The Fundamental Theorem of Algebra

This theorem, going back to Gauss, says any polynomial of degree n has n roots (counting with multiplicity).

<u>Thm</u>: Given $P(z) = a_n z^n + \cdots + a_1 z + a_0$

where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Then

$$P(z) = a_n (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Thus $P(\alpha_1) = P(\alpha_2) = \cdots = P(\alpha_n) = 0$.

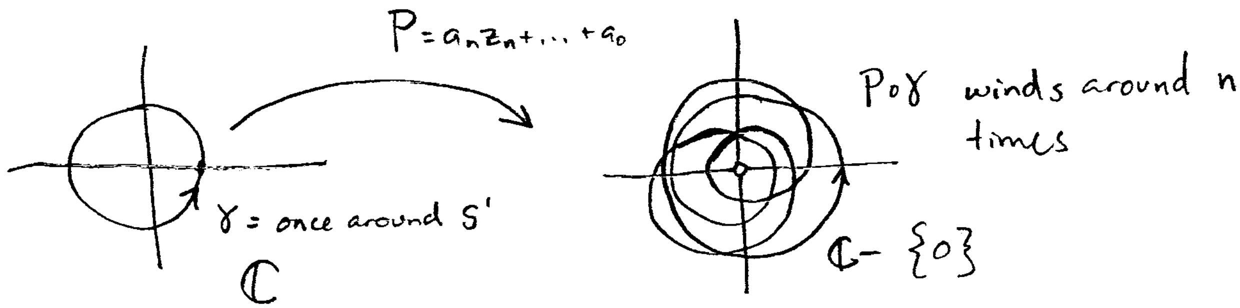
Using pure algebra, we can derive this from

<u>Thm 56.1</u>: Given the same assumptions, if $n \geq 1$ then P has at least one root $\alpha \in \mathbb{C}$, ie. $P(\alpha) = 0$.

The idea is to show that $(z-\alpha)$ is a factor of P, so $\frac{P}{z-\alpha}$ is a polynomial of degree $n-1$; then repeat.

Ironically, the "fundamental theorem of algebra" <u>can't</u> be proved without using topology, contrary to Munkres' claim on p. 354. The definition of $\mathbb{R}$ (and thus $\mathbb{C}$) involves topology, namely that $\mathbb{R}$ is the unique <u>complete</u> ordered field. There's a proof using complex analysis and one using π_1:

Proof.



If P had no root, we'd have $P: S^1 \to \mathbb{C} - \{0\}$; but P extends to D^2, so P must be <u>nulhomotopic</u>, contradicting the fact that the winding number is n

In detail:

Step 1. The map

$$\begin{aligned} f: S^1 &\longrightarrow S^1 \\ z &\longmapsto z^n \end{aligned} \qquad \text{(where } S^1 \subset \mathbb{C} \text{ is the unit circle)}$$

gives

$$\begin{array}{ccc} \pi_1(f) : \pi_1(S^1, 1) & \longrightarrow & \pi_1(S^1, 1) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \\ k & \longmapsto & nk \end{array}$$

To see this, recall that our isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$ carries $[\gamma_k]$ to $k \in \mathbb{Z}$ where

$$\gamma_k(z) = z^k \quad , \quad z = e^{i\theta} \in S^1$$

Note:

$$\pi_1(f)[\gamma_k] = [f \circ \gamma_k] = [(z^k)^n] = [z^{nk}] = [\gamma_{nk}]$$

so

$$\pi_1(f)[\gamma_k] = n[\gamma_k]$$

$\square$ step 1.

Step 2 - The map

$$g: S^1 \longrightarrow \mathbb{C} - \{0\}$$
$$z \longmapsto z^n$$

is not nulhomotopic.

If it were, $\pi_1(g): \pi_1(S^1, 1) \longrightarrow \pi_1(\mathbb{C} - \{0\}, 1)$ would be the zero homomorphism. But

$$\begin{tikzcd}
 & S^1 \arrow[dr, hook, "i"] & \\
S^1 \arrow[ur, "f"] \arrow[rr, "g"'] & & \mathbb{C} - \{0\}
\end{tikzcd}$$

commutes, so (getting lazy and omitting the basepoint)

$$\begin{tikzcd}
 & \mathbb{Z} \cong \pi_1(S_1) \arrow[dr, "\pi_1(i)"] & \\
\mathbb{Z} \cong \pi_1(S_1) \arrow[ur, "k \mapsto nk = \pi_1(f)"] \arrow[rr, "\pi_1(g) = 0"'] & & \pi_1(\mathbb{C} - \{0\}) \cong \mathbb{Z}
\end{tikzcd}$$

Both $\pi_1(f)$ and $\pi_1(i)$ are 1-1 (the latter is true since the retract

$$r: \mathbb{C} - \{0\} \longrightarrow S^1$$
$$z \longmapsto \frac{z}{|z|}$$

exists, so Im $\pi_1(i)$ is a subgroup of $\mathbb{Z}$) which contradicts the assertion that $\pi_1(g)$ is the zero homomorphism.

□ Step 2

Step 3 — Starting with a special case: if $a_n = 1$

$$P(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$$

and

$$|a_{n-1}| + |a_{n-2}| + \dots + |a_0| < 1$$

then

$$\exists\, \alpha \in D^2 \in \mathbb{C} \quad \text{s.t. } P(\alpha) = 0:$$

Suppose $P(\alpha) \neq 0$ for $\alpha \in D^2$. Then we'd have

$$\begin{aligned} P: D^2 &\longrightarrow \mathbb{C} - \{0\} \\ z &\longmapsto P(z) \end{aligned}$$

So $h = P|_{S^1}$ extends to $P: D^2 \to \mathbb{C} - \{0\}$, so h is nulhomotopic. But it can't be, since in step 2 we saw that

$$\begin{aligned} g: S^1 &\longrightarrow \mathbb{C} - \{0\} \\ z &\longmapsto z^n \end{aligned}$$

is nontrivial and g is homotopic to h:

$$F(z,t) = z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$$
$$F(z,0) = z^n = g(z)$$
$$F(z,1) = P(z) = h(z)$$

and

$$\begin{aligned} |F(z,t)| &\geq |z^n| - |t(a_{n-1} z^{n-1} + \dots + a_0)| \\ &\geq 1 - t(|a_n||z^{n-1}| + \dots + |a_0|) \\ &\geq 1 - t(|a_n| + \dots + |a_0|) \\ &> 1 - 1 \cdot 1 \\ &> 0 \end{aligned}$$

$\square$ Step 3

Step 4 - More generally: if

$$P(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$$

then we can reduce to the previous case using $y = cz$ for some $c > 0$

$$\frac{P(z)}{c^n} = y^n + \frac{a_{n-1}}{c} y^{n-1} + \dots + \frac{a_0}{c^n}$$

by making c big enough that

$$\left|\frac{a_{n-1}}{c}\right| + \dots + \left|\frac{a_0}{c^n}\right| < 1$$

□ Step 4

Step 5 - Most general: find roots of

$$\frac{P(z)}{a_n}$$

which is covered by step 4

□ theorem.