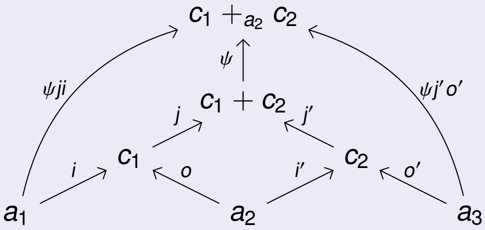
Theorem (B. Fong continued)

To compose two morphisms:

$$a_1 \xrightarrow{\;i\;} c_1 \xleftarrow{\;o\;} a_2 \qquad\qquad a_2 \xrightarrow{\;i'\;} c_2 \xleftarrow{\;o'\;} a_3$$

$$d_1 \in F(c_1) \qquad\qquad\qquad d_2 \in F(c_2)$$

we take the pushout in A*:*



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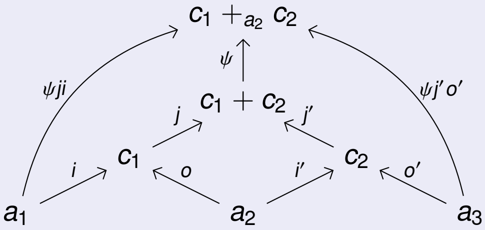

$$d_1 \odot d_2 \colon 1 \xrightarrow{d_1 \times d_2} F(c_1) \times F(c_2) \xrightarrow{\phi_{c_1,c_2}} F(c_1 + c_2) \xrightarrow{F(\psi)} F(c_1 +_{a_2} c_2)$$

For example, if we let $F\colon \mathrm{Set} \to \mathrm{Set}$ be the symmetric lax monoidal functor that assigns to a set N the (large) *set* of all graph structures having N as its set of vertices:

$$F(N) = \{E \underset{t}{\overset{s}{\rightrightarrows}} N\}$$

For an example of this example, if we take $N = \{n_1, n_2, n_3\}$ to be a three element set, then some elements of the (large) set $F(N)$ are given by:

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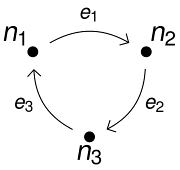
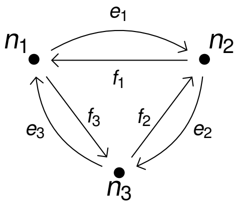
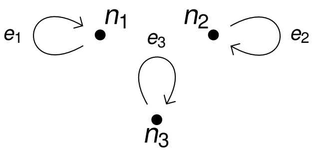


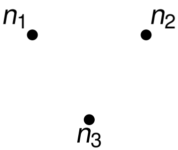
$d_1 \in F(N)$



$d_2 \in F(N)$



$d_3 \in F(N)$



$d_4 \in F(N)$

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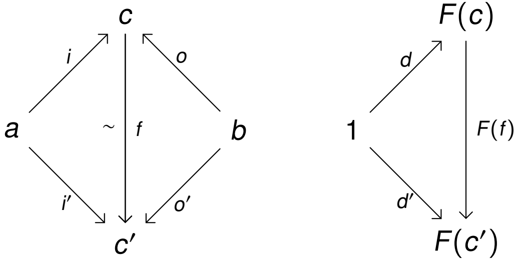

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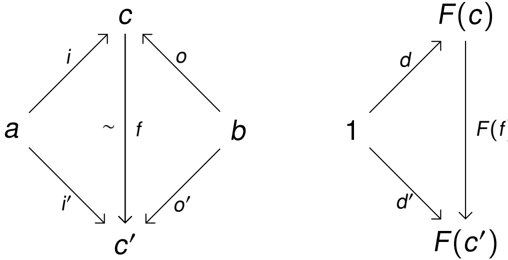
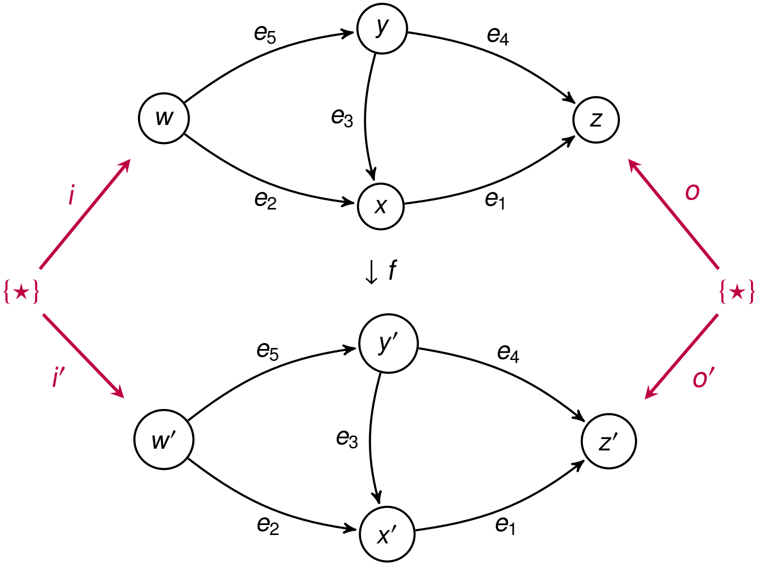


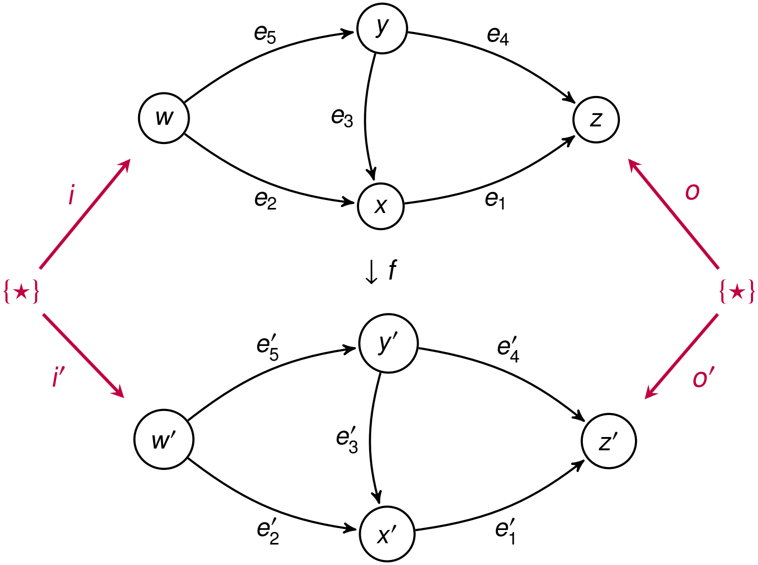
The triangle on the right is in Set and commutes on the nose.

This means that a decoration $d \in F(c)$ together with a bijection $f\colon c \to c'$ determines what the decoration $d' \in F(c')$ must be.

In the context of open graphs, the following two open graphs would be in the same isomorphism class:



But the following two open graphs would *not* be in the same isomorphism class:



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Let A *be a category with finite coproducts,* X *a category with finite colimits and* $L\colon \mathsf{A} \to \mathsf{X}$ *a finite coproduct preserving functor. Then there exists a category* ${}_L\mathbf{Csp}(\mathsf{X})$ *which has:*

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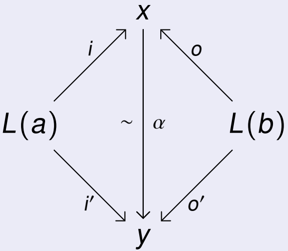
$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

Theorem (Baez, C. continued)

Two structured cospans are in the same isomorphism class if the following diagram commutes:



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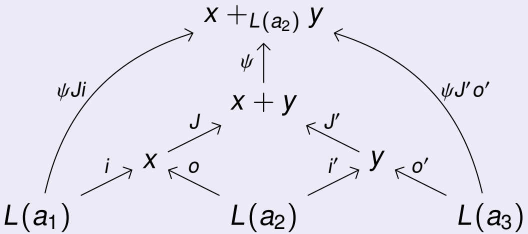

In the context of open graphs, we take $L \colon \mathrm{Set} \to \mathrm{Graph}$ to be the discrete graph functor which assigns to a set N the edgeless graph with vertex set N.

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Both Set and Graph have finite colimits and L is a left adjoint, so we get the following:

Corollary

Let $L\colon \mathsf{Set} \to \mathsf{Graph}$ *be the discrete graph functor. Then there exists a category* ${}_L\mathsf{Csp}(\mathsf{Graph})$ *which has:*

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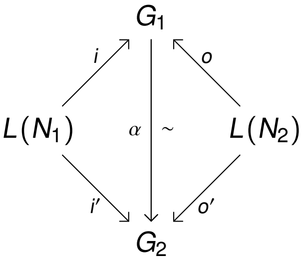
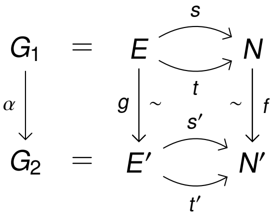
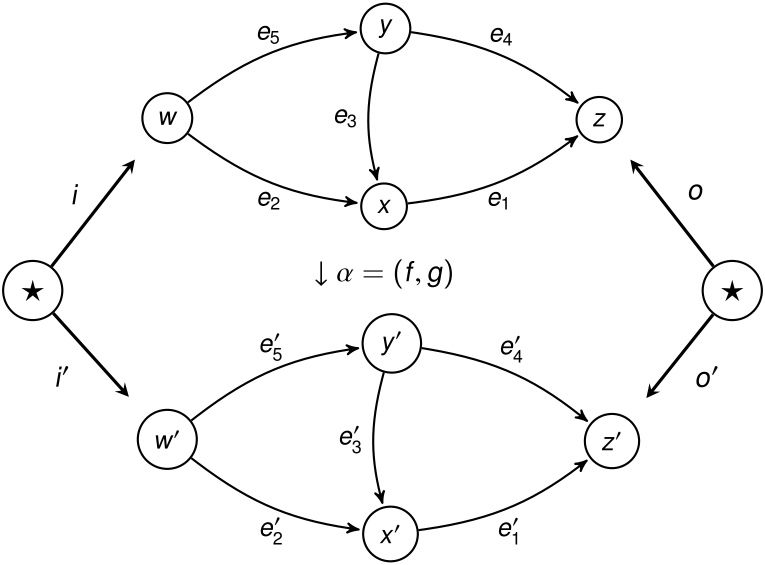
Now, two open graphs are in the same isomorphism class if there exists an isomorphism of graphs $\alpha\colon G_1 \to G_2$ making the following diagram commute:



Here, $\alpha\colon G_1 \to G_2$ is an isomorphism of graphs which is a *pair* of bijections (f, g) making the following squares commute:



And now, the following two open graphs are in the same isomorphism class.



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You might be thinking that we should then use a bicategory... and we *could* do this.

But instead, we're going to use a 'double category'!

A double category has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\big\downarrow & \Downarrow \alpha & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow{N} & D \end{array}$$

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and morphisms between horizontal 1-cells, called 2-morphisms, here denoted as α.

These 2-morphisms can be composed both vertically and horizontally.

$$
\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\
C & \xrightarrow{N} & D
\end{array}
\qquad\qquad
\begin{array}{ccc}
B & \xrightarrow{M'} & E \\
{\scriptstyle g}\downarrow & \Downarrow \beta & \downarrow{\scriptstyle h} \\
D & \xrightarrow{N'} & F
\end{array}
$$

$$
\begin{array}{ccc}
C & \xrightarrow{N} & D \\
{\scriptstyle f'}\downarrow & \Downarrow \alpha' & \downarrow{\scriptstyle g'} \\
G & \xrightarrow{P} & H
\end{array}
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\qquad
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D & \xrightarrow{N'} & F \\
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\end{array}
$$

$$(\alpha \odot \beta)(\alpha' \odot \beta') = (\alpha\alpha') \odot (\beta\beta')$$

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$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(a')$$

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Theorem (Baez, C. continued)

2-morphisms as maps of cospans in X *given by commutative diagrams of the form:*

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(a') \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle L(g)} \\ L(b) & \xrightarrow{i'} & y & \xleftarrow{o'} & L(b') \end{array}$$

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$$

The horizontal composite of two 2-morphisms:

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L(a) & \xrightarrow{i_1} & x & \xleftarrow{o_1} & L(b) \\
{\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle L(g)} \\
L(a') & \xrightarrow{i'_1} & x' & \xleftarrow{o'_1} & L(b')
\end{array}
\qquad
\begin{array}{ccccc}
L(b) & \xrightarrow{i_2} & y & \xleftarrow{i_2} & L(c) \\
{\scriptstyle L(g)}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle L(h)} \\
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is given by

$$\begin{array}{ccccc} L(a) & \xrightarrow{J\psi i_1} & x +_{L(b)} y & \xleftarrow{J\psi o_2} & L(c) \\ {\scriptstyle L(f)}\downarrow & & {\scriptstyle \alpha +_{L(g)} \beta}\downarrow & & \downarrow{\scriptstyle L(h)} \\ L(a') & \xrightarrow{J\psi i_1'} & x' +_{L(b')} y' & \xleftarrow{J\psi o_2'} & L(c'). \end{array}$$

Theorem (Baez, C. continued)

Monoidal structure:

$$
\begin{array}{ccccc}
L(a_1) & \xrightarrow{i_1} & x_1 & \xleftarrow{o_1} & L(b_1) \\
{\scriptstyle L(f)}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle L(g)} \\
L(a_2) & \xrightarrow[i_2]{} & x_2 & \xleftarrow[o_2]{} & L(b_2)
\end{array}
\otimes
\begin{array}{ccccc}
L(a_1') & \xrightarrow{i_1'} & x_1' & \xleftarrow{o_1'} & L(b_1') \\
{\scriptstyle L(f')}\downarrow & & {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle L(g')} \\
L(a_2') & \xrightarrow[i_2']{} & x_2' & \xleftarrow[o_2']{} & L(b_2')
\end{array}
$$

$$
= \begin{array}{ccccc}
L(a_1 + a_1') & \xrightarrow{(i_1 + i_1')\phi^{-1}} & x_1 + x_1' & \xleftarrow{(o_1 + o_1')\phi^{-1}} & L(b_1 + b_1') \\
{\scriptstyle L(f+f')}\downarrow & & {\scriptstyle \alpha+\alpha'}\downarrow & & \downarrow{\scriptstyle L(g+g')} \\
L(a_2 + a_2') & \xrightarrow[(i_2 + i_2')\phi^{-1}]{} & x_2 + x_2' & \xleftarrow[(o_2 + o_2')\phi^{-1}]{} & L(b_2 + b_2')
\end{array}
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We could also address the defect with decorated cospans more directly by instead of using a functor $F\colon \mathsf{A} \to \mathsf{Set}$, using a *pseudofunctor* $F\colon \mathsf{A} \to \mathit{Cat}$.

Theorem (Baez, Vasilakopoulou, C.)

Given a category A *with finite colimits and a symmetric lax monoidal pseudofunctor* $F\colon \mathsf{A} \to \mathit{Cat}$*, there exists a symmetric monoidal double category* $F\mathbb{C}\mathrm{sp}$ *which has:*

- *objects as those of* A*,*
- *vertical 1-morphisms as morphisms of* A*,*
- *horizontal 1-cells as* F*-decorated cospans, which are again pairs:*

$$a \xrightarrow{i} c \xleftarrow{o} b \qquad d \in F(c)$$

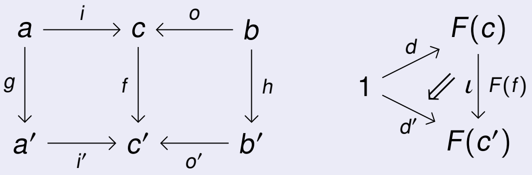
Theorem (Baez, Vasilakopoulou, C. continued)

- *2-morphisms given by maps of cospans in* A*:*

$$
\begin{array}{ccccc}
a & \xrightarrow{i} & c & \xleftarrow{o} & b \\
{\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\
a' & \xrightarrow[i']{} & c' & \xleftarrow[o']{} & b'
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \overset{d}{\nearrow} & F(c) \\
1 & & \downarrow{\scriptstyle F(f)} \\
 & \underset{d'}{\searrow} & F(c')
\end{array}
$$

Theorem (Baez, Vasilakopoulou, C. continued)

2-morphisms given by maps of cospans in A*:*



together with a 2-morphism ι *which can be viewed as a morphism*

$$\iota\colon F(f)(d) \to d'$$

in $F(c')$*.*

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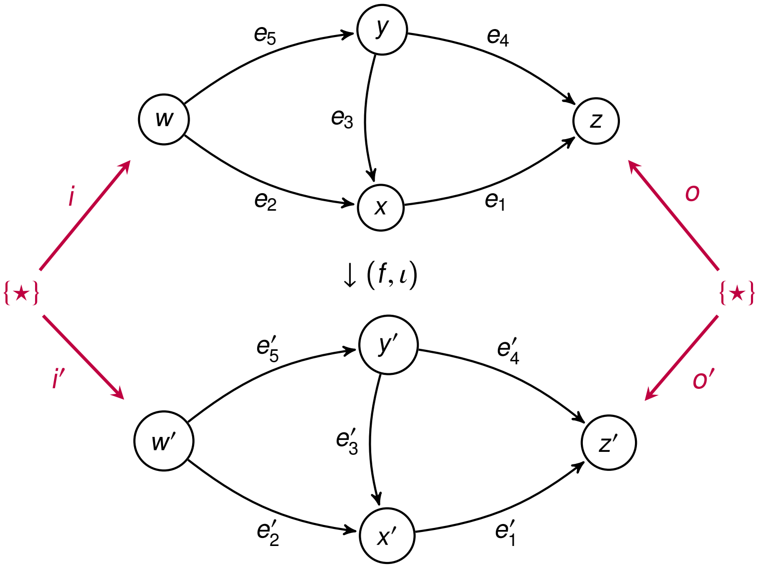
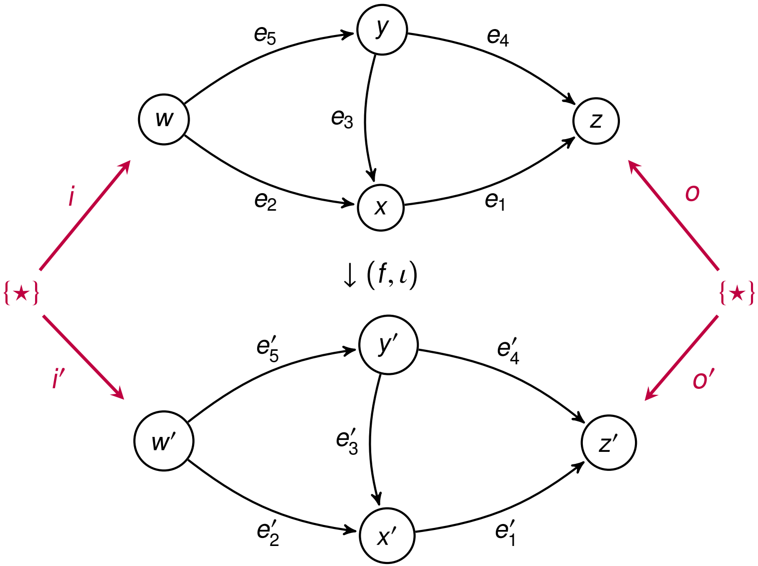


In the context of open graphs:



the morphism $\iota \colon F(f)(d) \to d'$ is the map of edges.

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Theorem (Baez, Vasilakopoulou, C.)

Given a finitely cocomplete category A *and a symmetric lax monoidal pseudofunctor* $F\colon \mathsf{A} \to \mathsf{Cat}$, *if each category* $F(a)$ *is also finitely cocomplete, then there is an equivalence of symmetric monoidal double categories*

$$ {}_L\mathbb{C}sp(\textstyle\int F) \simeq F\mathbb{C}sp. $$

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The functor L used to obtain the structured cospans double category is left adjoint to the Grothendieck construction of the pseudofunctor F:

$$ R \colon \textstyle\int F \to \mathsf{A}. $$

We've used the framework of structured cospans to create syntax categories for black box functors.

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There exists a left adjoint $L\colon \mathrm{FinSet} \to \mathrm{Circ}$ which we can use to obtain a symmetric monoidal category

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of finite sets and open electrical circuits.

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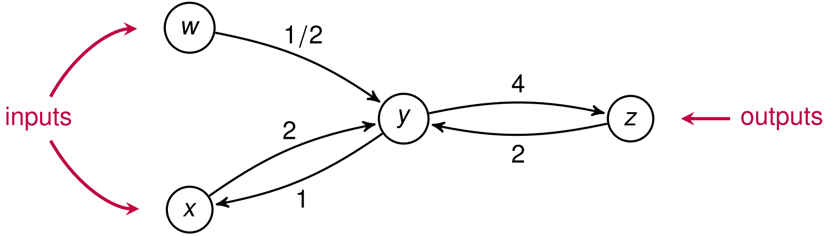

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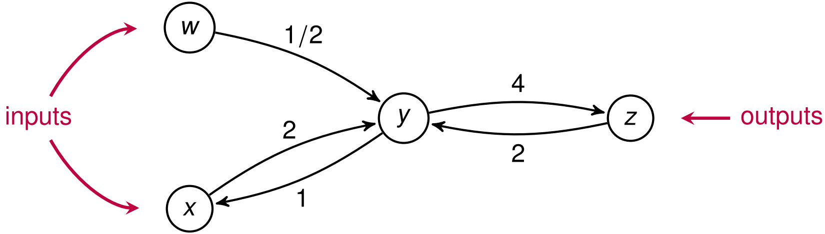
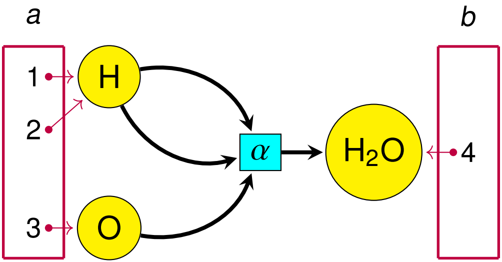


From this, we can obtain a black box functor

$$ \blacksquare\colon {}_L\mathit{Csp}(\mathrm{Circ}) \to \mathrm{Rel}. $$

And likewise for open Petri nets.



$$L\colon \mathrm{Set} \to \mathrm{Petri}$$

$$\blacksquare\colon {}_L Csp(\mathrm{Petri}) \to \mathrm{Rel}.$$

For more, see my thesis on Dr. Baez's website:

https://tinyurl.com/courser-thesis