5 Jan 07

HW due Friday Jan 12

Munkres 52.1-4

Guess: 1. Which capital English letters are simply connected

2. Which have $\pi_1 = \mathbb{Z}$

1. C E F G H I J K L M N S T U V W X Y Z
2. A D O P Q R (B has $\mathbb{Z} \times \mathbb{Z}$)

ALGEBRAIC TOPOLOGY

Turn topology into algebra problems & solve them using algebra

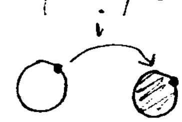
Topology is about spaces + maps:

A <u>space</u> is a topological space.

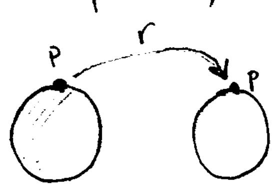
A <u>map</u> $f: X \to Y$ between spaces X, Y is a continuous fn.

Examples:

1) The <u>(unit) circle</u> S^1 is $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$
2) The <u>(closed unit) disc</u> D^2 is $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$
3) The inclusion map $i: S^1 \to D^2$
$p \longmapsto p$



A classic puzzle: is there a <u>retraction</u>, i.e. a map $r: D^2 \to S^1$ s.t. $\forall p \in S^1, r(p) = p$?

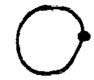


Answer: NO. But how to prove it? <u>Seems</u> obvious!

We prove it using algebra!

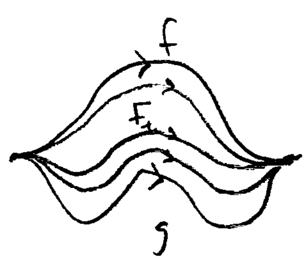
Sketch: we'll turn spaces into groups and maps into homomorphisms!

Well, actually pointed spaces: a pointed space is a pair (X, x_0) consisting of a top. space X and a point $x_0 \in X$. (x_0 sometimes called the basepoint)

Examples: $(D^2, (1,0))$  , $(S^1, (1,0))$ 

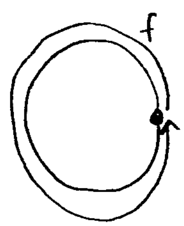
Defn. A loop f in a pointed space (X, x_0) is a map $f: I \to X$ with $f(0) = f(1) = x_0$ and $I = [0,1]$,

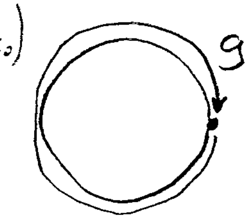
Wilhelm told you what a path homotopy was. In pictures:



for each value of t we have a path F_t where $F_0 = f$ and $F_1 = g$, and F_t is continuous.

We say two paths are (path) homotopic if there's a path homotopy between them. This is an equivalence relation. We use $[f]$ to represent an equivalence class of paths.

Examples: 

a loop in (S^1, x_0) 

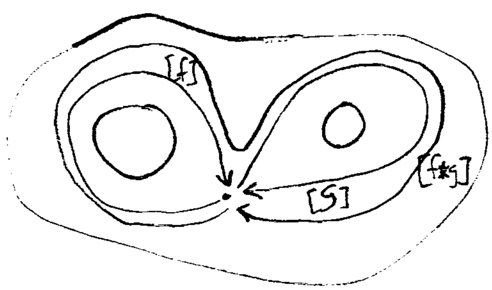
another one

$[f] \neq [g]$ f is not homotopic to g.

Defn: for any pointed space (X, x_0) we let $\pi_1(X, x_0)$ be the set of (path homotopy) equivalence classes of loops.

$\pi_1(X, x_0)$ is also a group! (Derek will prove it in detail next week.) It's the <u>fundamental group</u> of (X, x_0),

The group operation is concatenation of paths:



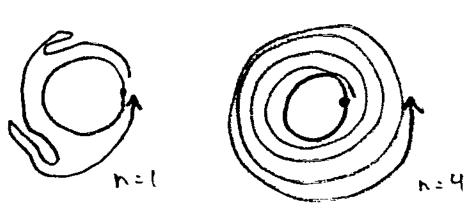
We define $[f] * [g] = [f*g]$.

What's $\pi_1(D^2, (1,0))$?

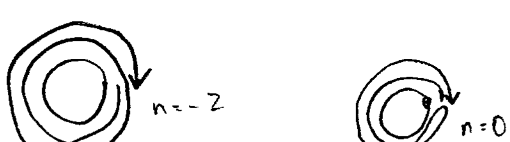
We can retract any loop to $(1,0)$, so all paths are homotopic, and $\pi_1(D^2, (1,0)) = 0$, the trivial group.

Boring!

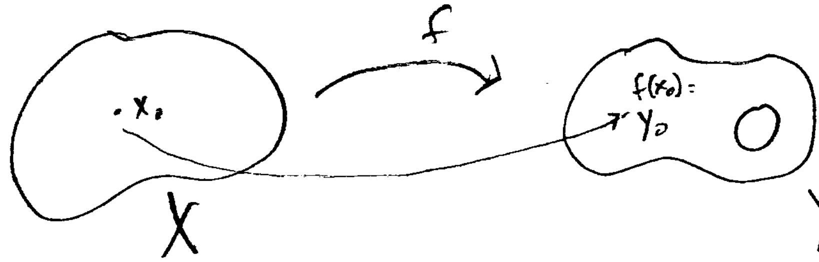
$\pi_1(S^1, (1,0))$?



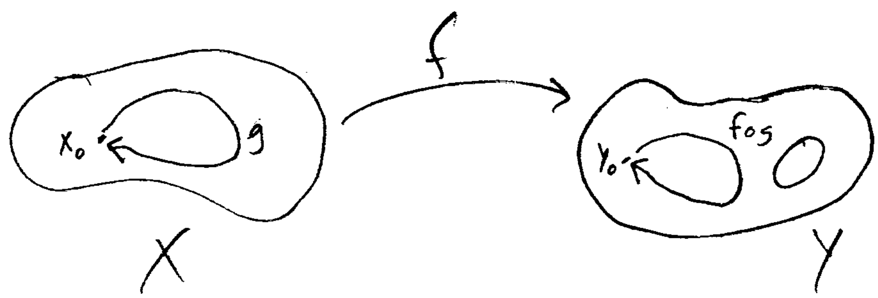
$\pi_1(S^1, (1,0)) = \mathbb{Z}$ "the winding number"



<u>Defn</u>: A <u>map</u> $f:(X,x_0) \to (Y,y_0)$ between pointed spaces is a cont. fn. mapping x_0 to y_0.



A map of pointed spaces sends loops in the first to loops in the second:



and gives a group homomorphism

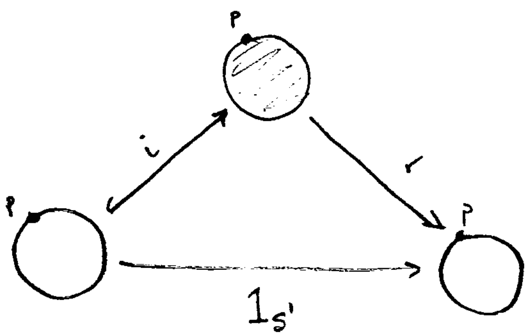
$$\pi_1(f) : \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$$

Facts: A) $1_X : (X,x_0) \to (X,x_0)$ sends any loop to itself

$$\pi_1(1_X) = 1_{\pi_1(X,x_0)}$$

B) $\pi_1(f \circ f') = \pi_1(f) \circ \pi_1(f')$

Now we can solve the problem!



$r \circ i = 1_{S'}$

$\pi_1 \Downarrow$

$$\begin{tikzcd}
 & 0 \arrow[dr, "\pi_1(r)"] & \\
\mathbb{Z} \arrow[ur, "\pi_1(i)"] \arrow[rr, "\pi_1(1_S) = 1_{\mathbb{Z}}"'] & & \mathbb{Z}
\end{tikzcd}$$

this is unique: $0 \to 0$.
But the diagram doesn't commute!
The top route takes everything to zero while the bottom takes everything to itself.

Therefore, r doesn't exist.

QED.