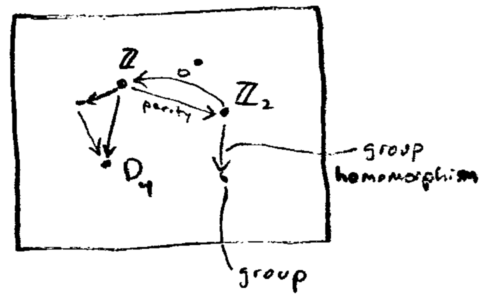
31 Jan 07

THE BIG PICTURE

Each branch of mathematics is about some category:

ALGEBRA

- group theory – groups & group homomorphisms form the category Grp



- ring theory – rings & ring homs: the category Ring
- linear algebra – vector spaces & linear maps: the category Vect
- ⋮

ANALYSIS

- measure space & measurable functions: the category Meas
- ⋮

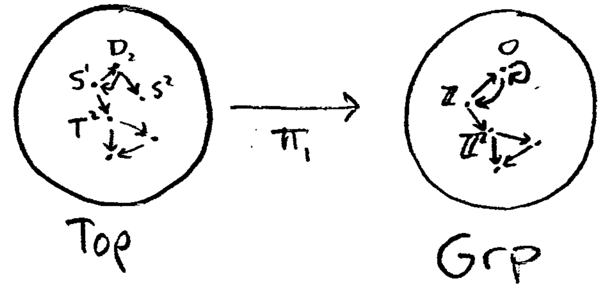
TOPOLOGY

- (topological) spaces & (continuous) maps: the category Top
- pointed spaces & pointed maps: the category Top_*
- ⋮

These categories are part of a single big picture, because there are things going between categories: functors!

Eg.

$$\pi_1 : Top_* \longrightarrow Grp$$



HW – 58.2 no proofs needed
4,5 proofs

Classify capital letters according to
1) homeomorphism type (no proofs)
2) homotopy type

In any category, it's important to try to tell when two objects are isomorphic. (In Top, we say "homeomorphic.") Alas, it's an endless task to classify spaces and see when they're homeomorphic. This is sad, compared to, say, linear algebra: in Vect, two vector spaces V, W are isomorphic if there are linear maps $f: V \to W$ and $g: W \to V$ which are inverses; V and W are isomorphic if and only if they have the same dimension.

For Top, there's no simple test for when things are homeomorphic. But there are easy tests to show they aren't. This is where functors come in.

<u>Defn</u>: A <u>category</u> $\mathcal{C}$ consists of

- a collection of <u>objects</u>; if x is an object of $\mathcal{C}$ we write $x \in \mathcal{C}$
- given two objects $x, y \in \mathcal{C}$, a set of <u>morphisms</u> from x to y; if f is a morphism from x to y, we write $f: x \to y$ or $x \xrightarrow{f} y$ (even though f is not necessarily a function and x, y aren't necessarily sets)
- given morphisms $f: x \to y$ and $g: y \to z$ there's a unique <u>composite morphism</u> $g \circ f: x \to z$
- given an object $x \in \mathcal{C}$, there's a unique <u>identity</u> morphism $1_x: x \to x$. (Sometimes we get lazy and write just 1.)
- The associative law holds for composition: given $f: w \to x$, $g: x \to y$, $h: y \to z$
$$(h \circ g) \circ f = h \circ (g \circ f)$$
- The left and right unit laws hold: given $f: x \to y$,
$$1_y \circ f = f = f \circ 1_x$$

For example:

Top is a category
Top_* is a category
Grp is a category
Diff (manifolds + smooth maps) is a category

<u>Defn</u>: Given categories $\mathcal{C}, \mathcal{D}$, a functor $F: \mathcal{C} \to \mathcal{D}$ is

- a "function" sending each object $x \in \mathcal{C}$ to an object $F(x) \in \mathcal{D}$
- for each $x, y \in \mathcal{C}$ a function sending every morphism $f: x \to y$ to a morphism $F(f): F(x) \to F(y)$

such that

- composition is preserved: given

$$f: x \to y \quad \text{and} \quad g: y \to z$$

we get

$$F(g \circ f) = F(g) \circ F(f)$$

- identities are preserved:

$$F(1_x) = 1_{F(x)}$$

For example:

$$\pi_1 : Top_* \longrightarrow Grp$$