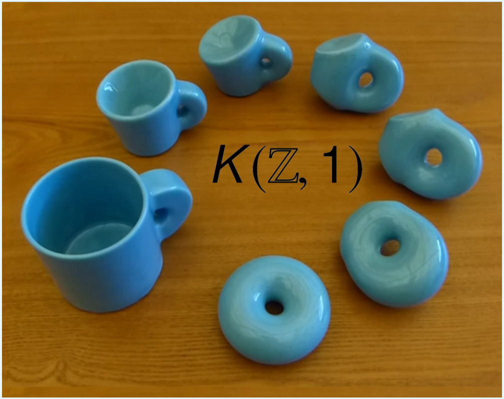
The Rise and Spread of Algebraic Topology



John Baez

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We now say the ith Betti number of a topological space X is the rank of $H_i(X)$. But that's not what Enrico Betti said!

Betti defined his numbers in 1871. In 1895 Poincaré recalled them in his *Analysis Situs*, saying:

Meanwhile, the field is by no means exhausted.

Poincaré said what it means for two oriented submanifolds of a manifold X to be homologous. He showed how to add and subtract homology classes. He essentially said that the ith Betti number of X is β_i if X has at most β_i linearly independent homology classes of i-dimensional submanifolds.

Only 20 years later, in 1915, did Alexander prove that Betti numbers are topological invariants.

In the summers of 1926–1928, Alexandroff and Hopf lectured on algebraic topology in Goettingen. Emmy Noether attended and pointed out that ith Betti number is the rank of an abelian group

$$H_i(X) = \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$$

where

$$\partial_i \colon C_i(X) \to C_{i-1}(X)$$

is a map between 'chain groups'. She also noticed that a map of simplicial complexes induced a map of homology groups. *All this was new!*

Noether never published a single paper about these ideas, and they spread slowly.

> *All these ideas were a very long time in the making because the people doing homology and homotopy theory were not algebraists and the algebraists didn't take any interest. The only person who took any interest was Emmy Noether. — Peter Hilton*

Around 1945, Eilenberg and Mac Lane realized that homology groups define *functors* between *categories*:

$$H_i \colon \mathrm{Top} \to \mathrm{Ab}$$

We now realize that the nicest invariants are functors, thus applying to morphisms as well as objects.

But they only invented these concepts to surmount this challenge: when do two ways of constructing homology groups count as “the same”? Answer: when they are *naturally isomorphic*.

> *I didn’t invent categories to study functors; I invented them to study natural transformations. — Saunders Mac Lane*

To really understand this well, we need to realize that Cat is a *2-category*, with

- categories as objects
- functors as morphisms
- natural transformations as 2-morphisms

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Y
\arrow[from=U, to=D, phantom, "\Downarrow\alpha"]
\end{tikzcd}
$$

Thus: the birth of categories laid the groundwork for the birth of 2-categories... though these were only invented around 1965, by Ehresmann and others.

But the impetus towards 2-categories also comes from the heart of topology itself! There is a 2-category Top_2 with:

- topological spaces as objects
- continuous maps as morphisms
- homotopies between maps as 2-morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

The really good invariants of topological spaces are *2-functors* from Top_2 to other 2-categories!

For example, taking the chain complex $C_\bullet(X)$ of a topological space X extends to a 2-functor from Top_2 to the 2-category Ch_2 with:

- chain complexes of abelian groups as objects
- chain maps as morphisms
- chain homotopies as 2-morphisms

And this doesn't stop! There is really an ∞-category Top_∞ with

- 'nice' topological spaces (CW complexes) as objects
- continuous maps as 1-morphisms
- homotopies as 2-morphisms
- homotopies between homotopies as 3-morphisms
- etcetera...

An object in here is called a **homotopy type**.

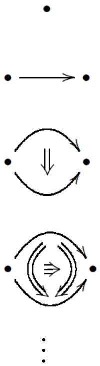
Similarly, there is an ∞-category Ch_∞ with

- chain complexes of abelian groups as objects
- chain maps as 1-morphisms
- chain homotopies as 2-morphisms
- chain homotopies between chain homotopies as 3-morphisms
- etcetera...

Taking the chain complex of a space extends to an ∞-functor

$$C_\bullet : \mathrm{Top}_\infty \to \mathrm{Ch}_\infty$$

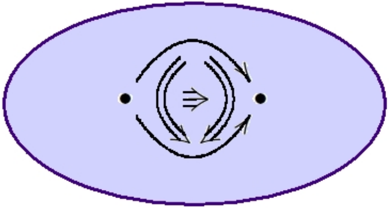
The dream of doing topology with ∞-categories was advocated by Grothendieck in his 600-page text *Pursuing Stacks*, written around 1983.

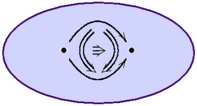


Only in the 1990s did mathematicians take it up in earnest.

Why all these higher morphisms in topology? Because any topological space X gives an ∞-category $\Pi_\infty(X)$, with

- points of X as objects
- paths in X as 1-morphisms
- paths of paths as 2-morphisms
- paths of paths of paths as 3-morphisms, etc....





$\Pi_\infty(X)$ is called the **fundamental ∞-groupoid** of X.

An **∞-groupoid** is an ∞-category where for all j, all j-morphisms are invertible up to higher morphisms.

The **Homotopy Hypothesis**, due to Grothendieck, says that ∞-groupoids are 'the same' as homotopy types. This can be made precise in many ways, some of which are theorems, and some of which are still ongoing projects.

This hypothesis *makes topology algebraic*, by suitably generalizing 'algebra' and limiting 'topology'.

A chain complex of abelian groups gives an ∞-groupoid with:

- 0-chains as objects
- 1-chains f with $df = y - x$ as morphisms from x to y
- 2-chains α with $d\alpha = g - f$ as 2-morphisms from f to g
- etc....

∞-groupoids coming from chain complexes are the simplest kind. Thanks to the homotopy hypothesis,

$$C_\bullet \colon \mathrm{Top}_\infty \to \mathrm{Ch}_\infty$$

can be reinterpreted as taking $\Pi_\infty(X)$ and simplifying it down to $C_\bullet(X)$.

The ∞-categories Top_∞ and Ch_∞ are not ∞-groupoids, because their 1-morphisms — continuous maps, and chain maps — are not all invertible, not even up to homotopy.

However, all their j-morphisms for $j > 1$ *are* invertible up to higher morphisms.

An ∞-category where all j-morphisms for $j > 1$ are invertible up to higher morphisms is called an **$(\infty, 1)$–category**.

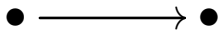
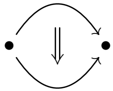
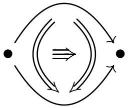
In 2006, Jacob Lurie put out a 735-page book *Higher Topos Theory*, which develops a powerful theory of $(\infty, 1)$–categories. It's free online!

The goal of this book, and related work by many other people, is the *homotopification of mathematics*.

In other words: redo as much math as possible, replacing the category of sets with the $(\infty, 1)$-category of ∞-groupoids — or by the Homotopy Hypothesis, homotopy types.

Instead of demanding that algebraic structures obey equations, make them obey equations only *up to coherent homotopy*: that is, up to higher morphisms, that themselves obey the right equations up to still higher morphisms, etc.

To make all this stuff precise, Lurie and others use simplices rather than globes:

objects	morphisms	2-morphisms	3-morphisms	$\cdots$
				Globes

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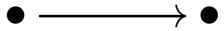
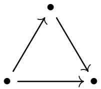
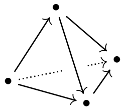
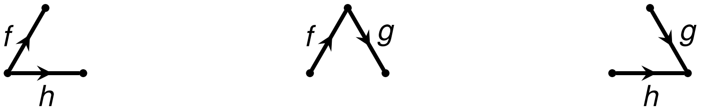
				Simplices

In the simplicial approach we define an '∞-groupoid' to be a **Kan complex**: a simplicial set where every horn has a filler:



The **Dold–Kan theorem** says there is an equivalence between chain complexes of abelian groups and simplicial abelian groups. Every simplicial abelian group is a Kan complex. This makes precise how chain complexes are special ∞-groupoids.

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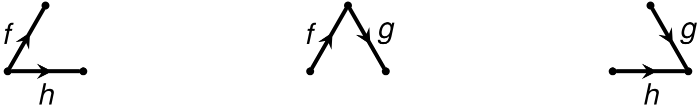

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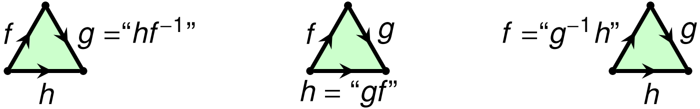

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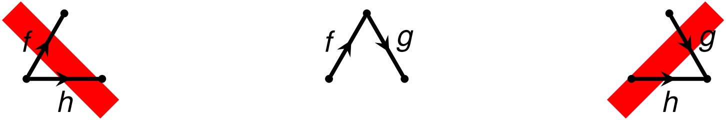
In the simplicial approach, we can define an ‘$(\infty, 1)$–category’ to be a **quasicategory**: a simplicial set where ‘inner’ horns have fillers:



Every Kan complex is a quasicategory. This makes precise how every ∞-groupoid is an $(\infty, 1)$–category.

There’s a quasicategory of all ‘nice’ topological spaces, Top_∞, and a quasicategory of all chain complexes, Ch_∞. Taking the chain complex of a space extends to a map

$$C_\bullet \colon \mathrm{Top}_\infty \to \mathrm{Ch}_\infty$$

as expected!

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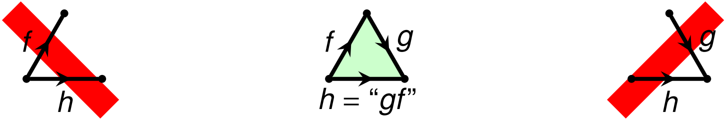

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In the simplicial approach, we can define $\Pi_\infty(X)$ to be the simplicial set consisting of all simplices mapped into X.

This is a Kan complex — our concept of ∞-groupoid — so it indeed deserves to be called the **fundamental ∞-groupoid** of X.

In fact, there's a quasicategory of all Kan complexes, Kan_∞, and an equivalence

$$\Pi_\infty \colon \mathrm{Top}_\infty \to \mathrm{Kan}_\infty$$

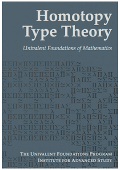
This makes the Homotopy Hypothesis into a theorem!

Back to the big picture:

> *I am pretty strongly convinced that there is an ongoing reversal in the collective consciousness of mathematicians: the right hemispherical and homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components of a space defined only up to homotopy. — Yuri Manin, 2009*

‘Homotopy type theory’ is an attempt to set up new axioms for math that take homotopy types rather than sets as fundamental. Instead of the category Set, these axioms apply to the $(\infty, 1)$-categories Top_∞ and Kan_∞... and many others, called ‘$(\infty, 1)$-topoi’.

Again there’s a free book to read: *Homotopy Type Theory: Univalent Foundations of Mathematics*.



But homotopy type theory does not yet solve all our problems: homotopy coherence is not completely ‘built in’. The revolution is not finished!

Where does topological data analysis fit into all of this?

A clue:

> *Instead of sets, clouds of discrete elements, we envisage some sorts of vague spaces, which can be very severely deformed, mapped one to another, and all the while the specific space is not important, but only the space up to deformation. — Yuri Manin, 2009*

Topological data analysis transforms a ‘cloud of discrete elements’ into a ‘sort of vague space’. But it does so in a subtle way. It creates not a homotopy type, but an object in some *other* $(\infty, 1)$-category to which the axioms of homotopy type theory apply!

We can think of these as ‘homotopy types that depend on a distance scale’.

For any metric space X and any $\epsilon \in [0, \infty]$, the **ϵ−Rips complex** is the simplicial set $R_\epsilon(X)$ where an n-simplex is an $(n+1)$-tuple of points in X with pairwise distances $\leq \epsilon$.

When $\epsilon \leq \delta$ we have an inclusion of simplicial sets

$$R_\epsilon(X) \subseteq R_\delta(X)$$

So, the Rips complex as a whole is a family of 'nested' simplicial sets, one for each $\epsilon \in [0, \infty]$, What kind of object is this? What world does it live in?

There is a category with numbers $\epsilon \in [0, \infty]$ as objects and a single morphism from ϵ to δ if $\epsilon \leq \delta$, none if $\epsilon > \delta$.

The **Rips complex** of a metric space X is a functor

$$R(X) \colon [0, \infty] \to \mathrm{SSet}$$

where SSet is the category of simplicial sets. This functor maps each object ϵ to the simplicial set $R_\epsilon(X)$, and each morphism $\epsilon \leq \delta$ to the inclusion of simplicial sets

$$R_\epsilon(X) \hookrightarrow R_\delta(X)$$

So, for any metric space, its Rips complex is an object in $\mathrm{SSet}^{[0,\infty]}$, the category with:

- functors $F\colon [0,\infty] \to \mathrm{SSet}$ as objects,
- natural transformations between these as morphisms.

Objects in $\mathrm{SSet}^{[0,\infty]}$ are 'homotopy types that depend on a distance scale'.

Indeed, there's a way to build an $(\infty, 1)$-category from $\mathrm{SSet}^{[0,\infty]}$. And this $(\infty, 1)$–category is an '$(\infty, 1)$–topos' in the sense of Lurie, so it obeys the axioms of homotopy type theory.

This sounds fancy, but it means that the Rips complex *lives in a world that's formally like the world of homotopy types... but different, and richer*.

How does this help topological data analysis? I don't know yet: I hope to find out this week!

But it shows applied mathematics is starting to enter a realm that algebraic topology reached by its own internal momentum: *the world where algebra and topology meet and merge*.

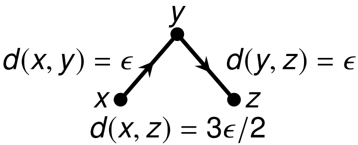
Realizing this should help us come up with new ideas.

Technical Details

To build a quasicategory containing Rips complexes, we use this recipe: for any model category M, the homotopy coherent nerve of the full subcategory M_{cf} consisting of fibrant and cofibrant objects is a quasicategory. We apply this to $\mathrm{SSet}^{[0,\infty]}$. As a category of simplicial presheaves, $\mathrm{SSet}^{[0,\infty]}$ has a number of different model structures with the same weak equivalences. All these give equivalent $(\infty, 1)$-categories, which are $(\infty, 1)$-topoi.

Let us consider the projective global model structure on $\mathrm{SSet}^{[0,\infty]}$, where fibrations are defined objectwise. In this model structure, a fibrant object is just a functor $F\colon [0,\infty] \to \mathrm{Kan}$, and conditions for an object to be cofibrant were given by Garner.

One can easily check using Garner's conditions that the Rips complex $R(X) \in \mathrm{SSet}^{[0,\infty]}$ is always cofibrant. Unfortunately it is rarely fibrant, because $R_\epsilon(X)$ is rarely a Kan complex. Here is a 2-dimensional horn in $R_\epsilon(X)$ that does not have a filler:



Only triangles whose two longer sides have equal length have fillers for all their horns! All horns of dimension $\neq 2$ have fillers.

Given this, we need to replace the Rips complex by a weakly equivalent fibrant and cofibrant object to make it into an object of the quasicategory obtained from $\mathrm{SSet}^{[0,\infty]}$ with its projective global model structure. I don't know if there's a model structure on $\mathrm{SSet}^{[0,\infty]}$ that makes the Rips complex of every metric space be both fibrant and cofibrant.