

Monoidal Grothendieck Construction

Joe Moeller Christina Vasilakopoulou

University of California, Riverside

Fourth Symposium on Compositional Structures
23 May 2019

Motivation

$k \in \text{Ring}$

Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

$\text{Mod}_{\text{all}}???$

Motivation

Grothendieck: Yes!

- ▶ objects (k, M) , where $M \in \text{Mod}_k$
- ▶ maps $(f, g): (k, M) \rightarrow (k', M')$
where $f: k \rightarrow k'$ and
 $g: M \rightarrow f^*(M')$



Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

$$\text{Mod}: \text{Ring} \rightarrow \text{Cat}???$$

Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

$$\text{Mod}: \text{Ring} \rightarrow \text{Cat}???$$

$$f: k \rightarrow k'$$

Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

$$\text{Mod}: \text{Ring} \rightarrow \text{Cat}???$$

$$f: k \rightarrow k' \rightsquigarrow f^*: \text{Mod}_{k'} \rightarrow \text{Mod}_k$$

Motivation

$$k \in \text{Ring} \rightsquigarrow \text{Mod}_k \in \text{Cat}$$

$$\text{Mod}: \text{Ring}^{\text{op}} \rightarrow \text{Cat}$$

$$f: k \rightarrow k' \rightsquigarrow f^*: \text{Mod}_{k'} \rightarrow \text{Mod}_k$$

Motivation

Given

$$\text{Mod}: \text{Ring}^{\text{op}} \rightarrow \text{Cat}$$

we defined Mod_{all} to have

- ▶ objects (k, M) , where $M \in \text{Mod}_k$
- ▶ maps $(f, g): (k, M) \rightarrow (k', M')$
where $f: k \rightarrow k'$ and $g: M \rightarrow f^*(M')$

Motivation

Given

$$\text{Mod}: \text{Ring}^{\text{op}} \rightarrow \text{Cat}$$

we defined Mod_{all} to have

- ▶ objects (k, M) , where $M \in \text{Mod}_k$
- ▶ maps $(f, g): (k, M) \rightarrow (k', M')$ where $f: k \rightarrow k'$ and $g: M \rightarrow f^*(M')$

Given

$$\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \text{Cat}$$

we define $\int \mathcal{F}$ to have

- ▶ objects (x, a) , where $x \in \mathcal{X}$, $a \in \mathcal{F}(x)$
- ▶ maps $(f, g): (x, a) \rightarrow (x', a')$ where $f: x \rightarrow x'$ and $g: a \rightarrow \mathcal{F}f(a')$

Indexed Categories

2-category $\mathbf{ICat}(\mathcal{X})$:

- ▶ an **indexed category** is a pseudofunctor $\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \mathbf{Cat}$.
- ▶ an **indexed functor** is a pseudonatural transformation $\alpha: \mathcal{F} \Rightarrow \mathcal{G}$
- ▶ an **indexed natural transformation** is a modification $m: \alpha \Rrightarrow \beta$

Fibrations

$$\begin{array}{ccc} A & & b \\ \downarrow P & & \\ \mathcal{X} & & x \xrightarrow{f} y \end{array}$$

Fibrations

cartesian lift

$$\begin{array}{ccc} \mathcal{A} & & a \xrightarrow{\phi} b \\ \downarrow P & & \\ \mathcal{X} & & x \xrightarrow{f} y \end{array}$$

Fibrations

pullback

$$\begin{array}{ccc} \mathcal{A} & & f^*(b) \xrightarrow{\phi} b \\ \downarrow P & & \\ \mathcal{X} & & x \xrightarrow{f} y \end{array}$$

Fibrations

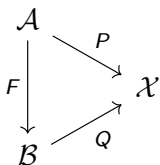
reindexing functor

$$\mathcal{A}_y \xrightarrow{f^*} \mathcal{A}_x$$

Fibrations

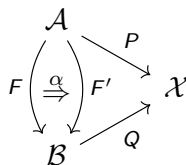
2-category $\text{Fib}(\mathcal{X})$:

- ▶ an object is a fibration
 $P: \mathcal{A} \rightarrow \mathcal{X}$
- ▶ a 1-morphism is a functor F



which preserves cartesian liftings

- ▶ a 2-morphism is a natural transformation



The Grothendieck Construction

For indexed category $F: \mathcal{X}^{\text{op}} \rightarrow \text{Cat}$, $\int F$ is naturally fibred over \mathcal{X} :

$$\begin{aligned} P_{\mathcal{F}}: \int \mathcal{F} &\rightarrow \mathcal{X} \\ (x, a) &\mapsto x \\ (f, k) &\mapsto f \end{aligned}$$

2-Equivalence

Theorem

The Grothendieck construction gives an equivalence:

$$\mathrm{ICat}(\mathcal{X}) \cong \mathrm{Fib}(\mathcal{X})$$

Fibre-wise Monoidal Grothendieck Construction

A **(fibre-wise) monoidal indexed category** is

- ▶ a pseudofunctor $\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \text{MonCat}$

Let $f\text{MonlCat}(\mathcal{X})$ denote the 2-category of fibre-wise monoidal indexed categories

Fibre-wise Monoidal Grothendieck Construction

A **(fibre-wise) monoidal fibration** is

- ▶ fibration $P: \mathcal{A} \rightarrow \mathcal{X}$
- ▶ the fibres \mathcal{A}_x are monoidal
- ▶ the reindexing functors are monoidal

Let $f\text{MonFib}(\mathcal{X})$ denote the 2-category of fibre-wise monoidal fibrations.

Fibre-wise Monoidal Grothendieck Construction

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to an equivalence:

$$f\text{MonFib}(\mathcal{X}) \simeq f\text{MonCat}(\mathcal{X})$$

Global Monoidal Grothendieck Construction

A **(global) monoidal indexed category** is

- ▶ an indexed category $\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \text{Cat}$
- ▶ \mathcal{X} is monoidal
- ▶ \mathcal{F} is lax monoidal $(\mathcal{F}, \phi): (\mathcal{X}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$

Let $g\text{MonlCat}$ denote the 2-category of global monoidal indexed categories.

Global Monoidal Grothendieck Construction

A **(global) monoidal fibration** is a fibration $P: \mathcal{A} \rightarrow \mathcal{X}$

- ▶ \mathcal{A} and \mathcal{X} are monoidal
- ▶ P is a strict monoidal functor
- ▶ $\otimes_{\mathcal{A}}$ preserves cartesian liftings.

Let $\mathit{gMonFib}(\mathcal{X})$ denote the 2-category of global monoidal fibrations.

Global Monoidal Grothendieck Construction

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to an equivalence:

$$g\text{MonFib}(\mathcal{X}) \simeq g\text{MonlCat}(\mathcal{X})$$

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi): (\mathcal{X}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$$

$$\phi: \mathcal{F}x \times \mathcal{F}y \rightarrow \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) =$$

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi): (\mathcal{X}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$$

$$\phi: \mathcal{F}x \times \mathcal{F}y \rightarrow \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) = (x \otimes y,$$

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi): (\mathcal{X}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$$

$$\phi: \mathcal{F}x \times \mathcal{F}y \rightarrow \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) = (x \otimes y, \phi_{x,y}(a, b))$$

Shulman's Monoidal Grothendieck Construction

Theorem (Shulman)

If \mathcal{X} is cartesian monoidal, then

$$g\text{MonFib}(\mathcal{X}) \simeq f\text{MonlCat}(\mathcal{X})$$

Cartesian Case

Theorem (Vasilakopoulou, M)

If \mathcal{X} is a cartesian monoidal category, then

$$\begin{array}{ccc}
 g\text{MonFib}(\mathcal{X}) & \xrightarrow{\cong} & g\text{MonICat}(\mathcal{X}) \\
 \downarrow \cong & & \downarrow \cong \\
 f\text{MonFib}(\mathcal{X}) & \xrightarrow{\cong} & f\text{MonICat}(\mathcal{X})
 \end{array}$$

Example: Modules

$$\begin{aligned} \text{Mod} &: \text{Ring}^{\text{op}} \rightarrow \text{Cat} \\ (f: k \rightarrow k') &\mapsto (f^*: \text{Mod}_{k'} \rightarrow \text{Mod}_k) \end{aligned}$$

Example: Modules

$$\text{Mod} : \text{Ring}^{\text{op}} \rightarrow \text{Cat}$$

$$(f : k \rightarrow k') \mapsto (f^* : \text{Mod}_{k'} \rightarrow \text{Mod}_k)$$

$\int \text{Mod} :$

$$(k, M) \xrightarrow{(f, g)} (k', N)$$

Example: Modules

$$(\text{Mod}, \mu): (\text{Ring}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$$

$$\begin{aligned} \mu: \text{Mod}_k \times \text{Mod}_{k'} &\rightarrow \text{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N \end{aligned}$$

Example: Modules

$$(\text{Mod}, \mu): (\text{Ring}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$$

$$\begin{aligned} \mu: \text{Mod}_k \times \text{Mod}_{k'} &\rightarrow \text{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N \end{aligned}$$

$$(\int \text{Mod}, \otimes_{\mu}):$$

$$(k, M) \otimes_{\mu} (k', N) = (k \otimes k', M \otimes_{\mathbb{Z}} N)$$



John Baez, John Foley, Joseph Moeller, and Blake Pollard.

Network models.

[arXiv:1711.00037 \[math.CT\]](#), 2017.



Joe Moeller and Christina Vasilakopoulou.

Monoidal grothendieck construction.

[arXiv:1809.00727 \[math.CT\]](#), 2019.



Michael Shulman.

Framed bicategories and monoidal fibrations.

Theory Appl. Categ., 20:No. 18, 650–738, 2008.