2 Feb 07

Homotopy Equivalence & Homotopy Type

Our functor

$$\pi_1 : Top_* \longrightarrow Grp$$

tells us when spaces & maps are <u>different</u>. If you have two maps $f, g : (X, x_0) \longrightarrow (Y, y_0)$, you get

$$\pi_1(f), \pi_1(g) : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

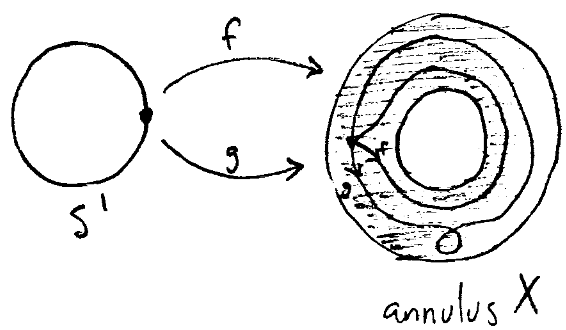
Clearly "if

$$f = g \text{ then } \pi_1(f) = \pi_1(g)\text{"}$$

and its contrapositive, "if

$$\pi_1(f) \neq \pi_1(g) \text{ then } f \neq g\text{"}$$

are true. But the converse isn't necessarily true:



f, g homotopic, not equal, but map to same element of $\pi_1(X, \cdot)$

Really π_1 only tells us about whether maps are homotopic, not equal. The book is jumping back + forth between Top and Top_* in an old-fashioned way.

We're going to stay in Top_*, so

<u>Def</u>: Given pointed maps $f, g : (X, x_0) \longrightarrow (Y, y_0)$, a <u>pointed</u> <u>homotopy</u> F from f to g is a map

$$F : X \times I \longrightarrow Y$$

such that:

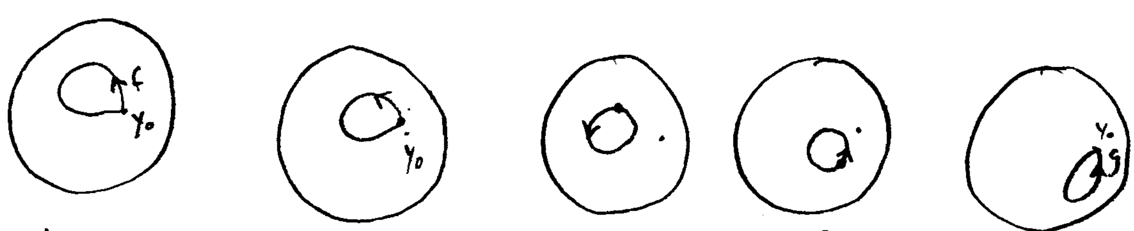
1) it's a <u>homotopy</u> from f to g:

$$\forall x \in X, \quad \begin{array}{l} F(x,0) = f(x) \\ F(x,1) = g(x) \end{array}$$

2) it's <u>pointed</u>:

$$\forall t \in I, \ F(x_0, t) = y_0.$$

<u>Example</u> of a <u>non</u>-pointed homotopy:



$t=0$ $\qquad$ $t=\frac{1}{4}$ $\qquad$ $t=\frac{1}{2}$ $\qquad$ $t=\frac{3}{4}$ $\qquad$ $t=1$

$\underbrace{t=\tfrac14,\ t=\tfrac12,\ t=\tfrac34}$

fail criterion 2:
these aren't elements
of $\pi_1(X, x_0)$

Defn: Given $f, g : (X, x_0) \longrightarrow (Y, y_0)$, we write $f \simeq g$ to mean $\exists$ pointed homotopy from f to g.

WARNING: Munkres uses the same symbol to denote "plain old" homotopy in Top.

Lemma 58.1: If $f, g : (X, x_0) \to (Y, y_0)$ and $f \simeq g$ then $\pi_1(f) = \pi_1(g)$.

Pf.

Given $[\gamma] \in \pi_1(X, x_0)$, show $\pi_1(f)[\gamma] = \pi_1(g)[\gamma]$.

Since $f \simeq g$, $\exists$

$$F : X \times I \to Y$$

with

$$\forall x \in X \quad F(x,0) = f(x) \;\&\; F(x,1) = g(x)$$

$$\forall t \in I \quad F(x_0, t) = y_0.$$

So

$$\phi(s,t) = F(\gamma(s), t) \qquad s, t \in I$$

is a path homotopy from $f \circ \gamma$ to $g \circ \gamma$:

$$\left.\begin{aligned} F(\gamma(s), 0) &= (f \circ \gamma)(s) \\ F(\gamma(s), 1) &= (g \circ \gamma)(s) \end{aligned}\right] \text{homotopy from } f\circ\gamma \text{ to } g \circ \gamma$$

$$\left.\begin{aligned} F(\gamma(0), t) &= F(x_0, t) = y_0 \\ F(\gamma(1), t) &= F(x_0, t) = y_0 \end{aligned}\right] \text{path homotopy}$$

So:

$$\pi_1(f)[\gamma] = [f \circ \gamma]$$
$$= [g \circ \gamma] \quad \text{since they're homotopic}$$
$$= \pi_1(g)[\gamma]$$

$\square$

What can we learn about pointed spaces by applying π_1?

1) $(X, x_0) = (Y, y_0) \Rightarrow \pi_1(X, x_0) = \pi_1(Y, y_0)$

2) If there's a pointed homeomorphism $f : (X, x_0) \xrightarrow{\sim} (Y, y_0)$, i.e. a pointed map with a pointed inverse map, then

$$\pi_1(f) : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(Y, y_0)$$

is an isomorphism.

Pf.

$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism

$\pi_1(f^{-1}) : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is a homomorphism

$$\pi_1(f) \circ \pi_1(f^{-1}) = \pi_1(f \circ f^{-1}) = \pi_1(1_Y)$$
$$\pi_1(f^{-1}) \circ \pi_1(f) = \pi_1(f^{-1} \circ f) = \pi_1(1_X)$$

So $\pi_1(f)$ is an isomorphism.

$\square$

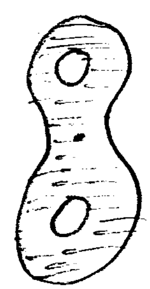
3) If (X, x_0) and (Y, y_0) are "pointed homotopy equivalent," then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

<u>Defn</u>: A pointed map $f:(X, x_0) \to (Y, y_0)$ is a <u>pointed homotopy equivalence</u> if it has a <u>homotopy inverse</u> $g:(Y, y_0) \to (X, x_0)$ s.t.

$$g \circ f \simeq 1_X \qquad f \circ g \simeq 1_Y$$

<u>Defn</u>: (X, x_0) is pointed homotopy equivalent to (Y, y_0) if $\exists f$ as above.

Example:

 is not homeomorphic to 

since removing the basepoint of the one on the left causes it to become disconnected, but there's no such point on the right.

But they are pointed homotopy equivalent.